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PURE MATHEMATICS,

INCLUDING

ARITHMETIC, ALGEBRA, GEOMETRY,

AND

PLANE TRIGONOMETRY.

BY

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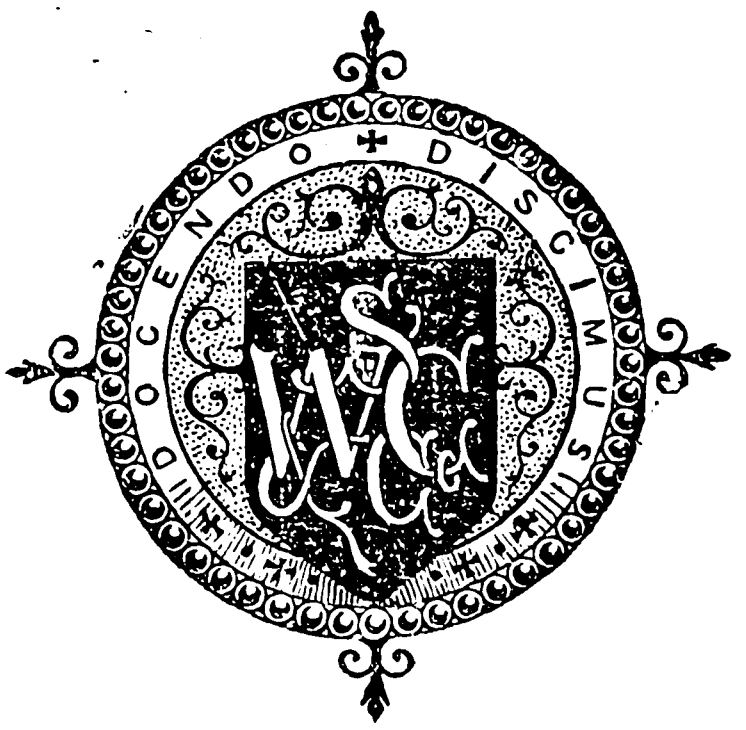

LONDON AND GLASGOW:

WILLIAM COLLINS, SONS, & COMPANY.

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TO THE REVEREND

WILLIAM FRY, M.A.,

HONORARY CANON OF PETERBOROUGH,

AND SECRETARY OF THE

LEICESTER ARCHIDIACONAL BOARD OF EDUCATION,

This Work is Offered,

AS A TRIBUTE OF RESPECT FROM ONE OF HIS

OLD PUPILS.

PREFACE.

THE special object of the present work is to meet the requirements of the Science and Art Department's Examinations in the first three stages of Pure Mathematics as set down in the Syllabus of the Science Directory. This will account for the arrangement of the subject matter. I hope, however, that it will be found not unsuitable as a general class-book in Elementary Mathematics.

In the Arithmetical Section my object has been to deduce the rules from first principles, avoiding as much as possible algebraical considerations.

The Geometry consists simply of the first three books of Euclid's Elements with exercises, the only point calling for remark being the marginal notes. I have found it useful in class teaching to put down upon a blackboard the chief steps of the proposition—just those points, in fact, which it is necessary to retain in the memory; and to encourage the pupil to depend upon himself for supplying the connecting links. This skeleton, as it were, of the proposition is placed in the margin, with the hope that it will be specially appreciated by many students of the industrial classes to whom the language of Euclid is ordinarily an insuperable barrier. Particular reference is here made to such of those classes as have grown up to almost manhood without any mathematical training.

In the Algebraical Section of Stage I., the more difficult examples are set particularly for the exercise of those

students who take up the work of Stage II. without having previously used this book for Stage I.

The amount of Plane Trigonometry included in this volume is small, extending only to the solution of triangles and the simpler cases of heights and distances. There is, however, sufficient to fully cover the requirements of Stage II. of the Government Syllabus. The treatment of the higher parts of Algebra and Plane Trigonometry, as well as that of Spherical Trigonometry, is reserved for a second volume.

E. A.

LEICESTER, *November, 1873.*

CONTENTS.

STAGE I.

SECTION I.—ARITHMETIC.

CHAP. I.—THE FUNDAMENTAL PRINCIPLES AND RULES APPLIED TO WHOLE NUMBERS AND DECIMAL FRACTIONS, ... 9

,, II.—THE TREATMENT OF FRACTIONS CONSIDERED AS RATIOS, ... 16

,, III.—APPLICATION TO CONCRETE QUANTITIES, . . 42

,, IV.—THE METRIC SYSTEM, ... 49

,, V.—PROPORTION, ... 61

,, VI.—APPLICATION TO ORDINARY QUESTIONS OF COMMERCE AND TRADE, ... 68

MISCELLANEOUS EXAMPLES, ... 85

SECTION II.—GEOMETRY.

EUCLID'S ELEMENTS, BOOK I., ... 89

SECTION III.—ALGEBRA.

CHAP. I.—ELEMENTARY PRINCIPLES, ... 141

,, II.—ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION, ... 155

,, III.—INVOLUTION AND EVOLUTION, ... 178

,, IV.—GREATEST COMMON MEASURE, AND LEAST COMMON MULTIPLE, ... 195

,, V.—FRACTIONS, ... 204

,, VI.—SIMPLE EQUATIONS AND PROBLEMS PRODUCING THEM, ... 209

STAGE II.

SECTION I.—GEOMETRY.

EUCLID'S ELEMENTS, BOOK II., ... 234

EUCLID'S ELEMENTS, BOOK III., ... 254

SECTION II.—ALGEBRA.

CHAP. I.—QUADRATIC EQUATIONS, ... 296

,, II.—PROBLEMS PRODUCING QUADRATIC EQUATIONS, . 312

,, III.—INDICES, ... 316

,, IV.—SURDS, ... 319

,, V.—RATIO AND PROPORTION, ... 327

SECTION III.—PLANE TRIGONOMETRY.

CHAP. I.—MODES OF MEASURING ANGLES BY DEGREES AND GRADES, ... 333

,, II.—THE GONIOMETRIC FUNCTIONS, ... 336

,, III.—CONTRARIETY OF SIGNS—CHANGES OF MAGNITUDE AND SIGN OF THE TRIGONOMETRICAL RATIOS THROUGH THE FOUR QUADRANTS, . . . 342

,, IV.—TRIGONOMETRICAL RATIOS—CONTINUED.—ARITHMETICAL VALUES OF THE TRIGONOMETRICAL RATIOS OF 30°, 45°, 60°, &C., . . . 346

,, V.—LOGARITHMS, ... 354

,, VI.—THE USE OF TABLES, ... 358

,, VII.—PROPERTIES OF TRIANGLES, ... 366

,, VIII.—SOLUTION OF RIGHT-ANGLED TRIANGLES, . 374

,, IX.—SOLUTION OF OBLIQUE-ANGLED TRIANGLES, . 377

,, X.—HEIGHTS AND DISTANCES, ... 381

ANSWERS, ... 387

MATHEMATICS.

FIRST STAGE.

SECTION I.

ARITHMETIC.

CHAPTER I.

THE FUNDAMENTAL PRINCIPLES AND RULES APPLIED TO WHOLE NUMBERS AND DECIMAL FRACTIONS.

Notation and Numeration.

1. We learn from elementary books on Arithmetic, that figures have a local as well as an intrinsic value, and that the local value of a figure increases tenfold, or diminishes tenfold, according as its position is changed from right to left, or from left to right. Thus, commencing with the right hand figure of an ordinary number, the respective figures of the number stand for units, tens, hundreds, thousands, &c.; or, beginning with the left hand figure, which, we will suppose, stands for thousands, the respective figures represent thousands, hundreds, tens, units. Let us carry this principle a little further. Take the figures 68754, and suppose that 7 represents 7 units; the question then arises as to the number represented by 68754. Now, as 7 is the units' figure, we have evidently, by the above principle, 6 hundreds, 8 tens, 7 units; and further, remembering that the

local value of a figure decreases tenfold for every remove to the right, the 5, on our supposition, must represent 5 tenths, and the 4 must represent 4 hundredths. Let us, as is usual in numbers thus represented, mark the units' figure by placing a dot to the right of it. Thus, 357·2605 will then represent 3 hundreds, 5 tens, 7 units, 2 tenths, 6 hundredths, 5 ten-thousandths; and to take one other example, ·3065, where the units' figure, though not expressed, is actually 0, will represent 3 tenths, 6 thousandths, 5 ten-thousandths. The dot is called the decimal point, and the digits to the right are called decimals, because they represent portions of the unit obtained by cutting it up into a number of equal parts, which is always some *power* of 10. It may be remarked, that 10 is called the first power of 10; 100, or 10 × 10, the second power, sometimes written 10^2; 1000, or 10 × 10 × 10, the third power, written 10^3, and so on.

To make the subject clear, let us see what the decimals, ·237, ·2370, ·0237 respectively represent. Now, the digits 2, 3, 7, in the first two decimals, are in exactly the same position with regard to the decimal point, and the respective digits in each have the same absolute value; moreover, the cipher affixed to the right of the decimal ·2370, has no intrinsic value, and hence the two decimals, ·237, and ·2370, have the same absolute value. And since the reasoning is the same, no matter how many ciphers are *affixed* to the right, we get the following important principle:—

The value of a decimal is not altered by affixing ciphers to the right.

We will now compare the first and third of our examples, namely, the decimals ·237 and ·0237. The cipher which is here *prefixed* to the left, has again no intrinsic value; but it has removed the digits 2, 3, 7, one stage to the right, and has, therefore, diminished their local value tenfold. The effect of prefixing the cipher, is therefore to diminish the absolute value of the decimal tenfold, and as every additional cipher so prefixed has a similar effect, we get another fundamental principle, as follows:—

The value of a decimal is diminished tenfold for every cipher prefixed.

After the above it is easy to see that we have only to remove the decimal point one place to the right or left, in order to increase or diminish respectively the value of a decimal tenfold. And if we allow the term decimal to include numbers which are greater than unity, as 35·721, we may extend the principle thus:—

A decimal may be divided by 10, by removing the dot *one* place to the left; by 100 or 10^2 by removing the dot *two* places to the left; by 1000 or 10^3 by removing the dot *three* places to the left, and so on; and further, a decimal may be multiplied by 10, 100, 1000, &c., that is, by 10, 10^2, 10^3, &c., by removing the dot 1, 2, 3, &c., places respectively to the right. Thus, 6872·3476 divided by 10, 100, 1000 respectively, becomes 687·23476, 68·723476, 6·8723476; and multiplied by the same becomes 68723·476, 687234·76, 6872347·6 respectively.

Addition and Subtraction.

2. If the student has understood the preceding article, he will at once perceive that, provided we keep the units' figure under the units' figure in every case, there is no difference between the addition and subtraction of decimals, and the addition and subtraction of ordinary integers. All he has to take care of is that the decimal points are kept under each other.

Ex. 1.—Add together 325·02, ·647, 5·6073, ·00214, 290, and 4·7001. Proceeding as in ordinary addition:

325·02
·647
5·6073
·00214
290·
4·7001
625·97654

Ex. 2.—Take 6·291 from 18·3064. Proceeding as in ordinary subtraction:

18·3064
6·291
12·0154

Ex. 3.—Find the difference between 15·02 and ·6732.

$$\begin{array}{r} 15\cdot02 \\ \cdot6732 \\ \hline 14\cdot3468 \end{array}$$

NOTE.—In an example of this kind, where the number of decimal figures in the lower line exceeds the number in the upper, it is advisable to mentally supply ciphers to make up the deficiency in the upper line. This may be done, as we have seen, without altering the value of the upper line.

Multiplication.

3. Suppose we have to multiply 2·935 by 6·34, and let us suppose the dot in each case removed to the extreme right. Then (Art. 1), we have multiplied the number 2·935 by 1000. and the number 6·34 by 100, and we have obtained the numbers 2935· and 634· respectively. As these numbers are integers, we may omit the dot, and write them 2935 and 634. Now $2935 \times 634 = 1860790$, but as we increased our original numbers one thousand and one hundredfold respectively, it is evident that our product is increased 1000×100, or one hundred thousandfold. Dividing, therefore, the above result, 1860790 by 100000, or what is the same thing (Art. 1), writing it 1860790· and removing the dot 5 places to the left, we get for our product of the numbers 2·935 and 6·34 the result, 18·60790. We may remark that the number of decimal figures in the product, namely, 5, is the sum of the numbers of decimal figures in the two given numbers.

We have, therefore, the following rule for multiplication :—Multiply the given numbers exactly as integers, regardless of the decimal points, and after the operation is finished, point off as many decimal figures in the product as there are together in the multiplier and multiplicand.

Ex. 1.—Multiply 6·35 by ·1703.

$$\begin{array}{r} \cdot1703 \\ 6\cdot35 \\ \hline 8515 \\ 5109 \\ 10218 \\ \hline 1081405 \end{array}$$

Now, the number of decimal figures in the multiplier and multiplicand together, is (4 + 2), or 6, and therefore we mark off 6 decimal figures in our product. This gives us 1·081405.

Ex. 2.—Multiply ·0063, by ·017.

$$\begin{array}{r} \cdot 0063 \\ \cdot 017 \\ \hline 441 \\ 63 \\ \hline 1071 \end{array}$$

And pointing off (4 + 3), or 7 decimal figures, we obtain for our product ·0001071.

Division.

4. Suppose we have to divide ·76875 by 6·25. We will proceed as in the case of multiplication, by imagining the decimal points in each number removed to the extreme right. The numbers will then be 76875·, and 625·, or, omitting the dot, as they are now integers, they will be 76875, and 625.

Proceed now as in ordinary division (which operation it is unnecessary to explain), and we get for our quotient 123.

Now we must remember that we have increased our dividend 100,000 or 10^5-fold, and that, consequently, our quotient will require to be diminished 10^5-fold. This is done (Art. 1) by removing the dot of the number 123· five places to the *left*. But, before doing that, we know that the divisor has been increased 100 or 10^2-fold, and on this account our quotient must be increased 10^2-fold. This is done (Art. 1) by removing the dot two places to the *right*. Hence, to get the true quotient of ·76875 by 6·25, we must remove the dot from the extreme right of the number 123·, five minus two, or three places to the left. Now three is the excess of the number of decimal figures in the dividend over the number in the divisor. We hence arrive at the following rule :—

Proceed as in ordinary division, and when all the figures of the dividend have been brought down, and the remainder, if any, obtained, cut off as many decimal figures in the

quotient, as the number of decimal figures in the dividend exceeds the number in the divisor.

NOTE.—When the number of decimal figures in the dividend is less than the number in the divisor, affix a sufficient number of ciphers to make the number of decimals in the dividend *equal* to the number in the divisor. After finishing the operation of ordinary division, there will be *no* decimal figures to cut off in the quotient. If there be a remainder, and the division carried on further, by affixing ciphers to the successive remainders, all the quotient figures thus obtained will be decimals.

Ex. 1.—Divide 117·85088 by 6·272.

```
6·272)117·85088(1879
      6272
     55130
     50176
      49548
      43904
       56448
       56448
```

We see that there are five decimal figures in the dividend, and three in the divisor, and so we cut off (5 – 3) or 2 in the quotient. The answer is therefore 18·79.

Ex. 2.—Divide 527·2 by ·0008.

Here it will be necessary to affix three ciphers to the dividend, and the operation will stand thus—

```
·0008)527·2000
      659000
```

As there is no remainder, and the number of decimal figures in the dividend is equal to that in the divisor, we have none to cut off. The answer is therefore 659000.

Ex. 3.—Divide 463·7 by 2·769 to four places of decimals.

Here we must affix two ciphers to the dividend, and the operation, as far as the ordinary remainder of long division, stands thus :—

$$
\begin{array}{l}
2{\cdot}769)463{\cdot}700(167 \\
\quad 2769 \\
\quad \overline{18680} \\
\quad 16614 \\
\quad \overline{{\cdot}20660} \\
\quad 19383 \\
\quad \overline{1277}
\end{array}
$$

The quotient up to this point is integral; but, as we have a remainder, we must continue the operation of division, first placing a dot at the right of the figures in the quotient, and affixing a cipher to the present and each successive remainder, until we have the requisite number of decimals in the quotient. By thus proceeding, it is easy to see we arrive at an answer—167·4611.

Ex. I.

1. Increase the numbers 4·523, 29, ·02367, ·07 respectively 10, 100, 1,000, 10,000-fold.

2. Divide by inspection the numbers 0·05, 1111, 4·0020, 45 respectively by 100, 10,000, 1,000, 10.

3. Express in words ·3467, 34·67, ·0003467, 3·467; and compare the values of the last three with the first.

4. Add together—
   (1.) 6·732, 14·9, ·0064, 14·27006.
   (2.) ·00291, ·291, 29, 29100·9.
   (3.) ·821, 29·60, 29·6, ·0029.

5. By how much does 5 exceed 4·2763, and 16·021 exceed 12·70009?

6. Find the value of—
   (1.) 74·25 + ·0067 − 3·0298 + 1·032 − 2·73.
   (2.) 3·276 − 8·2409 + 10·0326 − ·00091.
   (3.) 2·5 − ·00029 − 7·364 + 5·2791.

7. What number added to four thousandths will give three hundredths, and what number subtracted from 8,000 units will give 291 units 29 hundredths?

8. Find value of—

(1.) ·3 × ·3. (2.) 9·001 × 27·06. (3.) 0·403 × ·009. (4.) ·17 × ·017 × 100. (5.) ·3 × ·005 × 6·4. (6.) $(·4)^2 \times (·032)^2$.

9. Find the quotient of—

(1.) 79·4 by ·397. (2.) 5·928 by 4742·4. (3.) 28 by ·007. (4.) ·6426 by 2·8. (5.) $(·24)^2$ by 9·6. (6.) 1·806 by $(1·9)^2$.

10. Given the quotient ·00073, the dividend 124·1, find the divisor when there is no remainder.

11. What is the value of—

(1.) $\{2 - ·815\} \div \{·201 + ·039 - ·002\}$?

(2.) $\{(·693)^2 - (·307)^2\} \div \{·693 - ·307\}$?

12. If I add ·061 to a certain number, and then divide the result by 290, I get ·0009 for a quotient; what is the number?

CHAPTER II.

THE TREATMENT OF FRACTIONS CONSIDERED AS RATIOS.

5. A **Fraction** is a part or parts of a whole. It is generally expressed by two numbers, the one placed above the other and separated by a line. The lower number expresses the number of equal parts into which the whole quantity has been divided, and the upper number, how many of those parts are taken. Thus $\frac{3}{5}$ is a fraction, and tells us that unity has been divided into 5 equal parts, and that we have taken 3 of those parts. The fraction $\frac{3}{5}$ is read three fifths; each of the equal parts into which unity has been divided being called a fifth.

The denominator of a fraction is the lower number, and therefore shows the number of equal parts into which we have divided the unit.

The numerator is the upper number, and tells us how many of these equal parts are taken.

When the numerator is less than the denominator, the quantity expressed is actually less than a whole. The quantity is therefore a real or proper fraction. Again, when

the numerator and denominator are both integral numbers, the fraction is termed a simple fraction.

Thus $\frac{3}{7}$, $\frac{7}{9}$, $\frac{10}{11}$, are both proper and simple fractions.

It is however usual to include in the term fraction every expression which contains one or more simple fractions, with or without integral numbers.

Thus $\frac{2}{3}$, $\frac{4}{1}$, $\frac{12}{5}$, $6\frac{1}{2}$, $\frac{3}{2\frac{1}{9}}$, $\frac{4\frac{1}{5}}{7\frac{1}{3}}$, $4\frac{1}{7}$ of $\frac{5}{6}$ of $\frac{2\frac{1}{8}}{\frac{11}{16}}$, are all included in the term fraction.

They are, moreover, called **vulgar fractions** to distinguish them from decimals, which, as will be shown further on, may be looked upon as fractions, according to the above definition, whose denominators are powers of 10, and not expressed but understood.

It is convenient to classify fractions as follows :—

(1.) A **proper fraction** is one whose numerator is less than its denominator, as $\frac{2}{7}$, $\frac{3}{5}$, $\frac{12}{13}$, $\frac{2\frac{1}{4}}{9}$.

(2.) An **improper fraction** is one whose numerator is not less than its denominator, as $\frac{9}{4}$, $\frac{8}{8}$, $\frac{12}{11}$, $\frac{5\frac{1}{4}}{2\frac{1}{6}}$.

(3.) A **simple fraction** is one whose numerator and denominator are both integral numbers, as $\frac{5}{3}$, $\frac{2}{7}$, $\frac{6}{19}$.

(4.) A **mixed number** is a fraction expressed by an integer and a simple fraction, as $2\frac{1}{3}$, $4\frac{2}{5}$, $3\frac{5}{7}$.

(5.) A **complex fraction** has its numerator, or denominator, or both, in a fractional form, as $\frac{3}{5\frac{1}{7}}$, $\frac{2\frac{1}{3}}{6\frac{1}{9}}$, $\frac{\frac{5}{7}}{11}$.

(6.) A **compound fraction** is a fraction of a quantity which is itself fractional, as $\frac{5}{7}$ of $2\frac{1}{3}$, $2\frac{1}{9}$ of $6\frac{1}{15}$ of $\frac{3}{7\frac{1}{6}}$.

6. In the preceding article we have spoken of fractions in the ordinary way. We will now approach them from a different point of view.

By the term ratio we understand the result of the comparison of two quantities with regard to magnitude. There are two kinds of ratios—ratio by difference or subtraction,

and ratio by quotient or division. Thus we may consider how much one quantity exceeds another, or we may consider how many times one quantity contains another. The former kind is called the **arithmetical ratio**, and the latter the **geometrical ratio.** We shall speak only of the latter.

DEFINITION.—The ratio between two quantities is that multiple, part, or parts which the former is of the latter.

It is evident that a ratio can exist only between quantities of the same kind; thus, we may compare 12 horses and 6 horses, but not 9 men and 4 miles. And if the quantities are reduced to the same denomination, we may treat the quantities as abstract, just as we find the quotient of one concrete quantity by another, by reducing them both to the same denomination, and dividing as if they were abstract quantities.

Now, according to what has been stated above, the ratio of 12 to 7, or, as it is usually written, 12 : 7, is obtained by dividing 12 by 7; and this is the same thing as dividing unity or 1 into 7 equal parts, and computing how much 12 of such parts amount to. It hence follows that the fraction $\frac{12}{7}$ is properly expressed by the ratio 12 : 7.

The first term of a ratio is called the **antecedent**, and the second is called the **consequent;** and hence we may consider a fraction as a ratio, the numerator being the antecedent of the ratio, and the denominator the consequent.

When the antecedent is *equal* to the consequent, the ratio is said to be a ratio of equality; and it is said to be a ratio of *less or greater inequality* according as the antecedent is less or greater than the consequent.

Thus, 6 : 6 is a ratio of equality.
3 : 4 is a ratio of less inequality.
11 : 9 is a ratio of greater inequality.

The student will therefore have no difficulty in assenting to the following definitions:—

(1.) A proper fraction is a ratio of less inequality.

(2.) An improper fraction is a ratio of equality or of greater inequality.

(3.) A simple fraction is a ratio whose terms are integers.

Thus, $\frac{3}{5}$ = 3 : 5 is a simple fraction.

(4.) A mixed number is a ratio of greater inequality, whose antecedent has been actually divided by its consequent, and the result expressed as an integer and simple fraction.

Thus, $2\frac{3}{7} = {}^{1}\!/_{7}{}^{7} = 17 : 7.$

(5.) A complex fraction is a ratio, whose antecedent, or consequent, or both, are not integers.

Thus, $\frac{3\frac{1}{2}}{1\frac{3}{4}}, \frac{2}{7\frac{1}{6}}, \frac{9\frac{1}{8}}{5}$ = respectively to $3\frac{1}{2} : 1\frac{3}{4}$, $2 : 7\frac{1}{6}$, $9\frac{1}{8} : 5$ are complex fractions.

(6.) A compound fraction is an expression containing two or more ratios to be compounded together.

Thus $\frac{3}{4}$ of $\frac{7}{9}$ contains the ratios $3 : 4$ and $7 : 9$ to be compounded; $\frac{2\frac{1}{2}}{7}$ of $\frac{3}{6\frac{1}{4}}$ of $\frac{7\frac{1}{2}}{9}$ contains the ratios $2\frac{1}{2} : 7$, $3 : 6\frac{1}{4}$, $7\frac{1}{2} : 9$ to be compounded.

7. *A fraction whose numerator and denominator are multiplied or divided by the same quantity is not altered in value.*

Suppose, for example, we multiply the numerator and denominator of the fraction $\frac{3}{7}$ each by 4, we get $\frac{3}{7} = \frac{1}{2}\frac{2}{8}$. Now the ratio of $3 : 7$ is, from the definition of a ratio, four times as small as the ratio $(3 \times 4) : 7$ or $12 : 7$; and the ratio $12 : 7$ is, for the same reason, four times as great as the ratio $12 : (7 \times 4)$ or $12 : 28$. It therefore follows that the ratio $3 : 7$ is exactly equal to the ratio $12 : 28$, and consequently $\frac{3}{7} = \frac{1}{2}\frac{2}{8}$.

Again, suppose we divided each term of the fraction $\frac{1}{2}\frac{5}{7}$ by 3, we get $\frac{1}{2}\frac{5}{7} = \frac{5}{9}$. Now the ratio of $15 : 27$ is, from the definition of a ratio, three times as great as the ratio $(15 \div 3) : 27$ or $5 : 27$; and, again, the ratio $5 : 27$ is three times as small as the ratio $5 : (27 \div 3)$ or $5 : 9$. It therefore follows that the ratio $15 : 27$ = the ratio $5 : 9$, and consequently $\frac{1}{2}\frac{5}{7} = \frac{5}{9}$.

Cor.—An integer may be expressed as a fraction with any given denominator.

For we may consider an integer as a ratio whose consequent is 1, and we may multiply each term of this ratio by any given number without altering its value.

$$\text{Thus—}6 = \frac{6}{1} = \frac{6 \times 7}{1 \times 7} = \frac{42}{7}.$$

$$\text{Or,} \quad 6 = \frac{6}{1} = \frac{6 \times 9}{1 \times 9} = \frac{54}{9}.$$

8. To multiply a fraction by a whole number, we may either multiply the numerator by the number, or divide the denominator by it.

Ex. $\frac{8}{9} \times 3 = \frac{8 \times 3}{9} = \frac{24}{9}$; or we may proceed thus—

$$\frac{8}{9} \times 3 = \frac{8}{9 \div 3} = \frac{8}{3}.$$

As to the first method—

The ratio 8 : 9 will be evidently increased three times if we multiply its antecedent by 3; this follows from the definition of a ratio. We thus get the ratio (8 × 3) : 9 or 24 : 9.

It therefore follows that $\frac{8}{9} \times 3 = \frac{24}{9}$.

As to the second method—

The ratio 8 : 9 will be evidently increased three times if we make the consequent three times as small; and we thus get the ratio 8 : (9 ÷ 3) or 8 : 3; and hence it follows that $\frac{8}{9} \times 3 = \frac{8}{3}$.

It may be remarked that the two results, $\frac{24}{9}$ and $\frac{8}{3}$, are of exactly the same value (Art. 7), since the latter may be obtained from the former by dividing each of its terms by 3.

In actual practice we sometimes pursue the first method, and sometimes the second. If the denominator of the given fraction contains the multiplier as a factor, it is more convenient to use the second method, thus :—

$$\frac{11}{12} \times 3 = \frac{11}{12 \div 3} = \frac{11}{4}.$$

On the other hand, when the denominator does not contain the multiplier as a factor, we use the first method, thus :—

$$(1.) \quad \frac{7}{13} \times 5 = \frac{7 \times 5}{13} = \frac{35}{13}.$$

$$(2.) \quad \frac{9}{55} \times 10 = \frac{9 \times 10}{55}.$$

Here we see that the numerator and denominator have a common factor 5, and therefore, by Art. 7, if we divide them both by it, we have :—

$$\frac{9}{55} \times 10 = \frac{9 \times 2}{11} = \frac{18}{11}.$$

9. To divide a fraction by a whole number, we may either multiply the denominator by the number, or we may divide the numerator by it.

Ex.—Divide $1\frac{2}{7}$ by 4.

We may proceed thus, (1.) $1\frac{2}{7} \div 4 = \frac{12 \div 4}{17} = \frac{3}{17}$.

(2.) $1\frac{2}{7} \div 4 = \frac{12}{17 \times 4} = \frac{1}{6}\frac{2}{8}$.

As to the first method—

The ratio of 12 : 17 will evidently be diminished 4 times if we divide its antecedent 12 by 4. We thus get the ratio $(12 \div 4) : 17$ or 3 : 17 ; and it therefore follows that $1\frac{2}{7} \div 4 = \frac{3}{17}$.

As to the second method—

The ratio of 12 : 17 can also be diminished 4 times by increasing its consequent or divisor 4 times, so that we thus get the ratio $12 : (17 \times 4)$ or 12 : 68. It therefore follows that $1\frac{2}{7} \div 4 = \frac{1}{6}\frac{2}{8}$.

It may be remarked, as in Art. 8, that the two results, $\frac{3}{17}$ and $\frac{1}{6}\frac{2}{8}$, have exactly the same value, for the latter can be obtained from the former by multiplying each of its terms by 4 (see Art. 7).

And again, in actual practice, we usually take the first method when the numerator contains the divisor as a factor, but not otherwise. Thus—

(1.) $1\frac{8}{9} \div 6 = \frac{18 \div 6}{19} = \frac{3}{19}$.

(2.) $1\frac{1}{3} \div 5 = \frac{11}{13 \times 5} = \frac{1}{6}\frac{1}{5}$.

(3.) $1\frac{2}{7} \div 8 = \frac{12}{17 \times 8}$.

Here it is convenient to divide the numerator and denominator by the common factor 4 (Art. 7).

We then have $1\frac{2}{7} \div 8 = \frac{12 \div 4}{17 \times (8 \div 4)} = \frac{3}{17 \times 2} = \frac{3}{34}$.

10. To reduce a mixed number to an improper fraction. Looking at our definition of a mixed number (Art. 6), the following rule is evident:

Multiply the integral part by the denominator of the fractional part, and add in the numerator ; this gives the required numerator, and the denominator of the fractional part is the required denominator.

Ex.—Reduce $5\frac{2}{9}$, $7\frac{3}{5}$ to improper fractions.

(1.) $5\frac{2}{9} = \frac{5\times 9+2}{9} = 4\frac{7}{9}$.

(2.) $7\frac{3}{5} = \frac{7\times 5+3}{5} = 3\frac{8}{5}$.

11. To reduce a complex fraction to its equivalent simple fraction. Before stating a rule, let us take an example. Suppose we have to reduce $\frac{3\frac{1}{5}}{5\frac{2}{9}}$ to an equivalent simple fraction.

Now, by the last Art., $\frac{3\frac{1}{5}}{5\frac{2}{9}} = \frac{\frac{3\times 5+1}{5}}{\frac{5\times 9+2}{9}} = \frac{1\frac{6}{5}}{4\frac{7}{9}}$.

Again, the ratio $1\frac{6}{5} : 4\frac{7}{9}$ will not be altered in value if we multiply both its terms by the same quantity. Let us multiply them by 9 and it becomes $1\frac{6}{5}\times 9 : 4\frac{7}{9}\times 9$. Now, by Art. 8, $1\frac{6}{5}\times 9 = \frac{16\times 9}{5}$ and $4\frac{7}{9}\times 9 = \frac{47}{9\div 9} = \frac{47}{1} = 47$. The ratio then becomes $\frac{16\times 9}{5} : 47$. We will again multiply the terms of this ratio by the same quantity, viz. by 5, and we get the ratio $\frac{16\times 9}{5}\times 5 : 47\times 5$. Now, by Art. 8, $\frac{16\times 9}{5}\times 5 = \frac{16\times 9}{5\div 5} = \frac{16\times 9}{1} = 16\times 9$. Hence the ratio $1\frac{6}{5} : 4\frac{7}{9}$ is equivalent to the ratio $16\times 9 : 47\times 5$, and hence the fraction $\frac{1\frac{6}{5}}{4\frac{7}{9}} = \frac{16\times 9}{47\times 5}$.

Now 16 and 9 are called the *extreme* terms of the complex fraction $\frac{1\frac{6}{5}}{4\frac{7}{9}}$, and 5 and 47 are called its *mean* terms.

We arrive then at the following rule :—

RULE.—Bring the numerator and denominator to the form of simple fractions, then multiply together the extreme terms for a new numerator, and the mean terms for a new denominator.

Thus, $\frac{7\frac{1}{5}}{2\frac{1}{9}} = \frac{\frac{7\times 5+1}{5}}{\frac{2\times 9+1}{9}} = \frac{\frac{36}{5}}{\frac{19}{9}} = \frac{36\times 9}{5\times 19} = 3\frac{24}{95} = 3\frac{9}{5}\frac{3}{9}$.

12. To reduce a compound fraction to its equivalent simple fraction.

Let it be required to find the simple fraction equivalent to the compound fraction $\frac{3}{4}$ of $\frac{5}{7}$.

Now $\frac{3}{4}$ of $\frac{5}{7}$ is the ratio 3 : 4, where the unit of this ratio is $\frac{5}{7}$. It is therefore, from the definition of ratio, equal to 3 times this unit divided by 4.

Now 3 times $\frac{5}{7} = \frac{5}{7} \times 3 = \frac{5 \times 3}{7}$ (Art. 8),

And $\therefore$ 3 times $\frac{5}{7} \div 4 = \frac{5 \times 3}{7} \div 4 = \frac{5 \times 3}{7 \times 4} = \frac{3}{4} \times \frac{5}{7}$.

Hence we arrive at the result—

$$\frac{3}{4} \text{ of } \frac{5}{7} = \frac{3 \times 5}{4 \times 7}.$$

And in the same way we might show that

$$\frac{7}{8} \text{ of } \frac{4}{9} \text{ of } \frac{3}{5} \text{ of } 11 = \frac{7 \times 4 \times 3 \times 11}{8 \times 9 \times 5 \times 1}.$$

Hence the rule :—

RULE.—Multiply together the several numerators for a new numerator, and the several denominators for a new denominator.

Ex. $3\frac{1}{8}$ of $2\frac{1}{2}$ of $6 = \frac{3 \times 8 + 1}{8}$ of $\frac{2 \times 2 + 1}{2}$ of $\frac{6}{1} =$
$\frac{25}{8}$ of $\frac{5}{2}$ of $\frac{6}{1} = \frac{25 \times 5 \times 6}{8 \times 2 \times 1}$.

Ex. II.

1. Reduce the following to improper fractions—

$3\frac{1}{7}$, $4\frac{7}{9}$, $100\frac{7}{10}$, $35\frac{12}{7}$, $1\frac{11}{99}$, $11\frac{25}{64}$.

2. Reduce the integer 19 to sixths, tenths, thirteenths, eighteenths, nineteenths, and twentieths.

3. Bring the following fractions to integers, and reduce them respectively to fourths, sixths, eighths, tenths, twelfths, and fourteenths—

$\frac{36}{6}$, $\frac{91}{13}$, $\frac{84}{7}$, $\frac{1331}{121}$, $\frac{27}{3}$, $\frac{70}{5}$.

4. Multiply the following fractions each by 10, 11, 12—

$\frac{5}{7}$, $\frac{3}{16}$, $\frac{9}{12}$, $1\frac{2}{3}$, $7\frac{1}{9}$, $3\frac{1}{8}$.

5. By how much does 8 times the fraction $\frac{29}{4}$ exceed the quotient of $\frac{666}{37}$ by 3 ?

6. Divide the following fractions each by 6, 7, 18—

$\frac{37}{7}$, $\frac{14}{17}$, $1\frac{5}{6}$, $3\frac{7}{13}$, $103\frac{1}{7}$, $11\frac{5}{11}$.

7. Diminish the following ratios respectively 6, 7, 8-fold—

$12 : 5$, $3\frac{3}{13} : 4$, $9\frac{7}{9} : 2\frac{1}{5}$.

8. Simplify the expressions—

(1.) $\frac{1}{2}$ of $\frac{3}{7}$ of $\frac{5}{8}$ of $1\frac{5}{9}$. (2.) $\frac{7}{3}$ of 12 of $\frac{1}{1}\frac{1}{4}$ of $\frac{3}{1\,1}$.

(3.) $\frac{4}{1\,1}$ of $3\frac{1}{7}$ of $\frac{7}{9}$ of $1\frac{1}{8}$. (4.) $\frac{1}{1}\frac{1}{0}$ of 2 of $\frac{1}{9}\frac{4}{}$ of $1\frac{1}{1\,1}$ of $\frac{5}{8}$ of $\frac{3}{7}$. (5.) $2\frac{1}{5}$ of $\frac{4}{9}$ of $1\frac{1}{2}$ of $4\frac{1}{1\,1}$.

9. Reduce to simple fractions—

$$\frac{1\frac{1}{4}}{2\frac{1}{2}}, \frac{2\frac{2}{7}}{3\frac{3}{1\,4}}, \frac{5\frac{1}{3}\text{ of }6\frac{1}{2}}{3\frac{5}{7}}, \frac{2\frac{1}{4}}{7\frac{1}{9}\text{ of }3\frac{1}{1}\frac{5}{6}}, \frac{2\frac{1}{9}\text{ of }3}{9\frac{1}{2}\text{ of }3\frac{1}{3}}, 7\frac{1}{9}\text{ of }\frac{3\frac{3}{5}}{4\frac{4}{7}}\text{ of }\frac{1}{1\,4}.$$

10. What is the difference between $2\frac{3}{7}$ of $3\frac{1}{9}$ of $2\frac{1}{3}\frac{3}{4}$ and $\frac{9\frac{6}{7}}{1\frac{1}{2}}$.

11. Find the integral value of $\frac{8\frac{1}{7}\text{ of }2\frac{1}{3} - 2\frac{4}{5}\text{ of }1\frac{3}{7}}{3\frac{1}{2}\text{ of }\frac{6}{7} + \frac{8}{1\,9}\text{ of }4\frac{3}{4}}$.

12. Add the sum of the fractions $\frac{4\frac{1}{5}}{\frac{3}{1\,0}}$, $\frac{8\frac{1}{3}}{2\frac{1}{1\,2}}$ to their difference.

13. To reduce a fraction to its lowest terms.

DEF. A fraction is in its lowest terms when its numerator and denominator have no common factor, or are prime to each other. (Among such common factors, we include either the numerator or denominator itself, when one of them happens to be a divisor of the other.)

When the numerator and denominator have a common factor, we may divide them both by it (Art. 7) without altering the value of the fraction. Now, the highest common factor of two or more numbers is called their greatest common measure, usually written G.C.M.

Hence we have the following rule:—

RULE.—To reduce a fraction to its lowest terms, divide the numerator and denominator by their G.C.M.

Ex.—Reduce $\frac{2\,4}{8\,4}$ to its lowest terms.

We can easily see that 12 is the G.C.M. of 24 and 84; hence, dividing each by 12, we get—

$$\frac{2\,4}{8\,4} = \frac{2\,4 \div 1\,2}{8\,4 \div 1\,2} = \frac{2}{7}, \text{ the fraction required.}$$

14. It is not, however, always easy to tell by inspection the G.C.M.; but, before giving a general method for determining it, it will be useful to make a few remarks as to the divisibility of numbers in certain cases.

A number is divisible as follows :—

By 2, when it is *even*.

By 3, when the sum of its digits is divisible by 3.

By 4, when the number formed by the last two figures is divisible by 4.

By 5, when it ends in 5 or 0.

By 6, when it is even, and is also divisible by 3.

By 8, when the number formed by the last three figures is divisible by 8.

By 9, when the sum of its digits is divisible by 9.

By 10, when it ends in 0.

By 11, when the sum of the digits in the odd places (that is, the sum of the 1st, 3rd, 5th, &c.) is equal to the sum of the digits in the even places, or the one exceeds the other by a multiple of 11.

By 12, when the number formed by its last two figures is divisible by 4, and the sum of its digits is a multiple of 3.

We may add also :—

(1.) A number is divisible by 37, when it is composed of digits which are repeated three times, or any multiple of three times, as 111, 333, 444444, &c.

(2.) A number which has three figures repeated in the same order is divisible by 7, 11, 13.

Thus 271271, 165165, 23023 are divisible by 7, 11, and 13 ; for the last may be written 023023.

(3.) A number which has four figures repeated in the same order is divisible by 73 and 137.

Thus 53245324, 2760276 are both divisible by 73 and 137 ; for the last may be written 02760276.

Hence a fraction may often then be reduced to its lowest terms by gradually striking out factors determined by inspection.

Ex.—Reduce $\frac{7\,9\,2}{2\,2\,4\,4}$ to its lowest terms.

Now 792 and 2244 are each divisible by 4, for the numbers 92 and 44, which are formed by the last two figures of each, are evidently so. Hence, dividing numerator and denominator by 4, we have $\frac{7\,9\,2}{2\,2\,4\,4} = \frac{7\,9\,2 \div 4}{2\,2\,4\,4 \div 4} = \frac{1\,9\,8}{5\,6\,1}$; each is evidently divisible by 3. Hence $\frac{7\,9\,2}{2\,2\,4\,4} = \frac{1\,9\,8 \div 3}{5\,6\,1 \div 3} = \frac{6\,6}{1\,8\,7}$;

and each is now evidently divisible by 11. Hence $\frac{792}{2244} = \frac{66 \div 11}{187 \div 11} = \frac{6}{17}$.

REMARK.—It may sometimes happen that, although we are able to tell by inspection some of the factors of numerator and denominator, none of them are common to both numerator and denominator. We cannot then strike them out, but we may use them to determine what would be left supposing they are struck out, and we may thus often come upon the G.C.M. of both numerator and denominator.

Ex.—Reduce $\frac{474}{2133}$ to its lowest terms.

Now 474 is even, and therefore divisible by 2; thus $474 \div 2 = 237$. Again, 2133 is divisible by 9, for the sum of its digits, viz., $(2 + 1 + 3 + 3)$ or 9 is so divisible; thus, $2133 \div 9 = 237$.

We have thus learned that, although 2 and 9 are not common factors, 237 is a common factor, and, in fact, the G.C.M. Dividing the numerator and denominator of the given fraction by 237, we get $\frac{474}{2133} = \frac{474 \div 237}{2133 \div 237} = \frac{2}{9}$.

We proceed now to give the general method of determining the G.C.M.

15. To find the G.C.M. of two numbers.

RULE.—Divide the greater number by the less, and if there be no remainder, the less is the G.C.M.; but if there be a remainder, make a divisor of this remainder, and a dividend of the first divisor; if there be a remainder again, make a divisor of it, and a dividend of the preceding divisor, and so on until there be no remainder. The last divisor will be the G.C.M.

Ex.—Find the G.C.M. of 282 and 799.

The operation will stand thus—

```
282)799(2
    564
    ---
235)282(1
    235
    ---
 47)235(5
    235
    ---
```

or thus—

282	799	2
	564	
235	282	1
	235	
47	235	5
	235	

The G.C.M. is therefore 47.

The reason of this rule is easy to see. 47 divides 235, and it therefore divides 235 + 47 or 282. Hence it is a common divisor of 282 and 235, and it is therefore a common divisor of 282 and 282 × 2 + 235 or of 282 and 799.

It is, moreover, the highest common divisor; for every number which divides 799 and 282 must also divide 799 − 282 × 2 or 235; and hence every number which is a measure of 799 and 282 is also a measure of 235 and 282. Similarly, every number which divides 235 and 282 will also divide 282 − 235 or 47, and hence every measure of 799 and 282 is a measure of 47. But no higher number than 47 can divide 47, and therefore 47 is the G.C.M of 799 and 282.

16. It is now easy to see how any fraction may be reduced to its lowest terms.

Ex.—Reduce $\frac{148}{703}$ to its lowest terms.

148	703	4
	592	
111	148	1
	111	
37	111	3
	111	

Hence 37 is the G.C.M, and dividing numerator and denominator by it, we get

$$\frac{148}{703} = \frac{4}{19}.$$

17. To find the G.C.M. of more than two numbers.

The following rule needs no explanation:—

RULE.—Find the G.C.M. of any two of the numbers, then the G.C.M. of this result and the third number, and so on. The last result will be the G.C.M. required.

Ex.—Find the G.C.M. of 282, 987, 658, 1128.
The operation will stand thus—

282	987		141	658	4	47	1128	24
	846			564			94	
141	282	2	94	141	1		188	
	282			94			188	
			47	94	2			
				94				

Hence the G.C.M. is 47.

Ex. III.

1. Resolve into prime factors 44, 64, 150, 252, 1269, 462.

2. Find the prime factor of 1386, 1720, 4608, 21175, 15972, 14256.

3. A number is said to be a perfect number when it is equal to the sum of its aliquot parts. Show that the following are each perfect numbers—6, 28, 496, 8128.

4. Show that 284 and 220 are a pair of amicable numbers; that is, such a pair that each is equal to the sum of the divisors of the other, unity being here counted as a divisor.

5. Reduce, by inspection, to their lowest terms, the following fractions:

(1.) $\frac{6}{15}$, $\frac{18}{24}$, $\frac{35}{40}$, $\frac{121}{990}$, $\frac{632}{3950}$, $\frac{548}{1233}$.
(2.) $\frac{95}{160}$, $\frac{75}{108}$, $\frac{165}{176}$, $\frac{1539}{2052}$, $\frac{185}{555}$, $\frac{1836}{2052}$.
(3.) $\frac{168}{216}$, $\frac{540}{900}$, $\frac{726}{1331}$, $\frac{204}{432}$, $\frac{16016}{35035}$, $\frac{1573}{71373}$.

6. Find the G.C.M. of

(1.) 304, 323. (2.) 413, 448. (3.) 377, 533. (4.) 1866, 2832. (5.) 1189, 1517. (6.) 4374, 5103. (7.) 168, 378, 602. (8.) 539, 616, 792. (9.) 780, 1092, 2145.

7. Reduce to their lowest terms the following fractions—

(1.) $\frac{975}{1625}$, $\frac{952}{1224}$, $\frac{702}{2574}$, $\frac{2068}{6721}$, $\frac{1392}{1972}$, $\frac{1555}{2799}$.
(2.) $\frac{12266}{18399}$, $\frac{474}{553}$, $\frac{1053}{1215}$, $\frac{194}{1746}$, $\frac{1302}{1344}$, $\frac{1599}{21106}$.

8. Show, without applying the rule for the G.C.M., that $\frac{7}{8}\frac{3}{2}\frac{4}{0}\frac{4}{8} = \frac{1}{1}\frac{7}{9}$, and that $\frac{2}{3}\frac{2}{6}\frac{5}{2}\frac{4}{6}\frac{2}{3}\frac{3}{7} = \frac{2}{3}\frac{3}{7}$.

9. It rains in a certain district 634 days out of every 2,219; express this fact in the simplest way possible.

10. Out of 1,659 men engaged in a battle, only 1,185 answered the roll-call in the evening; express by a ratio in its simplest terms the number missing in relation to the whole.

11. There are 40 numbers less than 100, and prime to it; what are they?

12. A, B, C can do a piece of work respectively in 318, 477, 795 hours; express the relative value of A, B, C as workmen by the simplest integral numbers possible.

The Least Common Multiple.

18. It is often necessary to express fractions as equivalent fractions, having a common denominator; and it is, moreover, convenient to have this denominator as small as possible. Now, there are always an infinite number of numbers which will contain each of the given denominators as a factor, and our problem is therefore to obtain the least of such numbers.

DEF.—The least common multiple (L.C.M.) of two or more numbers is the *least* number which contains each of the given numbers exactly.

RULE.—Arrange the numbers in a line, putting one of them as a divisor. Strike out the greatest factor common to this divisor, and each of the numbers separately, and place the several quotients in the line below; at the same time bring down every number prime to the divisor. Repeat the operation upon the second line, and so on until we have a line of numbers prime to each other. Multiply the several divisors and the numbers in the lowest line together, and their continued product will be the least common multiple.

Ex. 1.—Find the L.C.M. of 12, 16, 36, 45, 60.

36	12, 16, 36, 45, 60
5	1, 4, 1, 5, 5
	4, 1, 1

Hence the L.C.M. is $36 \times 5 \times 4 = 720$.

Ex. 2.—Find the L.C.M. of 15, 21, 60, 84, 140.

60	15, 21, 60, 84, 140
7	1, 7, 1, 7, 7
	1, 1, 1

Hence the L.C.M. is $60 \times 7 = 420$.

19. Reduction of fractions to equivalent fractions having the least common denominator.

It is evident that the least common denominator cannot be a less number than the L.C.M., and therefore the following rule needs no explanation:—

RULE.—Divide the L.C.M. of the given denominators by each denominator in turn, and multiply the corresponding numerator by the quotient. The product thus obtained is

the new numerator, and the L.C.M. is the least common denominator.

Ex. 1.—Reduce $\frac{2}{3}$, $\frac{5}{12}$, $\frac{7}{8}$, $\frac{11}{10}$ to equivalent fractions, having the least common denominator.

$$\begin{array}{r|l} 12 & 3,\ 12,\ 8,\ 10 \\ \hline & 1,\ \ 1,\ 2,\ \ 5 \end{array}$$

$\therefore$ The L.C.M. is $12 \times 2 \times 5 = 120$.

Dividing 120 by the respective denominators, we get as quotients 40, 10, 15, 12; and hence the given fractions become

$$\frac{2\times40}{120},\ \frac{5\times10}{120},\ \frac{7\times15}{120},\ \frac{11\times12}{120};$$

Or, $\frac{80}{120}$, $\frac{50}{120}$, $\frac{105}{120}$, $\frac{132}{120}$.

Ex. 2. Reduce to their least common denominator the following:—$\frac{11}{30}$, $\frac{13}{18}$, $\frac{52}{45}$, $\frac{29}{63}$.

$$\begin{array}{r|l} 45 & 30,\ 18,\ 45,\ 63 \\ \hline 2 & 2,\ \ 2,\ \ 1,\ \ 7 \\ \hline & 1,\ \ 1,\ \ \ \ \ 7 \end{array}$$

Hence the L.C.M. is $45 \times 2 \times 7 = 630$.

Dividing 630 by the respective denominators, we get for quotients 21, 35, 14, 10. Hence the required fractions are

$$\frac{11\times21}{630},\ \frac{13\times35}{630},\ \frac{52\times14}{630},\ \frac{29\times10}{630};\ \text{or,}\ \frac{231}{630},\ \frac{455}{630},\ \frac{728}{630},\ \frac{290}{630}.$$

Note 1.—The operation of dividing the L.C.M. of the denominators is often simplified by using the L.C.M. in its factorial form. Thus, in our present example, the L.C.M. is $45 \times 2 \times 7$. Now it is easy to see that the quotient of this by 30 or $3 \times 2 \times 5$ is 3×7 or 21, and that the quotient by 18 or 9×2 is 5×7 or 35, and so on. We thus avoid the process of long division.

Note 2.—It is sometimes necessary, and generally advisable, especially for beginners, to reduce the given fractions to their lowest terms before applying the rule for the least common denominator. Thus, the least common denominator of the fractions $\frac{2}{5}$, $\frac{9}{12}$, $\frac{11}{20}$, taken as they are, is 60; whereas, if we reduce the second fraction to its lowest terms by striking out the factor 3, common to both numerator and denominator, the fractions become $\frac{2}{5}$, $\frac{3}{4}$, $\frac{11}{20}$, and the least common denominator is 20. If, however, the denominator of any such fraction not in its lowest terms is contained in the L.C.M. of the denominators, when the fractions are all in their lowest terms, it is unnecessary to reduce the fraction to its lowest terms.

We strongly recommend the beginner, however, to always commence by reducing the given fractions to their lowest terms.

Ex. IV.

1. Find the L.C.M. of—

(1.) 2, 6, 8, 12. (2.) 4, 9, 10, 14. (3.) 15, 21, 40, 45. (4.) 12, 20, 35, 126. (5.) 39, 65, 52, 140. (6.) 37, 60, 222, 225.

2. Reduce to their least common denominator—

(1.) $\frac{3}{4}, \frac{7}{5}, \frac{2}{15}, \frac{11}{20}$. (2.) $\frac{2}{7}, \frac{5}{12}, \frac{13}{28}, \frac{11}{21}$. (3.) $\frac{4}{3}, \frac{2}{9}, \frac{11}{12}, \frac{13}{14}$. (4.) $\frac{1}{3}, \frac{4}{20}, \frac{1}{7}$. (5.) $\frac{3}{4}, \frac{7}{16}, \frac{38}{162}, \frac{175}{360}$. (6.) $\frac{77}{126}, \frac{19}{153}, \frac{51}{289}$.

3. A can run round a ring in three minutes, B in four minutes, and C in six minutes, and they start together. In how many minutes will they all be again at the starting point?

4. Arrange the fractions $\frac{7}{13}, \frac{11}{28}, \frac{15}{39}, \frac{17}{21}$, in order of magnitude.

5. Multiply the greatest of the fractions $\frac{22}{7}, \frac{33}{10}\frac{3}{6}, \frac{35}{11}\frac{5}{3}$ by 339.

6. Divide the least of the fractions $\frac{8}{43}, \frac{3}{14}, \frac{10}{49}$, by 6.

7. Reduce to a simple fraction the complex fraction having the greater of the fractions $\frac{3}{5}, \frac{2}{3}$ in the numerator, and the less in the denominator.

8. Which of the fractions $\frac{1}{4}, \frac{4}{19}$ is nearer to $\frac{1}{5}$?

9. Show that $\frac{3\frac{1}{9}}{9\frac{1}{3}}$ is less than $\frac{2\frac{1}{5}}{5\frac{1}{2}}$.

10. Arrange in order of magnitude the following:—

$$\frac{4\frac{3}{4} \text{ of } \frac{4}{9}}{6\frac{1}{3}}, \quad \frac{1\frac{1}{6} \text{ of } \frac{3}{5}}{1\frac{3}{4} \text{ of } 2\frac{4}{7}}, \quad \frac{1\frac{1}{3}}{13\frac{2}{3} \text{ of } \frac{1}{4}}.$$

11. Show that nine times the less of the fractions $\frac{\frac{1}{1}\frac{4}{5}}{1\frac{1}{5}}, \frac{3\frac{1}{2}}{4}$ is eight times the greater.

12. Show that the ratio 18 : 7 is a ratio of greater inequality than the ratio 41 : 16.

Addition of Fractions.

20. We have shown (Art. 6) that the numerator and denominator respectively represent the antecedent and consequent

of a ratio; and it is evident from the definition of a ratio (Art. 6) that the sum of two ratios having the same consequent is equal to a ratio whose antecedent is the sum of the given antecedents, and whose consequent is unaltered. Hence we have the following rule for addition of fractions—

RULE.—Bring the given fractions to their least common denominator, add together the numerators thus obtained, and place under the sum the least common denominator.

Ex. 1.—Add together $\frac{2}{3}$, $\frac{10}{21}$, $\frac{9}{14}$, $\frac{5}{6}$.

The least common denominator is easily found to be 42. Dividing this by each of the given denominators, we get as quotients, 14, 2, 3, 7.

$$\text{Hence, } \frac{2}{3} + \frac{10}{21} + \frac{9}{14} + \frac{5}{6} = \frac{2\times14}{42} + \frac{10\times2}{42} + \frac{9\times3}{42} + \frac{5\times7}{42}$$

$$= \frac{28}{42} + \frac{20}{42} + \frac{27}{42} + \frac{35}{42} = \frac{28+20+27+35}{42}$$

$$= \frac{110}{42} = 2\frac{26}{42} = 2\frac{13}{21}.$$

Ex. 2.—Add together $3\frac{1}{5}$, $2\frac{7}{9}$, $1\frac{11}{30}$, $4\frac{7}{12}$.

The sum of the integral parts of the given fractions = $3 + 2 + 1 + 4 = 10$. Hence, $3\frac{1}{5} + 2\frac{7}{9} + 1\frac{11}{30} + 4\frac{7}{12} = 10 + \frac{1}{5} + \frac{7}{9} + \frac{11}{30} + \frac{7}{12}$.

The least common denominator is easily found to be 180. Dividing this by each of the given denominators we get as quotients 36, 20, 6, 15.

Hence the required sum

$$= 10 + \frac{1\times36}{180} + \frac{7\times20}{180} + \frac{11\times6}{180} + \frac{7\times15}{180}$$

$$= 10 + \frac{36}{180} + \frac{140}{180} + \frac{66}{180} + \frac{105}{180}$$

$$= 10 + \frac{36+140+66+105}{180} = 10 + \frac{347}{180}$$

$$= 10 + 1\frac{167}{180} = 11\frac{167}{180}.$$

Subtraction of Fractions.

21. After the preceding article there will be no difficulty in comprehending the following rule—

RULE.—Bring the given fractions to their least common denominator, subtract the numerators thus obtained, and under the difference place the least common denominator.

Ex. 1.—Subtract $\frac{3}{5}$ from $\frac{7}{8}$.

$$\frac{7}{8} - \frac{3}{5} = \frac{7\times5}{40} - \frac{3\times8}{40} = \frac{35}{40} - \frac{24}{40} = \frac{35-24}{40} = \frac{11}{40}.$$

Ex. 2.—Take $6\frac{7}{9}$ from $9\frac{1}{6}$.

$$9\tfrac{1}{6} - 6\tfrac{7}{9} = 3\tfrac{1}{6} - \tfrac{7}{9} = 3 + \frac{1\times3}{18} - \frac{7\times2}{18}$$
$$= 3 + \tfrac{3}{18} - \tfrac{14}{18} = 3 + \frac{3-14}{18}.$$

Here the number to be subtracted is the greater. We shall, however, take the *less* from the *greater*, and put the *negative* sign to the remainder, meaning by this that the remainder has yet to be subtracted.

Hence, $9\frac{1}{6} - 6\frac{7}{9} = 3 - \frac{11}{18} = 2 + (1 - \frac{11}{18})$; now $\frac{11}{18}$ taken from unity or $\frac{18}{18}$ evidently leaves $\frac{7}{18}$.

$$\therefore \text{required result} = 2 + \tfrac{7}{18} = 2\tfrac{7}{18}.$$

We will now give an example involving both addition and subtraction.

Ex. 3.—Find the value of $6\frac{1}{3} - 7\frac{2}{9} + 4\frac{5}{12} - 1\frac{5}{18}$.

Whenever we have an expression involving both + and − signs, the simplest method is to add together all the quantities affected with the *plus* sign, and likewise those affected with the *minus* sign; then taking the difference between these two sums, we place the sign of the greater sum before the result.

Thus, taking first the integers, we have $6 - 7 + 4 - 1 = 10 - 8 = 2$ (it must be remembered that the sign + is understood before a number which appears without a sign when it stands alone or at the head of an expression).

Hence the given expression $= 2 + \frac{1}{3} - \frac{2}{9} + \frac{5}{12} - \frac{5}{18}$
$= 2 + \frac{1\times12}{36} - \frac{2\times4}{36} + \frac{5\times3}{36} - \frac{5\times2}{36} = 2 + \frac{12-8+15-10}{36}$
$= 2\frac{27-18}{36} = 2\frac{9}{36} = 2\frac{1}{4}$.

We shall give one more example in order to show how brackets are to be treated.

Ex. 4.—Find the value of $7\frac{1}{8} - (2\frac{1}{5} - 3\frac{1}{2}) + (4\frac{1}{6} - 1\frac{1}{3}) - (2\frac{3}{10} + 3\frac{1}{12})$.

The general rule, which the student will better understand when we come to Algebra, is this—

When a *minus* sign stands before a bracket, it changes all the signs within on removing the bracket; but when a *plus* sign stands before the bracket, the latter may be removed *without* changing any of the signs within.

Thus, taking the expression $-(2\frac{1}{5} - 3\frac{1}{2})$:—

We must remember that the sign of $2\frac{1}{5}$ when within the bracket is, according to a remark made in Ex. 3, *understood* to be *plus*. In fact, though it is unusual, we might write the expression thus: $-(+2\frac{1}{5} - 3\frac{1}{2})$.

Now, remove the bracket, and it becomes $-2\frac{1}{5} + 3\frac{1}{2}$.

Again $+(4\frac{1}{6} - 1\frac{1}{3}) = +4\frac{1}{6} - 1\frac{1}{3}$.

And $-(2\frac{3}{10} + 3\frac{1}{12}) = -2\frac{3}{10} - 3\frac{1}{12}$.

Hence our given expression—

$$= 7\tfrac{1}{8} - 2\tfrac{1}{5} + 3\tfrac{1}{2} + 4\tfrac{1}{6} - 1\tfrac{1}{3} - 2\tfrac{3}{10} - 3\tfrac{1}{12}$$

$$= 14 - 8 + \tfrac{1}{8} - \tfrac{1}{5} + \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{3} - \tfrac{3}{10} - \tfrac{1}{12}$$

$$= 6 + \frac{1\times15}{120} - \frac{1\times24}{120} + \frac{1\times60}{120} + \frac{1\times20}{120} - \frac{1\times40}{120} - \frac{3\times12}{120} - \frac{1\times10}{120}$$

$$= 6 + \frac{15-24+60+20-40-36-10}{120} = 6 + \frac{95-110}{120}$$

$$= 6 - \tfrac{15}{120} = 6 - \tfrac{1}{8} = 5 + (1 - \tfrac{1}{8}) = 5\tfrac{7}{8}.$$

Ex. V.

1. Add together (1.) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$; (2.) $\frac{3}{5}$, $\frac{7}{10}$, $\frac{11}{15}$; (3.) $\frac{4}{9}$, $\frac{5}{12}$, $\frac{7}{18}$.

2. Find the sum of $3\frac{4}{5}$, $17\frac{11}{60}$, $2\frac{3}{10}$, $1\frac{7}{15}$, $\frac{9}{8}$, $3\frac{2}{3}$.

3. Add together $2\frac{7}{9}$, $\dfrac{3\frac{1}{2}}{4}$, $\dfrac{5}{2\frac{1}{7}}$, $2\frac{2}{3}$ of $1\frac{1}{8}$, $5\frac{1}{5}$ of $1\frac{1}{13}$.

4. Find the difference between (1.) $\frac{7}{8}$ and $\frac{3}{4}$; (2.) $\frac{5}{6}$ and $\frac{4}{5}$; (3.) $\frac{6}{7}$ and $\frac{7}{8}$.

5. Subtract (1.) $6\frac{1}{3}$ from $8\frac{7}{12}$; (2.) $3\frac{7}{9}$ from $4\frac{1}{15}$; (3.) $2\frac{7}{10}$ from $6\frac{1}{3}$.

6. Take $\dfrac{1 - \frac{13}{16}}{1\frac{2}{5}}$ from $\dfrac{1\frac{9}{16}}{2 - 1\frac{1}{4}}$

7. Find the value of $1\frac{3}{28} - 2\frac{6}{7} + 3\frac{1}{8} - \frac{17}{14}$.

8. Simplify the expression $(2\frac{1}{2} - 1\frac{1}{3}) - (3\frac{5}{6} - 7\frac{3}{4})$.

9. By how much does $3\frac{4}{13} - 1\frac{3}{5}$ exceed $2\frac{10}{26} - \frac{7}{10}$?

10. Take the difference of $6\frac{7}{17}$ and $1\frac{11}{35\,11}$ from their sum.

11. Add the difference of the same two fractions to their sum.

12. Find the value of the expression $\dfrac{3\frac{1}{3} \text{ of } 1\frac{2}{5}}{1\frac{3}{4}} - \left(\dfrac{6}{2\frac{1}{2}} - \dfrac{3\frac{3}{4}}{5}\right)$

Multiplication of Fractions.

22. RULE.—Multiply together the numerators of the fractions for a new numerator, and the denominators for a new denominator.

The reason of this rule is easily seen. Let it be required to find the product of $\frac{3}{5}$ and $\frac{7}{8}$, or the value of $\frac{3}{5} \times \frac{7}{8}$.

Now, what is the meaning of multiplying the ratio 3 : 5 by the ratio 7 : 8? It means evidently that the ratio 3 : 5 is to be multiplied by 7, and the result divided by 8.

Now (Art. 8) the ratio 3 : 5, when multiplied by 7, becomes $3 \times 7 : 5$; and (Art. 9) the ratio $3 \times 7 : 5$, when divided by 8, becomes $3 \times 7 : 5 \times 8$; and we have

$$\frac{3}{5} \times \frac{7}{8} = \frac{3 \times 7}{5 \times 8}.$$

And so on for any number of fractions. Hence the above rule.

Ex 1.—Multiply together the fractions $\frac{5}{6}$, $\frac{3}{8}$, $\frac{12}{25}$.

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{5 \times 3 \times 12}{6 \times 8 \times 25}$$

Before actually performing the operation of multiplication, it is advisable to strike out any factor common to both numerator and denominator. We see that 5 is common to 5 and 25, 3 common to 3 and 6, 4 common to 12 and 8, and we then have

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{1 \times 1 \times 3}{2 \times 2 \times 5} = \frac{3}{20}.$$

The whole operation is sometimes written thus—

$$\frac{5}{6} \times \frac{3}{8} \times \frac{12}{25} = \frac{\overset{1}{\cancel{5}} \times \overset{1}{\cancel{3}} \times \overset{3}{\cancel{12}}}{\underset{2}{\cancel{6}} \times \underset{2}{\cancel{8}} \times \underset{5}{\cancel{25}}} = \frac{3}{20}.$$

Ex. 2.—Multiply together $2\frac{3}{5}$, $3\frac{13}{29}$, $1\frac{1}{20}$, $\frac{87}{91}$.

$$2\frac{3}{5} \times 3\frac{13}{29} \times 1\frac{1}{20} \times \frac{87}{91} = \frac{13}{5} \times \frac{100}{29} \times \frac{21}{20} \times \frac{87}{91}$$

$$= \frac{\overset{1}{\cancel{13}} \times \overset{20}{\cancel{100}} \times \overset{3}{\cancel{21}} \times \overset{3}{\cancel{87}}}{\underset{1}{\cancel{5}} \times \underset{1}{\cancel{29}} \times \underset{1}{\cancel{20}} \times \underset{7}{\cancel{91}}} = \frac{9}{1} = 9.$$

Division of Fractions.

23. RULE.—Invert the divisor, and proceed as in multiplication.

To explain this rule, let us endeavour to divide $\frac{7}{9}$ by $\frac{5}{8}$. We may evidently consider the required quotient as nothing

else than the ratio $\frac{7}{9} : \frac{5}{8}$, and this, by the reasoning of Art. 11, is equivalent to the ratio $7 \times 8 : 9 \times 5$, and we hence get

$$\frac{7}{9} \div \frac{5}{8} = \frac{7 \times 8}{9 \times 5}.$$

Now, $\frac{8}{5}$ is the divisor $\frac{5}{8}$ when inverted, and hence the above rule.

Ex. 1.—Divide $\frac{9}{10}$ by $\frac{3}{8}$.

$$\frac{9}{10} \div \frac{3}{8} = \frac{9}{10} \times \frac{8}{3} = \frac{\overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\underset{5}{\cancel{10}} \times \cancel{3}} = \frac{12}{5} = 2\frac{2}{5}.$$

Ex. 2.—Divide $1\frac{2}{3}$ of $7\frac{1}{8}$ by $3\frac{1}{3}$ of $3\frac{2}{9}$.

$$1\tfrac{2}{3} \text{ of } 7\tfrac{1}{8} \div 3\tfrac{1}{3} \text{ of } 3\tfrac{2}{9} = \frac{5}{3} \times \frac{57}{8} \div \left(\frac{10}{3} \times \frac{29}{9}\right)$$

$$= \frac{5}{3} \times \frac{57}{8} \times \frac{\overset{5}{\cancel{10}}}{\underset{2}{\cancel{29}}} \times \frac{9}{29} = \frac{513}{464} = 1\frac{49}{464}.$$

We have introduced a bracket on the right side of the first equality, for otherwise the sign $\div$ affects only the first fraction $\frac{10}{3}$. On the other side a bracket is unnecessary, for the sign $\div$ standing before a *compound* fraction (not *two* fractions) affects the whole.

Ex. 3.—Simplify the expression

$$1\frac{2}{15} \div 6\frac{2}{5} \times 4\frac{1}{4} \div 2\frac{2}{3} \text{ of } 24\frac{1}{12}.$$

$$\text{The given expression} = \frac{17}{15} \div \frac{32}{5} \times \frac{17}{4} \div \left(\frac{8}{3} \times \frac{289}{12}\right)$$

$$= \frac{\cancel{17}}{\cancel{15}} \times \frac{\cancel{5}}{\cancel{32}} \times \frac{\cancel{17}}{\cancel{4}} \times \frac{\cancel{8}}{\cancel{3}} \times \frac{\overset{3}{\cancel{12}}}{\cancel{289}} = \frac{3}{256}.$$

Ex. VI.

1. Find the sum, difference, and product of $2\frac{1}{8}$ and $1\frac{3}{4}$.

2. Multiply the sum of the fractions $3\frac{3}{5}$, $2\frac{7}{10}$ by their difference.

3. Simplify the expression $\left\{\left(3\frac{2}{17}\right)^2 - \left(1\frac{1}{1}\frac{5}{7}\right)^2\right\} \div \left\{3\frac{2}{17} - 1\frac{1}{1}\frac{5}{7}\right\}$.

4. Reduce to a simple fraction each of the following expressions—

(1.) $1\frac{5}{13} \div 7\frac{1}{5} \times 8\frac{2}{3} \div 2\frac{2}{3} \text{ of } 20\frac{1}{4}$.

(2.) $1\frac{5}{13} \div 7\frac{1}{5} \text{ of } 8\frac{2}{3} \div 2\frac{2}{3} \div 20\frac{1}{4}$.

5. What is the difference between $\left(8\frac{1}{4} - 3\frac{7}{8}\right)$ and $\left(5\frac{1}{6} - 4\frac{1}{12}\right)$?

6. Divide $\dfrac{4}{3 + \dfrac{2\frac{8}{21}}{7\frac{1}{7}}}$ by $\dfrac{\frac{6}{7} \text{ of } 1\frac{2}{5}}{4}$.

7. Reduce to a simple fraction $3 + \dfrac{1}{7 + \frac{1}{11}}$.

8. Simplify the expressions (1.) $\dfrac{\frac{3}{4} - \frac{5}{12}}{\frac{5}{6} - \frac{5}{8}} \div \dfrac{2\frac{1}{2} - 1\frac{5}{6}}{1\frac{3}{4} + \frac{7}{9}}$.

(2.) $\dfrac{6\frac{1}{8}}{\frac{1}{3} \text{ of } 1\frac{2}{7} \text{ of } 2\frac{1}{3}} \times \left\{3\frac{1}{2} - \left(2\frac{1}{6} - 1\frac{1}{8}\right)\right\}$.

9. Find the quotient of $103\frac{6}{9}$ by $30\frac{1}{3}$ of $\frac{9}{203}$.

10. The cost of $7\frac{3}{5}$ articles is £$65\frac{2}{5}$, what is the cost of each article?

11. Find the cost of $89\frac{1}{7}$ articles, when one cost £$4_{11}\frac{1}{2}$.

12. The sum of two quantities is $34_{11}\frac{7}{2}$, and their difference is $6\frac{1}{11}\frac{1}{2}$; required the greater.

Reduction of Fractions to Decimals.

24. If we place a decimal point to the right of an integer, and add as many ciphers as we please, it is clear, from Art. 1, that we do not alter its value. And hence a given ratio, as 3 : 8, is not altered in value by writing it 3·000 : 8; and further, dividing each of its terms by 8, according to the rule for division of decimals, it becomes ·375 : 1. It therefore follows, putting each of these ratios in a fractional form, that

$$\frac{3}{8} = \frac{3{\cdot}000}{8} = \frac{{\cdot}375}{1} = {\cdot}375.$$

We get, therefore, the following rule:—

RULE.—Place a decimal point to the right of the numerator, and add as many ciphers as may be thought necessary. Divide the new numerator by the given denominator, according to the rule for division of decimals, and, if necessary, add ciphers to the successive remainders until the division terminates, or until we have obtained as many decimal figures as required.

Ex. 1.—Reduce $\frac{5}{32}$ to a decimal.

32)5·0(·15625
32
180
160
200
192
80
64
160
160

Hence $\frac{5}{32} = \cdot\cdot15625$.

Ex. 2.—Reduce $\frac{5}{296}$ to a decimal.

296)5·00(·016891̇
296
2040
1776
2640
2368
2720
2664
560
296
264

It will be seen that we have arrived at a remainder, 264, exactly the same as the second remainder; and that, therefore, the quotient figures 891 will continually repeat, and that the division will never terminate. We call 891 the recurring period of the decimal, and it is usual to indicate the fact of its recurrence by placing dots over its first and last figures, as above.

We have, therefore, as a result, $\frac{5}{296} = \cdot016\dot{8}9\dot{1}$.

NOTE.—It is easy to see that no fraction, *reduced to its lowest terms*, whose denominator contains any prime factor, other than 2 or 5, can be expressed as a terminating decimal. For every terminating decimal is an exact number of tenths, hundredths, &c., and may, therefore, be transformed into a fraction, having some power of 10 as its

denominator. Now, if we wish to bring a fraction already in its lowest terms to an equivalent fraction having some power of 10 for its denominator, it can only be done by *multiplying* its numerator and denominator by some integer; and it is impossible to obtain any power of 10 by multiplication only, from a number which contains any prime factor, other than 2 or 5.

Reduction of Terminating Decimals to Fractions.

25. Remembering (Art. 1) that any given terminating decimal may be considered as derived from an integer by diminishing it 10, or 10^2, 10^3, &c. fold, we have

$$\cdot 345 = 345 \div 10^3.$$

Hence, $\cdot 345$ is the value of the ratio $345 : 10^3$; and we have, also, $\cdot 345 = \frac{345}{10^3} = \frac{345}{1000}$.

The following rule is, therefore, clear:—

RULE.—Make a numerator of the integer, formed by taking away the decimal point; and for a denominator put 1, followed by as many ciphers as there are given decimal figures.

Ex. 1.— $\cdot 625 = \frac{625}{1000} = \frac{5 \times 125}{8 \times 125} = \frac{5}{8}$.

$3 \cdot 95 = 3\frac{95}{100} = 3\frac{19 \times 5}{20 \times 5} = 3\frac{19}{20}$.

$\cdot 0025 = \frac{25}{10000} = \frac{25}{400 \times 25} = \frac{1}{400}$.

Reduction of Recurring or Circulating Decimals to Fractions.

26. There are two kinds, and it is convenient to treat them separately.

(1.) *Pure circulating decimals*, where the whole of the decimal figures recur.

RULE.—Take away the decimal point and the dots, make a numerator of the integer thus obtained, and place under this as denominator as many *nines* as there are recurring figures.

The following example will make this rule clear.

Ex.—Reduce $\cdot\dot{2}\dot{0}\dot{7}$ to a fraction.

The value of the decimals is evidently not altered by writing it $\cdot 207\dot{2}0\dot{7}$. Let us remove the decimal point three

places to the right, or, what is the same thing (Art. 1), let us multiply the given decimal by 1000.

We then get $207{\cdot}\dot{2}0\dot{7}$ as the value of 1000 times the given decimal. Now the number $207{\cdot}\dot{2}0\dot{7}$ includes the integer 207, and the given decimal $\cdot\dot{2}0\dot{7}$; and it therefore follows that the integral part 207 is (1000 − 1), or 999 times the value of the given decimal.

Hence, dividing it by 999, we get

$$\cdot\dot{2}0\dot{7} = 207 \div 999 = \cdot\tfrac{2}{9}\tfrac{0}{9}\tfrac{7}{9}.$$

(2.) *Mixed circulating decimals.*—Where part only of the figures recurs.

RULE.—Take away the decimal point and the dots, subtract from the integer thus obtained the integer formed by the figures which do not recur, and make a numerator of the result. Then, for a denominator, place underneath as many *nines* as there are recurring figures, followed by as many ciphers as there are figures which do not recur.

We shall make this rule clear by the following example:—

Ex.—Reduce $\cdot 24\dot{5}7\dot{3}$ to a fraction.

Let us remove the decimal point *two* places to the right, thus, by Art. 1, multiplying the given decimal by 100; we then get -

$$100 \text{ times the value of } \cdot 24\dot{5}7\dot{3} = 24{\cdot}\dot{5}7\dot{3}.$$

Now, by case (1) above, $24{\cdot}\dot{5}7\dot{3} = 24\tfrac{5}{9}\tfrac{7}{9}\tfrac{3}{9}$; or reducing to an improper fraction, and noticing that $24 \times 999 = 24000 - 24$, we have

$$24{\cdot}\dot{5}7\dot{3} = \frac{24000 - 24 + 573}{999} = \frac{24573 - 24}{999}.$$

Hence, dividing this result by 100, we get

$$\cdot 24\dot{5}7\dot{3} = \frac{24573 - 24}{999} \div 100 = \frac{24573 - 24}{99900} \text{ (Art. 24.)}$$

Ex. 1.—$\cdot\dot{4}2857\dot{1} = \tfrac{4}{9}\tfrac{2}{9}\tfrac{8}{9}\tfrac{5}{9}\tfrac{7}{9}\tfrac{1}{9} = \dfrac{3 \times 142857}{7 \times 142857} = \tfrac{3}{7}$.

Ex. 2.—$\cdot 27\dot{0}\dot{9} = \dfrac{2709 - 27}{9900} = \tfrac{2}{9}\tfrac{6}{9}\tfrac{8}{0}\tfrac{2}{0} = \dfrac{149 \times 2 \times 9}{550 \times 2 \times 9} = \tfrac{1}{5}\tfrac{4}{5}\tfrac{9}{0}$.

27. In arithmetical operations involving circulating decimals, and, indeed, any decimals having a large number of decimal figures, it is generally sufficient to obtain a result correct to a given number of decimals.

1. In *addition* and *subtraction* we obtain this result most easily by using in our operation *one* or *two* more figures of the given decimals than are required in the result.

Ex. 1.—Add together (correct to five places) the following:—

$\cdot 3\dot{0}2\dot{6}$, $6\cdot 729\dot{4}$, $\cdot 0\dot{1}\dot{6}$, $\cdot 4163729$.

$\cdot 3026026$
$6\cdot 7294444$
$\cdot 0166666$
$\cdot 4163729$

$7\cdot 4650865$. Ans. $7\cdot 46508$.

NOTE.—If our object is to obtain a decimal of five places which shall give the *approximate* sum of the given decimals we must write $7\cdot 46509$ as the approximate sum; for $7\cdot 46509$ is nearer to the true value of $7\cdot 465086$ &c. than $7\cdot 46508$. The general rule is to increase by 1 the decimal figure at which we stop, *when the next figure is 5 or above 5*.

Ex. 2.—Find the difference (correct to six places) of $3\cdot 07\dot{4}\dot{5}$ and $4\cdot \dot{2}6\dot{3}$, and express the approximate difference by a decimal of five places.

$4\cdot 26326326$
$3\cdot 07454545$
$1\cdot 18871781$

Hence the difference correct to six places is $1\cdot 188717$, and the required approximate difference $1\cdot 18872$.

2. In *multiplication* and *division* of circulating decimals it is generally preferable to reduce the given decimals to fractions, bring out the result in a fractional form, and afterwards reduce this to a decimal.

Ex. VII.

1. Express as a decimal the sum of the following fractions:—

$\frac{3}{8}$, $\frac{15}{96}$, $\frac{7}{112}$, $\frac{35}{28}$, $\frac{7}{64}$, $\frac{1}{4}$.

2. Reduce to fractions the following decimals:—

$\cdot 35$, $\cdot 026$, $\cdot \dot{1}\dot{6}$, $\cdot \dot{1}42857\dot{}$, $\cdot 1\dot{6}$, $\cdot 4285714\dot{}$.

3. Find the value (correct to six places) of $\cdot 23\dot{7} + 3\cdot 8\dot{1}\dot{6} - 6\cdot 023\dot{5} + 4\cdot 2\dot{9} - \cdot 00\dot{2} + 1\cdot 37\dot{4}$.

4. Add together $\cdot 6\dot{2}$, $\cdot\dot{0}3\dot{7}$, $2\cdot 47\dot{6}\dot{1}$, $\cdot 8\dot{1}0\dot{6}$, $\cdot\dot{7}$, $\cdot 375$.

5. Find the product and quotient of $3\cdot\dot{5}\dot{4}$ by $4\cdot\dot{3}$.

6. Simplify the expression—

$$(4\cdot\dot{6} \times \cdot\dot{4}28571\dot{} - 2\cdot 2 \times \cdot 3\dot{6})\ (1 - \cdot 1\dot{6}).$$

In the next six examples the dots are signs of multiplication, and you are required to give the values of the expressions correct to six places.

7. $1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \&c.$

8. $\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \&c.$

9. $\frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \&c.$

10. $1 + \frac{1}{8^2} + \frac{1}{8^4} + \frac{1}{8^6} + \&c.$

11. $16\left\{\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \&c.\right\} - 4\left\{\frac{1}{239} - \frac{1}{3\cdot 239^3} + \&c.\right\}$

12. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{6\cdot 7}$.

CHAPTER III.

APPLICATION OF THE PRECEDING ARTICLES TO CONCRETE QUANTITIES.

To find the Value of a Fraction of a Concrete Quantity.

28. RULE.—Multiply the concrete quantity by the numerator of the fraction, and divide the product by the denominator.

Ex. 1.—Find the value of $\frac{3}{7}$ of 2 tons, 3 cwt. 21 lbs.

$\frac{3}{7}$ of 2 tons, 3 cwt. 21 lbs. $= \frac{(2 \text{ tons}, 3 \text{ cwt. } 21 \text{ lbs.}) \times 3}{7} = \frac{6 \text{ tons}, 9 \text{ cwt. } 2 \text{ qrs. } 7 \text{ lbs.}}{7}$

$= 18$ cwt. 2 qrs. 1 lb. *Ans.*

Ex. 2.—Find the value of $3\frac{2}{9}$ of £12. 6s. $2\frac{1}{4}$d.

		£.	s.	d.
3 times £12. 6s. $2\frac{1}{4}$d.	$= (£12.\ 6s.\ 2\frac{1}{4}d.) \times 3$	36	18	$6\frac{3}{4}$
$\frac{2}{9}$ of £12. 6s. $2\frac{1}{4}$d.,	$= \frac{(£12.\ 6s.\ 2\frac{1}{4}d.) \times 2}{9}$			
	$= \frac{£24.\ 12s.\ 4\frac{1}{2}d.}{9}$	2	14	$8\frac{1}{2}$
Hence, adding, the value required		£39	13	$3\frac{1}{4}$

Ex. 3.—Find the value of $\frac{3}{7}$ of 4 miles + $\frac{7}{9}$ of 3 fur. + $\frac{5}{21}$ of 8 poles.

		Mile.	Fur.	Poles.	Yards.
$\frac{3}{7}$ of 4 miles	$= \frac{(4 \times 3) \text{ miles}}{7} = \frac{12 \text{ miles}}{7}$	1	5	28	$3\frac{1}{7}$
$\frac{7}{9}$ of 3 fur.	$= \frac{(3 \times 7) \text{ fur.}}{9} = \frac{21 \text{ fur.}}{9}$	0	2	13	$1\frac{5}{6}$
$\frac{5}{21}$ of 8 poles	$= \frac{(8 \times 5) \text{ poles}}{21} = \frac{40 \text{ poles}}{21}$	0	0	1	$4\frac{4}{42}$
Hence, adding, the value required		2	0	3	$4\frac{9}{42}$

Note.—The addition of the yards is thus effected :—

$(3\frac{1}{7} + 1\frac{5}{6} + 4\frac{41}{42})$ yds. $= (8 + \frac{6+35+41}{42})$ yds. $= (8 + 1\frac{20}{21})$ yds.

$= 9\frac{20}{21}$ yds. $= 1$ pole, $(3\frac{1}{2} + \frac{20}{21})$ yds.

$= 1$ pole, $(3 + \frac{21+40}{42})$ yds. $= 1$ pole, $4\frac{19}{42}$ yds.

Ex. VIII.

Find the values of—

1. $\frac{2}{3}$ of £1; $\frac{3}{5}$ of 1s.; $\frac{2}{9}$ of 12s.; $\frac{3}{7}$ of £3.
2. $\frac{7}{8}$ of £5; $\frac{2}{7}$ of a guinea; $\frac{5}{11}$ of 2s. 6d.; $\frac{3}{4}$ of a crown.
3. $3\frac{2}{3}$ of £1. 12s.; $2\frac{1}{2}$ of £3. 10s.; $7\frac{9}{11}$ of £3. 4s. $5\frac{1}{2}$.
4. $\frac{6}{7}$ of 1 ton; $\frac{3}{5}$ of 1 qr.; $\frac{4}{9}$ of 1 stone; $\frac{3}{11}$ of 9 lbs.
5. $3\frac{2}{7}$ mile; $\frac{4}{9}$ of 3 fur.; $\frac{2}{7}$ of 15 poles; $3\frac{1}{7}$ of $2\frac{1}{5}$ of 12 yds.
6. $\frac{8}{20\frac{1}{3}}$ leap year; $\frac{2}{1\frac{3}{4}}$ lunar month; $\frac{5\frac{1}{9}}{10\frac{2}{3}}$ of 10 h. 15′ 12″.
7. 12 lbs. 3 oz. 7 dwt. 5 grs. × $3\frac{1}{7}$; 10 oz. 4 gr. ÷ $16\frac{1}{3}$.

8. $8\frac{3}{8}$ of 4 ac. 3 po. $-\frac{7}{176}$ of 1 sq. mile $+ 1\frac{2}{9}$ of 3 r. 20 sq. yds.

9. $(5\frac{3}{4}$ of $\frac{9}{38} \div 1\frac{4}{19})$ of $35° 36' 25''$; $(2\frac{3}{176} \div 1\frac{1}{8}$ of $1\frac{2}{3})$ of $30°$.

10. $\frac{3}{170}$ of 1 lb. Troy $+ \frac{5}{6}$ of 1 lb. Troy $- \frac{3}{576}$ of 1 lb. Avoirdupois (= 7000 grains).

11. $\frac{2}{9}$ of $\frac{5\frac{1}{2} \text{ of 3 guineas}}{\frac{3}{7} \text{ of £1. 3s. 10d.}}$; $\frac{2}{9}$ of 1 ton $+ \frac{3}{7}$ of 3 qrs. $- \frac{1}{168}$ of 7 cwt.

12. $2\frac{3}{133}$ of 15 h. $10' 13\frac{1}{5}'' - \frac{5}{27}$ of 1 day, 12 h. $11' 12\frac{1}{9}''$.

To Reduce a given Quantity to the Fraction of any other given Quantity of the same kind.

29. Rule.—Reduce both the given quantities to the same denomination, and the fraction required will have the number of units in the first quantity as numerator, and that of the second quantity as denominator.

Ex. 1.—Reduce 3s. 8d. to the fraction of £1.

3s. 8d. = 44d,
and £1 = 240d.

Hence, the fraction required $= \frac{44}{240} = \frac{11}{60}$.

Or, better, thus:

3s. 8d. = 11 fourpences,
and £1 = 60 fourpences.

$\therefore$ Fraction required $= \frac{11}{60}$.

Note.—It is always best to keep the denominations to which the given quantities are reduced as high as possible.

Ex. 2.—Reduce $\frac{1}{7}$ of a moidore to the fraction of $2\frac{1}{2}$ guineas.

$\frac{1}{7}$ of a moidore $= (\frac{1}{7} \times 27)$s., and $2\frac{1}{2}$ guineas $= (2\frac{1}{2} \times 21)$s.

Hence, the fraction required $= \frac{\frac{1}{7} \times 27}{2\frac{1}{2} \times 21} = \frac{\frac{1}{7} \times 27}{\frac{5}{2} \times 21} = \frac{1 \times 27 \times 2}{5 \times 21 \times 7}$

$= \frac{1 \times 9 \times 2}{5 \times 7 \times 7} = \frac{18}{245}$

Ex. 3.—Reduce 3 cwt. $2\frac{1}{9}$ qrs. to the fraction of 4 cwt. 2 qrs. 4 lbs.

3 cwt. $2\frac{1}{9}$ qrs. $= 14\frac{1}{9}$ qrs., and 4 cwt. 2 qrs. 4 lbs. $= 4$ cwt. $2\frac{1}{7}$ qrs. $= 18\frac{1}{7}$ qrs.

$\therefore$ Fraction required $= \frac{14\frac{1}{9}}{18\frac{1}{7}} = \frac{\frac{127}{9}}{\frac{127}{7}} = \frac{127 \times 7}{9 \times 127} = \frac{7}{9}$.

Ex. IX.

Reduce—

1. 1s. 8d. to the fraction of £1; $7\frac{1}{2}$d. to the fraction of 10s.

2. 2s. 4d. to the fraction of 10s. 8d.; 1s. $7\frac{1}{2}$d. to the fraction of 3s. $4\frac{1}{2}$d.

3. 3 qrs. 15 lbs. to the fraction of 1 ton; 2 stones 10 oz. to the fraction of 3 cwt.

4. 3 lbs. avoirdupois to troy weight; 10 lbs. 3 oz. 4 dwt. troy to avoirdupois.

5. 3 quires, 10 sheets to the fraction of 2 reams, 3 quires; 3 ft. $8\frac{1}{5}$ in. to the fraction of 3 yards.

6. 30° 3′ 12″ to the fraction of a right angle (= 90°); 57° 16′ $21\frac{9}{11}$″ to the fraction of two right angles.

What fraction is—

7. $\frac{3}{4}$ ac. of $3\frac{1}{2}$ ac.; $2\frac{1}{3}$ days of 17 weeks?

8. $\frac{\cdot125 + 1\cdot875}{\cdot0140625}$ acres of $19\frac{1}{5}$ poles; $\frac{3\cdot16 \times 1\cdot4}{2\cdot375 \times \cdot1}$ yds. of $3\frac{1}{7}$ m?

9. $\left(\frac{3}{4\frac{1}{5}}\text{ rood} + \frac{7\frac{1}{6}}{4\frac{7}{9}}\text{ poles} + \frac{1\frac{1}{2} \times 1\frac{4}{11}}{2\frac{1}{2}}\text{ yd.}\right)$ of 3 acres?

10. $\frac{\frac{5}{7}}{2\frac{8}{21}}$ gal. of $\frac{3\frac{7}{26}}{2\frac{6}{13}}$ pipes?

11. What fraction of his original income has a person left after paying a tax of 4d. in the £?

12. A garden roller is 2 ft. 6 in. wide, and it is rolled at the rate of 1 mile in 20 minutes: find in what fraction of a day a man will roll $\frac{1}{6}$ of an acre.

To Find the Value of a Decimal of a Concrete Quantity.

30. Rule.—Multiply the given decimal by the number of units in the concrete quantity when expressed in terms of one denomination, and the integral part of the result will be the number of units of this denomination. Then multiply the decimal part of this denomination by the number of units connecting it with the next lower and the integral part will be units of this latter denomination, and so on.

Ex. 1.—Find the value of ·325 of £3. 10s.

£3. 10s. = 70s.; proceeding then according to rule, we have:

·325
70
22·750s.
12
9·00d. Ans.: 22s. 9d. or £1. 2s. 9d.

Ex. 2.—Find the value of ·546875 of 3 tons.

·546875
3
1·640625 tons.
20
12·812500 cwt.
4
3·2500 qrs.
28
200
50
7·00 lbs. Ans.: 1 ton, 12 cwt. 3 qrs. 7 lbs.

Ex. 3.—Find the value of 6·66875 acres.

6·66875 acres.
4
2·67500 roods.
40
27·000 poles. Ans.: 6 acres, 2 roods, 27 poles.

Ex. 4.—Find the value of $\cdot 31\dot{6}$ of £1.

First Method.

£
·3166667 nearly.
20
6·3333340s.
12
4·000008d. Ans.: 6s. 4d.

Second Method.

$\cdot 31\dot{6} = \frac{316 - 31}{900} = \frac{285}{900}$

$= \frac{15 \times 19}{15 \times 60} = \frac{19}{60}$

Then by rule for reduction of fractions,

$£\frac{19}{60}$ = 6s. 4d. Ans.

The latter method is preferable when perfect accuracy is required.

Ex. X.

Find the value of—

1. £·375; £·98125; £·815625.
2. $\cdot4\dot{1}\dot{6}$ of £3; $\cdot\dot{4}28571\dot{4}$ of 6s. 5d.; $8\cdot\dot{5}71428\dot{5}$ of 3s. $0\frac{3}{4}$d.
3. ·625 of 1 ton; ·046875 of 3 tons; $4\cdot3\dot{9}$ of 1 cwt. 53 lbs.
4. $1\cdot6\dot{6}71875$ acres; ·3475 rood; $\cdot0\dot{7}6923\dot{0}$ of 5 acres, 6 poles.
5. $\cdot208\dot{3}$ of 1 ream; $\cdot458\dot{3}$ quire; $\cdot38\dot{3}$ of 3 reams, 12 sheets.
6. ·3078125 pipe; ·490625 tun; ·37125 bushel.
7. Express in grains $\cdot1428\dot{5}\dot{7}$ of 4 lbs. avoirdupois, and express the result in troy weight.

8. Express 10 oz. 3 dwt. 14 grs. as the decimal of 1 lb. avoirdupois.

9. What is the sum of $\cdot\dot{6}$ of 1 guinea, $\cdot08\dot{3}$ of 1 crown, and $\cdot0\dot{3}\dot{7}$ of £1. 0s. 3d.

10. Find the value of

$$\left(4 \div \frac{\cdot20\dot{4}1\dot{6} \times 7\cdot5}{\cdot21875}\right) \text{cwt.} + (\cdot95 \text{ ton} - \dot{7}69230\dot{} \text{ qrs.}) \times 26.$$

11. What is the value of

$$\left(3 + \cfrac{\cdot\dot{3}}{1 + \cfrac{\cdot\dot{1}}{1 + \cfrac{\cdot\dot{3}}{1 + \cfrac{\cdot\dot{1}}{\cdot\dot{3}}}}}\right) \text{ of 1 cwt. 35 lbs. ?}$$

12. Simplify $\left(\dfrac{1 + \cdot1\dot{6}}{1 - \cdot1\dot{6}} + \dfrac{1 - \cdot1\dot{6}}{1 + \cdot1\dot{6}}\right)$ of 1 oz. 15 dwts.

To Reduce from one Denomination to the Decimal of another Denomination of the same kind.

31. Rule.—Bring the given quantity to the fraction of the proposed denomination, and reduce this fraction to a decimal.

Ex. 1.—Reduce 3s. 3d. to the decimal of 8s. $1\frac{1}{2}$d.

3s. 3d. = 26 three-halfpence,

8s. $1\frac{1}{2}$d. = 65 three-halfpence,

and the fraction $\frac{26}{65} = \frac{2 \times 13}{5 \times 13} = \frac{2}{5} = \cdot 4$.

Hence ·4 is the decimal required.

Ex. 2.—Reduce 3 qrs. 21 lbs. to the decimal of 2 cwt. 3 qrs.

3 qrs. 21 lbs. = 105 lbs.

2 cwt. 3 qrs. = 308 lbs.

and the fraction $\frac{105}{308} = \frac{15 \times 7}{44 \times 7} = \frac{15}{44} = \frac{3 \cdot 75}{11} = \cdot 3409$.

Hence ·3409 is the decimal required.

Ex. 3.—Reduce 18s. $8\frac{1}{4}$d. to the decimal of £1.

The rule may be applied most concisely as follows:—

4	1·
12	8·25
20	18·6875
	·934375

∴ ·934375 is the decimal required.

The farthing is first reduced to the decimal of a penny, and the 8d. prefixed; then 8·25d. are reduced to the decimal of a shilling, and the 18s. prefixed; lastly, 18·6875s. are reduced to the decimal of £1.

Ex. 4.—Reduce 3 qrs. 21 lbs. to the decimal of 1 ton.

28 { 7	21
4	3·
4	3·75
20	·9375
	·046875

Hence ·046875 is the decimal required.

Ex. XI.

Reduce—

1. 12s. 6d., 10s. $7\frac{1}{2}$d., 11s. $1\frac{1}{2}$d., 18s. $6\frac{3}{4}$d., each to the decimal of £1.

2. 13s. $0\frac{3}{4}$d., 10s. $8\frac{1}{4}$d., 9s. 6d., each to the decimal of 15s. $5\frac{1}{4}$d.

3. 7s. $8\frac{3}{4}$d. to the decimal of a guinea; and 3s. $2\frac{1}{4}$d. to the decimal of a moidore.

4. 1 qr. 7 lbs. to the decimal of 1 ton; 3 cwt. 3 qrs. 20 lbs. to the decimal of 3 tons.

5. $3\frac{1}{2}$ lbs. to the decimal of 1υ cwt.; 14 oz. to the decimal of 3 cwt. 2 qrs.

6. 20 grs. to the decimal of 1 lb. Troy; 3 dwts. 16 grs. to the decimal of 4 oz. 11 dwts.

7. 1 rood, 10 poles to the decimal of 1 acre; 3 roods, $15\frac{1}{8}$ square yards to the decimal of 5 acres.

8. $\frac{3}{8}$ hours to the decimal of 10 weeks; 7 h. 18′ to the decimal of 1 year (365 days).

9. Bring the sum of $\frac{4}{15}$ of 9 hours, $\frac{5}{8}$ of $12\frac{1}{2}$ days, $\frac{2}{3}$ of $7\frac{1}{2}$ minutes, to the decimal of a week.

10. Express a pound troy as the decimal of a pound avoirdupois.

11. Reduce the sum of 6 lbs. $6\frac{6}{7}$ oz. avoirdupois, and 8 oz. 6 dwts. 16 grs. to the decimal of 1 ton.

12. Express $2{\cdot}3\dot{6}$ of 4s. − $\cdot5\dot{1}\dot{8}$ of 9s. 2d. + $1{\cdot}4583\dot{3}$ of 6d. as the decimal of £5.

CHAPTER IV.

THE METRIC SYSTEM.

32.—The fundamental unit of the metric system is the *metre*. A metre is the ten-millionth, or $\frac{1}{10^7}$ part of 90° of the earth's meridian, and measures 39·3708 English inches. In order to express multiples and sub-multiples of this unit, and, indeed, of *any* unit in the metric system, we make use of one or more of the following prefixes:—

Deka, . .	10 times.	**Deci,** . .	10th.
Hecto, .	100 ,,	**Centi,** . .	100th.
Kilo, . .	1,000 ,,	**Milli,** . .	1,000th.
Myria, .	10,000 ,,		

We will arrange these prefixes and the word *unit* in order according to their signification, thus—

Myria, Kilo, Hecto, Deka, Unit, Deci, Centi, Milli.

Now, as we read this line from left to right, it is evident

that the words have a signification decreasing tenfold in value; and as we read it from right to left, they have a signification increasing tenfold in value.

It therefore follows that figures placed under the above words have a local as well as an intrinsic value; and further, if when a figure is wanting to complete the series, its place be filled up by a cipher, it will be seen that the local value corresponds exactly with the ordinary decimal notation.

Moreover, we have only to place a mark (in fact, a decimal point) at the right of the figure standing under any of the words of the above memorial line, and the given quantity is at once expressed in the denomination corresponding to that figure.

Thus, taking the metre as our unit:

Myriam.	Kilom.	Dekam.	Metres.	Decim.	Millim.
3	2	5	4	7	8

Myriam.	Kilom.	Hectom.	Dekam.	Metre.	Decim.	Centim.	Millim.
= 3	2	0	5	4	7	0	8

= 3·2054708 myriametres.
= 32·054708 kilometres.
= 320·54708 hectometres.
= 3205·4708 dekametres.
= 32054·708 metres.
= 320547·08 decimetres.
= 3205470·8 centimetres.
= 32054708· or 32054708 millimetres.

The following rule for expressing any quantity in terms of any one multiple or sub-multiple of the unit, or of the unit itself, is therefore evident:—

RULE.—Put ciphers in the place of any multiple, unit, or sub-multiple absent in the series, and write the figures in close order, as in the ordinary decimal notation. Then place a decimal point at the *right* of the figure corresponding to the denomination in which we wish to express the given quantity.

Ex. 1.—Express 5 myriam. 3 hectom. 6 decim. as metres.

Filling up with ciphers the vacant spaces, we have—

Myriam.	Kilom.	Hectom.	Dekam.	Metres.	Decim.
5	0	3	0	0	6

= 50300·6 metres.

Ex. 2.—Express 3 dekam. 4 decim. as myriametres.

Filling up with ciphers, we have—

Myriam.	Kilom.	Hectom.	Dekam.	Metres.	Decim.	Centim.	Millim.
0	0	0	3	0	4	0	0

= 0·0030400 myriametres = 0·00304 myriametres.

(The student will see that it was unnecessary here to extend the series beyond decimetres.)

Ex. 3.—Express 13 metres 502 millimetres as kilometres.

We may write the given quantity thus—

Kilom.	Hectom.	Deckam.	Metres.	Decim.	Centim.	Millim.
0	0	1	3	5	0	2

= 0·013502 kilometres.

We have hitherto spoken only of the fundamental unit, and its multiples and sub-multiples. We shall hereafter (Art. 35) explain the principal derived units, viz., the Gram, the Are, the Stere, the Litre, and the Franc; but as the multiples and sub-multiples of these derived units bear the same relation respectively to the corresponding derived unit, as in the case of the fundamental unit, all the preceding remarks relative to the multiples and sub-multiples of the fundamental unit apply equally to those of the Gram, the Are, the Stere, the Litre, and the Franc.

With regard to the units, multiples, and sub-multiples of square and cubic measure, properly so called, it is necessary to make a few remarks.

33. Square Measure.—The unit of square measure is the *square metre*; and since the series *myriam.*, *kilom.*, *hectom.*, *dekam.*, *metre.*, &c., decrease in value tenfold when read from left to right, and increases similarly when read from right to left, it follows that the series square myriam., square kilom., square hectom., square dekam., square metre, &c., will decrease or increase 10^2 or 100-fold. Hence we see that in square measure the multiples and sub-multiples increase or decrease successively 100-fold, and, therefore, when quantities in square measure are expressed by the ordinary decimal notation, each multiple or sub-multiple must occupy the place of two figures, a cipher being supplied when we have less than ten of any multiple or sub-multiple, and two ciphers when there is any blank in the series

	Sq. kilom.	Sq. hectom.	Sq. dekam.	Sq. metre.	Sq. centim.
Ex.	10	3	15	3	5

	Sq. kilom.	Sq. hectom.	Sq. dekam.	Sq. metre.	Sq. decim.	Sq. centim.
=	10	03	15	03	00	05

= 10·0315030005 square kilometres.
= 1003·15030005 „ hectometres.
= 100315·030005 „ dekametres.
= 10031503·0005 „ metres.
= 1003150300·05 „ decimetres.
= 100315030005 „ centimetres.

34. Cubic Measure.—The unit of cubic measure is the *cubic metre*, and hence after the remarks in the last article, since $10^3 = 1000$, when quantities in cubic measure are expressed by the ordinary decimal notation, the units, multiples, and sub-multiples must respectively occupy the place of three figures, ciphers being supplied to fill up blank spaces when necessary.

Ex. 1. 325 cubic metres 51 cubic decimetres.
= 325 „ „ 051 „ „
= 325·051 cubic metres.
= 325051 cubic decimetres.

Ex. 2.—25 cubic metres 3 cubic decim. 40 cubic centim.
= 25 cubic metres 003 cubic decim. 040 cubic centim.
= 25·003040 cubic metres.
= 25003·040 „ decimetres.
= 25003040 „ centimetres.

35. Derived Units.—The principal derived units of the metric system are—

1. The **Gram**, for measures of weight.

The *gram* is the weight of a cubic centimetre of distilled water at the temperature of 4° C.

1 gram = 15·4323 grains, 1 grain = ·0648 gram.

2. The **Are**, for land measure.

The *are* is a square whose side measures 10 metres ; it is therefore equal to a square dekametre, or 100 square metres.

1 are = 119·6033 square yards, 1 hectare = 2·471 acres, 1 acre = ·405 hectare.

3. The **Stere,** for fire-wood.

The *stere* is equivalent to a cubic metre. It is therefore the solidity of a cube whose edge measures 1 metre.

4. The **Litre,** for measures of capacity.

The *litre* is a capacity equal to the volume of a cube whose edge measures a decimetre or 10 centimetres. It is therefore equal to a cubic decimetre or 1,000 cubic centimetres, and 1,000 litres are equivalent to a cubic metre.

1 litre = ·2201 gallon, 1 gallon = 1·543 litres, 11 gallons = 50 litres nearly.

5. The **Franc,** for money.

The *franc* is a coin weighing 5 grams, and composed of an alloy, nine-tenths of which are silver and one-tenth copper.

The following table exhibits at a glance the fundamental unit and the above derived units, together with the multiples and sub-multiples at present in use :—

TABLE OF THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

MULTIPLES.				UNITS.	SUB-MULTIPLES.		
10,000	1,000	100	10		10th.	100th.	1,000th.
Myria.	Kilo.	Hecto.	Deka.	METRE, Long Measure.	Deci.	Centi.	Milli.
Myria.	Kilo.	Hecto.	Deka.	GRAM, Weight.	Deci.	Centi.	Milli.
		Hecto.		ARE, Land Measure.		Centi.	
			Deka.	STERE, Solid Measure.	Deci.		
		Hecto.	Deka.	LITRE, Capacity.	Deci.	Centi.	Milli.
				FRANC, Money.	Decime.	Centime.	

A quintal = 100 kilog. = 2 cwt. nearly; a millier, or tonneau de mer, = 10 quintals = 20 cwt. nearly.

Ex. XII.

Examples upon the Multiples and Sub-Multiples of the Units.

1. Express each of the following as metres—
15 myriam.; 20 kilom.; 1 hectom.; 27 dekam.; 25 decim.; 100 centim.; 345 centim.; 5294 millim.

2. How many centimetres in the following—
46 myriam.; 30 kilom.; 295 hectom.; 1·5 dekam.; 3·95 metres, 295 millim.?

3. Express according to the metric table—
20 kilog. 29 dekag.; 18 kilog. 85 decig.; 123 hectog. 13 centig.; 12 dekag. 296 millig.; 153 centig. 3 millig; 3427 millig.

4. How many decigrams in the following—
16 kilog. 12 centig; 25 hectog. 10 grams; 39,645 millig.; 20 kilog. 35 dekag. 5 grams?

5. Express in ares—
10 hectar.; 296 centiar.; 29 hectar. 3 centiar.; 3 hectar. 12 centiar.; 376,543 centiar.

6. How many square decimetres are there in the following—
100 sq. kilom.; 10 sq. hectom.; 5 sq. dekam.; 3498 sq. met.; 46 sq. met.?

7. How many steres in the following quantities—
15 dekas.; 394 decis.; 9 dekas. 2 decis.; 186 dekas. 3 decis.; 3764 decis.; 4 decis.?

8. Express as cubic metres—
10,000 cubic decim.; 1,234,567 cubic centim.; 372,456,126 cubic millim.; 1,000,000 cubic centim.; 639 cubic centim.; 293 cubic decim.

9. Express as litres—
3 kilol. 2 hectol. 3 litres; 4 decil.; 2 kilol. 3 millil.; 76,384 millil.; 2934 centil.; 830 dekal.; 34,576 decil.

10. Express as litres, as dekalitres, and as centilitres—
18 kilol. 3 hectol. 4 decil. 5 centil. 3 millil.

11. How many francs in the following sums of money—
100 cent.; 736 dec.; 24,645 cent.; 5 cent. 25 dec.; 1695 cent.?

12. How many centimes and how many decimes are expressed by the following—

13 francs; 7 fr. 13 c.; 12 fr. 3 dec. 5 c.; 29 c.; 3 fr. 2 dec.; 18 fr. 4 c.?

Addition, Subtraction, Multiplication, and Division in the Metric System.

36. Since (Art. 33) all quantities in the metrical system may be expressed as one denomination by figures whose local as well as intrinsic values follow the decimal system of notation, it is evident that when they are so expressed we may add, subtract, multiply, or divide them exactly as ordinary integers and decimals.

Ex. 1.—Add together 49 metres 36 centim.; 3 kilom. 2 dekam. 3 decim.; 2 hectom. 3 metres 25 centim.; and 13 dekam. 327 millim.

Metres.		Millimetres.
49·36		49360
3020·3		3020300
203·25	or thus, by integers—	203250
130·327		130327
3403·237		3403237

Ans.: 3 kilom. 4 hectom. 3 metres, 2 decim. 3 centim. 7 millim.

Ex. 2.—Subtract 3 kilog. 2 dekag. 37 millig. from 10 myriag. 25 grams, 369 millig.

Kilograms.		Milligrams.
100·025369		100025369
3·020037	or thus, by integers—	3020037
97·005332		97005332

Ans.: 9 myriag. 7 kilog. 5 grams, 3 decig. 3 centig. 2 millig.

Ex. 3.—Multiply 12 dekasteres, 3 steres, 5 decisteres by 23.

Steres.		Decisteres.
123·5		1235
23		23
3705	or thus, by integers—	3705
2470		2470
2840·5 steres.		28405 decisteres.

Ans.: 284 dekasteres, 5 decisteres.

Ex. 4.—Multiply 455,602 cubic centimetres by 36.

Cubic Metres.		Cubic Centimetres.
0·455602		455602
36		36
2733612	or thus, by integers—	2733612
1366806		1366806
16·401672		16401672

Ans.: 16 cub. met. 401 cub. decim. 672 cub. centim.

Ex. 5.—Divide 1369 kilol. 35 lit. 36 centil. by 72.

Kilolitres.		Centilitres.
72 { 9 | 1369·03536		72 { 9 | 136903536
8 | 152·11504	or thus, by integers—	8 | 15211504
19·01438		1901438

Ans.: 19 kilol. 1 dekal. 4 lit. 3 decil. 8 centil., or 19 kilol. 12 lit. 38 centil.

Ex. 6.—How many times is 12 sq. dekam. 3 sq. met. 15 sq. decim. contained in 216 sq. dekam. 56 sq. met. 70 sq. decim.?

Sq. Dekam. Sq. Dekam.		Sq. Decim. Sq. Decim.
12·0315)216·5670(18		120315)2165670(18
120315		120315
962520	or thus, by integers—	962520
962520		962520

Ans.: 18.

Ex. XIII.

1. Add together—

(1.) 3 metres, 2 decim. 4 centim.; 18 metres, 219 millim.; 4 kilom. 2 hectom. 3 dekam. 14 centim.; 12 kilom. 36 metres.

(2.) 74006 hectom., 3216 kilom.; 12 myriam. 2167 metres.

(3.) 4 sq. met. 42 sq. decim.; 12 sq. dekam. 18 sq. decim.; 82 sq. met. 3250 sq. decim.; 3·271 sq. met.

(4.) 18 cub. met. 186 cub. decim.; 39·207365 cub. met. 30761 cub. centim.; 12 cub. met. 124·27 cub. decim.

(5.) 25 kilog. 235 grams; 3072 centig.; 13 kilog. 51 grams, 63 millig. 8132·07 decig.

(6.) 319 hectar. 4 ares, 51 centiar.; 93·712 hectar. 23756·27 ares; 6 hectar. 4 centiar.

(7.) 3 steres, 5 decis.; 209 steres, 4 decis.; 25·76 steres, 13·027 dekas.

(8.) 51 kilol. 126 lit. 32 centil.; 123 lit. 3 centil. 15·02703 kilol.; 12 kilol. 3·27602 hectol.

(9.) 161 fr. 35 c.; 32 fr. 4 c.; 8276 c.; 10·26 fr.; 16 decimes, 5 c.

2. Subtract—

(1.) 4 metres, 372 millim. from 16 hectom. 5·06 metres.
(2.) 30765 centim. from 12 kilom. 4 metres, 9 millim.
(3.) 3 sq. met. 89 sq. decim. from 1 sq. dekam. 7 sq. decim.
(4.) 12·0324 cub. met. from 18 cub. met. 29 cub. millim.
(5.) 39 grams, 65 millig. from 6 kilog. 12 grams.
(6.) 8 hectar. 19·08 ar. from 32 hectar. 70 ar. 2 centiar.
(7.) 9 dekas. 6 decis. from 50 dekas. 2 decis.
(8.) 6 kilol. 6 millil. from 700 kilol. 3 lit. 3 centil.
(9.) 65 c. from 3 fr.; and 2 fr. 4 c. from 100 fr. 60 c.

3. Multiply—

(1.) 10 metres, 35 millim. by 7, 11, 13.
(2.) 18 kilom. 3·07 metres by 27, 48, 64.
(3.) 3·0625 sq. met. by 16, 18, 35.
(4.) 4 cub. met. 10 cub. decim. 5 cub. millim. by 19, 23, 26.
(5.) 7364 hectog. 9·31 decig. by 15, 25, 20.
(6.) 12 hectar. 3 centiar. by 30, 50, 40.
(7.) 416 steres, 2·9 decis. by 100, 150, 60.
(8.) 612305·06 litres by 12, 14, 16.
(9.) 39 fr. 10 c. by 75, 105, 135.

4. A merchant owed 1500 fr., and he gave in payment 69 metres of cloth at 3 fr. 4 c. per metre, 48 metres of silk at 8 fr. 65 c., 13·5 metres of calico at 75 c. How much does he still owe?

5. Make out the following bill—

		fr.	c.		
44 hectol. of oil, . .	at	0	75	the	litre.
66 kilog. 125 gr. of sugar,	,,	1	25	,,	kilog.
375 gr. of pepper, . .	,,	3	5	,,	,,
128·75 hectog. of soap, .	,,	1	75	,,	,,
562 gr. 5 decig. of coffee, .	,,	0	30	,,	hectog.

6. Divide—

(1.) 17 metres, 16 centim. by 11, 12, 13.
(2.) 41 kilom. 82 dekam. by 15, 16, 17.
(3.) 29 sq. met., 2740 sq. centim. by 14, 21, 35.
(4.) 376·38 cub. met. by 9, 27, 45.
(5.) 4 kilog. 14 dekag. 18 decig. by 22, 33, 55.
(6.) 8 hectar. 58 ares by 65, 60, 55.
(7.) 12 dekas. 1·2 decis. by 12, 13, 91.
(8.) 36 myrial. 4 kilol. 16 lit. 7 decil. by 9, 18, 27.
(9.) 7339 fr. 50 c. by 25, 30, 75.

7. Find the price of—

(1.) A metre, when 2 met. 80 centim. cost 70 fr.
(2.) A square decim., when 30 sq. met. cost 450 fr. 30 c.
(3.) A cubic metre, when 15 cub. decim. cost 361 fr. 80 c.
(4.) A hectometre, when 3 kilom. 125 metres cost 10 fr. 25 c.
(5.) A kilog. of coffee, when 7 hectog. 50 grams cost 1 fr. 35 c.
(6.) A hectare, when 4265 fr. 2·50 c. is the price of 149 ares, 25 centiar.
(7.) A stere, when 125 dekas. 4 decis. cost 20631 fr. 60 c.
(8.) A decilitre, when 47 dekal. 5 litres cost 570 francs.
(9.) A cub. centim., when 1 cubic metre cost 10,000 fr.

8. How many times is—

(1.) 1 kilom. 470 met. 38 centim. contained in 36759·50 metres?
(2.) 12 sq. decim. 75 sq. centim. contained in 10 sq. met. 20 sq. decim.?
(3.) 13 sq. met. 25 sq. decim. contained in 318 sq. dekam.?

(4.) 31 cub. met. 725 cub. decim. contained in 45684 cub. met.?

(5.) 345 millig. contained in 165 kilog. 6 hectog.?

(6.) 275 centiar. contained in 396 hectar.?

(7.) 7 steres, 2·5 decis. contained in 29 dekas.?

(8.) 4 kilog. 5 grams, contained in 38 myriag. 4 kilog. 480 grams?

(9.) 8 centimes contained in 10 francs?

9. A merchant bought 95 litres of wine for 118 fr. 75 c., and sold it at a loss of 10 c. per litre. What was the selling price per kilolitre?

10. To make 12 suits of clothes, it required 40 metres of stuff 90 centim. wide. How much stuff will it take if the width is 80 centim.?

11. How many cubic decimetres of iron are there in a bar weighing 280 kilog. 368 grams, when one cubic centim. weighs 7 grams 788 millig.?

12. An iron wire, 126 metres long, is cut into pieces 3 centim. 2·5 millim. long. How many pieces are there?

Relation between the Metric Units and the English System of Weights and Measures.

37. We shall work a few examples to show how quantities expressed in the metric system may be expressed in the English system, and *vice versa*.

Ex. 1.—Reduce 10 kilom. 321 metres to English measure.

$$\begin{aligned} 10 \text{ kilom. } 321 \text{ metres} &= 10321 \text{ metres.} \\ &= (10321 \times 1{\cdot}094) \text{ yards.} \\ &= \frac{10321 \times 1{\cdot}094}{1760} \text{ miles.} \\ &= 6 \text{ miles } 731{\cdot}174 \text{ yards.} \end{aligned}$$

Ex. 2.—Express 2 miles, 309 yards in the metric system.

$$\begin{aligned} 2 \text{ miles, } 309 \text{ yards} &= (2 \times 1760 + 309) \text{ yards.} \\ &= 3829 \text{ yards} = \frac{3829}{1{\cdot}094} \text{ metres.} \\ &= 3500 \text{ metres.} \\ &= 3 \text{ kilom. } 500 \text{ metres.} \end{aligned}$$

Ex. 3.—Reduce 1 ton to kilograms, having given 1 gram = 15·4323 grains.

$$\begin{aligned} 1 \text{ ton} &= 20 \times 112 \times 7000 \text{ grains.} \\ &= \frac{20 \times 112 \times 7000}{15{\cdot}4323} \text{ grams.} \\ &= 1016050{\cdot}7507 \text{ grams nearly.} \\ &= 1016 \text{ kilog. } 50 \text{ grams, } 750{\cdot}7 \text{ millig. nearly.} \end{aligned}$$

Ex. 4.—Express £13. 17s. $4\frac{1}{2}$d. in the pound and mil system.

(£1 = 10 florins, 1 florin = 10 cents, 1 cent = 10 mils.)

Reducing the given sum to the decimal of a pound, we have—

$$\begin{aligned} £13.\ 17\text{s.}\ 4\tfrac{1}{2}\text{d.} &= £13{\cdot}86875. \\ &= £13.\ 8 \text{ fl. } 6 \text{ cent. } 8\tfrac{3}{4} \text{ mil.} \end{aligned}$$

Ex. XIV.

1. Express a mile in the metric system, having given that a metre = 39·3708 inches.

2. An are contains 1076·43 square feet. Reduce 53 ares 25 centiares to English measure.

3. The area of a room is 22 sq. met. 26 square decim. Express this in English measure (1 metre = 39·3708 inches).

4. A block of marble measures 3 feet, 3 inches in length, 2 feet, 6 inches in depth, and 3 feet, 9 inches in width. What is the solid content expressed in cub. centim.?

5. In 1235 litres how many gallons, when 50 litres = 11 gallons nearly?

6. Supposing a franc to be equivalent to $9\frac{1}{2}$d., reduce £44. 13s. to francs.

7. Taking £1 sterling as equal to 25·22 francs, reduce £2. 13s. $7\frac{1}{2}$d. to francs.

8. In 1852 France reaped about 47850000 hectol. of wheat. Express this in gallons, assuming 1 gallon = 4·543 litres.

9. The ceiling of a room contains 83 sq. met. 53·96 sq. decim. What will be the expense of painting it at 10d. a square yard (1 metre = 1·094 yard)?

10. Find the cost of 2000 kilog. of sugar at 0 fr. 50 c. per lb.

11. In England the unit of work is the foot-pound, and in the metric system it is the kilogram-metre. Reduce 62 metric units of work to English units, taking 1 gram = 15·4323 grains, and 1 metre = 39·3708 inches.

12. The pressure of the atmosphere is $14\frac{3}{4}$ lbs. upon the square inch. Find the pressure in kilograms upon the square centimetre.

CHAPTER V.

PROPORTION.

38. Proportion is the equality of ratios.

Thus, since the ratio $6 : 8 = \frac{6}{8} = \frac{3}{4} = \frac{15}{20}$,
we have ratio $6 : 8$ = ratio $15 : 20$;

and we say that the numbers 6, 8, 15, 20 form a proportion. We generally express the fact thus—

$$6 : 8 :: 15 : 20.$$

It is easy to find *by trial* that *the product of the extreme terms is equal to the product of the means.*

Thus, we have $6 \times 20 = 8 \times 15$.

We may prove this property of the terms of a proportion to hold generally as follows:—

Suppose we have given the proportion $12 : 21 :: 20 : 35$.

It follows, from our definition above, that $\frac{12}{21} = \frac{20}{35}$, and multiplying each of these fractions by the product of their denominators, viz., by 21×35, we have

$$\frac{12}{21} \times 21 \times 35 = \frac{20}{35} \times 35 \times 21.$$

Now (Art. 8), $\frac{12}{21} \times 21 = \frac{12}{1} = 12$, and $\frac{20}{35} \times 35 = \frac{20}{1} = 20$, and we hence have $12 \times 35 = 20 \times 21$.

Now, we have not in our reasoning taken into account the actual value of the terms of the given proportion; and it is therefore evident that a similar result will follow from every proportion, and we may hence conclude generally:—

In every proportion *the product of the extremes is equal to the product of the means.*

39. Having given any three terms of a proportion, to find the remaining one.

Since the product of the extremes is equal to the product of the means, the following rule is evident:—

RULE.—If the required term be a mean, divide the product of the extremes by the other mean; but if the required term be an extreme, divide the product of the means by the other extreme.

Ex. 1.—28, 24, 30 are respectively the 1st, 3rd, and 4th terms of a proportion, required the 2nd term.

We have— 28 : required term : : 24 : 30

$$\therefore \text{ required term } = \frac{28 \times 30}{24} = \frac{7 \times 5}{1} = 35.$$

Ex. 2.—10, 45, 16 are respectively the 1st, 2nd, and 3rd terms, required the 4th term.

We have— 10 : 45 : : 16 : required term

$$\therefore \text{ required term } = \frac{45 \times 16}{10} = \frac{9 \times 8}{1} = 72.$$

Ex. 3.—2 hours, 45 minutes, 8 men are respectively the 1st, 2nd, and 3rd terms, required the 4th term.

We must express (Art. 6) the 1st and 2nd terms in the same denomination, and the proportion will stand thus—

Min. Min. Men.
120 : 45 : : 8 : required term.

Now, the ratio of the first two terms is the same as the ratio of the abstract numbers 120 and 45; and the 4th term must be of the same denomination as the 3rd term, otherwise the 3rd and 4th terms could not form a ratio.

We have therefore—

$$\text{Required term } = \frac{45 \times 8}{120} \text{ men } = \frac{3 \times 1}{1} \text{ men } = 3 \text{ men.}$$

Simple Proportion.

40. In Arithmetic we divide Proportion into Simple and Compound. Simple Proportion is the equality of two simple ratios, and therefore contains four simple terms; and the usual problem is to find the fourth term, having given the *first three* terms.

When we know the exact order of the given terms, the fourth term is, of course (Art. 38), found thus—

RULE.—Multiply the 2nd and 3rd terms together, and divide by the 1st.

The formal arrangement of the three given terms in their proper order is called *the statement;* and the only difficulty, therefore, in working a sum in Simple Proportion, or Single Rule of Three, as it is called, consists in stating it.

We shall work a few examples to illustrate the mode of doing this.

Ex. 1.—If 12 men earn £18, what will 15 men earn under the same circumstances?

We have here *two kinds* of terms, *men* and *earnings*, and whatever ratio any given number of men bears to any second given number of men, it is evident that it must be equal to the ratio of the earnings of the first lot of men to the earnings of the second lot, and we may therefore write—

Men. Men.
12 : 15 = £18 : 2nd earnings.

Men. Men.
or 12 : 15 :: £18 : 2nd earnings.

As the first two terms are of the same denomination, their ratio is not altered by treating them as abstract quantities, and the denomination of the 4th term must be the same as that of the 3rd.

Hence we have—

$$\text{Ans.:} \quad = £\frac{18 \times 15}{12} = £\frac{3 \times 15}{2} = £22.\ 10\text{s.}$$

Ex. 2.—If 18 men do a piece of work in 25 days, in what time will 20 men do it?

The two kinds of terms we have here to consider are *men* and *time.* In doing work we know that the *time* will *diminish* exactly as the number of *men increases*, and hence the ratio of the *second* lot of men to the *first* lot will be equal to the ratio of the *given* time to the *time required.* We therefore have—

Men. Men. Days.
20 : 18 :: 25 : required time.

$$\therefore \text{ Ans.:} \quad = \frac{18 \times 25}{20} \text{ days} = \frac{9 \times 5}{2} \text{ days} = 22{\cdot}5 \text{ days.}$$

We have reasoned out the above examples thus to show that the working of problems in Rule of Three depends upon the principle of the equality of ratios. Practically, however, we proceed as follows :—

Ex. 1.—If 12 men earn £18, what will 15 men earn under the same circumstances?

We are required to find *earnings*, and we therefore put down for the 3rd term the given *earnings*, thus—

£
: :: 18

The question is with regard to 15 men instead of 12 men, and we know their earnings must be *greater*. We therefore place the *greater* of these terms in the 2nd place and the other in the 1st, and the statement becomes—

Men. Men. £.
12 : 15 :: 18 : required earnings.

∴ as before—

$$\text{Ans.} = £\frac{15 \times 18}{12} = £\frac{15 \times 3}{2} = £22.\ 10\text{s.}$$

Ex. 2.—If 18 men do a piece of work in 25 days, in what time will 20 men do it?

We are required to find *time*, and we place therefore the given time, viz., 25 days, in the 3rd place.

Again, the question is with regard to 20 men instead of 18 men. Now, we know that 20 men require *less time* than 18 men to do a piece of work, and we hence place the less of these terms in the 2nd place. The statement then becomes—

Men. Men. Days.
20 : 18 :: 25 : required time.

∴ as before—

$$\text{Ans.} = \frac{18 \times 25}{20} \text{ days} = \frac{9 \times 5}{2} \text{ days} = 22{\cdot}5 \text{ days.}$$

Ex. XV.

1. If 12 articles cost £15, what will 624 cost?
2. What is the price of 35 loaves, when 29 loaves cost 15s. $8\frac{1}{2}$d.?
3. If I get 140 metres of cloth for 541 fr. 70 c., what must I pay for 89 metres. 3 decim.?

4. If 4 cubic metres of water run into a cistern in 18 minutes, in what time will it be full, supposing it to be 4 metres long, 6 metres, 25 centim. deep, and 35 decim. wide?

5. If the carriage of a parcel for the first 50 miles be 1s. 3d., and if the rate be reduced by one-third for distances beyond, how far can the parcel be carried for 1s. 7d.?

6. If a half-kilogram of sugar cost 1 fr. 10 c., what will be the cost of 3 kilog. 625 grams.?

7. There are two pieces of the same kind of cloth, measuring 43 yards and 57 yards respectively, and the second costs £1. 9s. 2d. more than the first. What is the cost of the first?

8. A garrison of 720 men have provisions for 35 days, and after 7 days 120 more men arrive. How long will the provisions last?

9. After paying 4d. in the pound income-tax a person has £299. 18s. 4d. left. What was the amount of his original income?

10. Two clocks, one of which gains 3 minutes and the other loses 5 minutes per day, are put right at noon on Monday. What is the time by the second clock when the first indicates 4 P.M. on the following Thursday?

11. When will the hands of a clock be exactly 30 minute divisions apart between 2 and 3 o'clock?

12. If I lend a friend £120 for 9 months, how long ought he to lend me £270?

Compound Proportion.

41. Compound Proportion is an equality between ratios, one of which at least is a ratio compounded of two or more simple ratios.

Arithmetical questions depending on Compound Proportion are generally said to belong to the Double Rule of Three; and the proportion consists of an equality between a ratio, on the one hand, compounded of two or more simple ratios; and, on the other hand, a simple ratio, whose consequent is required.

The following examples will illustrate the method of working questions in this rule:—

Ex. 1.—If 12 horses eat 20 bushels of corn in 8 days, in what time will 24 horses eat 16 bushels?

$$\left.\begin{matrix}24 \text{ horses} : 12 \text{ horses} \\ 20 \text{ bushels} : 16 \text{ bushels}\end{matrix}\right\} :: 8 \text{ days} : \text{required time.}$$

Explanation.—We are required to find *time*; and so, as in simple proportion, we put in the 3rd place the given time, viz., 8 days.

Leaving, for the present, the quantity eaten out of consideration, we know that 24 horses require less time to consume a given quantity of food than 12 horses do; we therefore place the *less* of these two terms in the 2nd place, and the other in the 1st place.

(The statement up to this point is 24 horses : 12 horses : : 8 days : required time, and we might obtain 4 days as an answer, irrespective of the quantity eaten. We might now place this answer in the 3rd term of another simple proportion, and take the *quantity eaten* into consideration, irrespective of the number of horses, thus getting an answer depending both upon the *number of horses* and the *quantity eaten*. It is more convenient, however, to proceed thus :)

Again, taking into consideration the *quantity eaten*, and *leaving out* of consideration the other given pair of terms, we see that *less time* is required to eat 16 bushels than to eat 20 bushels. We, therefore, put the *less term* in the 2nd place, and the other in the 1st.

Now, treating the terms of the ratios which occupy the 1st and 2nd places as abstract quantities, and compounding them, we have:

$$24 \times 20 : 12 \times 16 :: 8 \text{ days} : \text{required time.}$$

$$\therefore \text{ required time} = \frac{12 \times 16 \times 8}{24 \times 20} \text{ days} = 3{\cdot}2 \text{ days.}$$

Ex. 2.—How much bread can I get for 9d. when wheat is at 18s. a bushel, if the fourpenny loaf weigh 3 lbs. when wheat is at 20s. a bushel?

Proceeding as in Example 1, we have

$$\left.\begin{matrix}4\text{d.} : 9\text{d.} \\ 18\text{s.} : 20\text{s.}\end{matrix}\right\} :: 3 \text{ lbs.} : \text{weight required.}$$

Or, $4 \times 18 : 9 \times 20 :: 3$ lbs. : weight required.

$$\therefore \text{ weight required} = \frac{9 \times 20 \times 3}{4 \times 18} \text{ lbs.} = 7{\cdot}5 \text{ lbs.}$$

Ex. XVI.

1. If 15 men can build a wall 81 feet long in 18 days, how many men can build 135 feet of the same kind of wall in 30 days?

2. In 4 days, 18 workmen can dig a ditch 162 yards long, 7 feet wide, and 12 feet deep. What must be the depth of a ditch which 45 workmen can dig in 7 days, supposing it to be 387 yards long and 5 feet wide?

3. A traveller, going 15 hours a day, walks 1500 kilometres in 20 days. How far will he go in 30 days, walking 12 hours a day with the same velocity? Express your answer in English miles.

4. Two men are partners; one puts in a capital of £800, and receives as 6 months' profit £120. What is the capital of the other, who receives £3375 as 9 months' profit?

5. Two tourists having spent £1. 16s. 8d. in $2\frac{1}{2}$ days, meet three others with whom they continue their tour, and they spend while together £21. 1s. 8d., at the same rate per day. Required how long they were in company.

6. If 16 men and 10 boys do a piece of work in 10 days, in how many days would 8 men and 18 boys do a piece 7 times as great, supposing the work of 5 boys equal that of 2 men?

7. Supposing the rate of carriage to be diminished one-third after the first 50 miles, find the cost of carrying 16 cwt. for 40 miles, when 12 cwt. can be carried 100 miles for 4s. 2d.

8. A cistern is 8 metres, 4 decim. long, 1 metre, 8 centim. wide, and 275 centim. deep. Find the depth of another cistern of equal capacity whose length is 7 metres, 2 decim., and width 11 decim.

9. Persons whose incomes are less than £300 per annum are taxed upon £80 less than their income. Supposing 3 persons, having equal incomes, to pay £7 in the aggregate, at 4d. in the pound, find the total tax upon 14 persons, each having incomes 3 times as great.

10. If 12 horses eat 10 acres of grass in 16 weeks, and 18 horses eat 10 acres in 8 weeks, how many horses would eat 40 acres in 6 weeks, the grass being supposed to grow uniformly?

11. A boat, propelled by 8 oars, which take 28 strokes per minute, goes at the rate of $9\frac{1}{3}$ miles per hour. Find the rate of a boat propelled by 6 oars, which take 36 strokes per minute, the work done by each stroke of the latter being one-sixth less than that by each stroke of the former.

12. If 4 men and 10 women can do a piece of work in 8 days, which 12 women and 20 children can do in 4 days, in what time will 6 men, 18 women and 5 children do a work three times as great?

CHAPTER VI.

APPLICATION TO ORDINARY QUESTIONS OF COMMERCE AND TRADE.

Interest.

42. Interest is the money paid for the use of money.

The **Principal** is the money lent, and the **Amount** is the sum of the interest and principal.

The **Rate** of interest is the money paid for a given sum for a given time. £100 is in practice the given sum, and one year the given time. Thus, if £4 be paid for the use of £100 for one year, the rate of interest is £4 per cent. per annum, or as we generally say, 4 per cent.

Simple Interest is interest calculated on the original principal only.

Compound Interest is the interest which arises from adding the interest for each year to the principal of that year, and calculating interest for the next year upon the amount so obtained.

Simple Interest.

43. RULE.—Multiply the principal by the rate per cent. and by the number of years; then divide the product by 100, and the quotient will be the simple interest.

Ex. 1.—Find the simple interest of £420 for 3 years at 5 per cent.

£
420
5
2100
3
£63·00 Ans.: £63.

Reason for this process—

£100 gains £5 in 1 year, and the question is to find the gain upon £420 in 3 years.

Hence, proceeding as in Double Rule of Three, we have—

100 × 1 : 420 × 3 :: £5 : interest required.

∴ interest required = $£\frac{420 \times 5 \times 3}{100}$, which is exactly as stated in the rule.

Ex. 2.—Find the simple interest of £352. 1s. 8d. from March 16 to August 21, 1873, at 4 per cent.

The following process will be easily understood:—

£	s.	d.
352	1	8
		4
£14·08	6	8
	20	
	1·66s.	
	12	
	8·00d.	

∴ £14. 1s. 8d. is the interest for 1 year.

Now, from March 16 to August 21 are 158 days; hence we have—

365 days : 158 days :: £14. 1s. 8d. : Interest required.

∴ interest required = £6. 1s. $11\frac{9}{73}$d.

Ex. XVII.

Find the simple interest of—

1. £350 for 4 years at 5 per cent.
2. £295. 2s. 1d. for $3\frac{1}{2}$ years at 4 per cent.

3. £375. 8s. 4d. for $2\frac{1}{3}$ years at $4\frac{1}{2}$ per cent.
4. £160 from Feb. 1 to June 12, 1872, at $7\frac{1}{2}$ per cent.
5. £48 for 7 months at $1\frac{1}{4}$ per cent. per month.
6. £219. 4s. 2d. for 6 years at $1\frac{2}{3}$ per cent.

In the six following examples, understand simple interest.

7. At what rate per cent. will £129. 8s. 4d. gain £6. 3s. $5\frac{1}{2}$d. in $2\frac{1}{2}$ years?

8. A certain sum amounts in 3 years at $7\frac{1}{2}$ per cent. to £289. 16s. $3\frac{1}{2}$d.; find the original sum.

9. In what time will £175. 6s. 3d. amount to £192. 7s. $10\frac{1}{2}$d. at $2\frac{1}{2}$ per cent?

10. What sum will amount in 2 years 9 months at 4 per cent. to £427. 7s?

11. If £320 gain £9 in 13 months, in what time will £480 gain £6 at the same rate?

12. Find the interest of £29. 7s. 5d. for 6 months at $5\frac{5}{19}$ per cent.

Compound Interest.

44. RULE 1.—Find the interest for one year as in Simple Interest, and add it to the principal; then find the interest for one year upon this amount reckoned as *principal* for the second year, and add it to the second year's principal, and so on. Subtract the original principal from the amount so obtained for the given number of years, and the result will be the compound interest required.

RULE. 2.—Divide the given rate per cent. by 100, putting the result in a decimal form, and place *unity* before the decimal point. Raise the number thus obtained to a power corresponding to the given number of years, multiply the principal by the result, and we get the *amount* for the given number of years.

Thus, supposing 5 to be the rate per cent., we have, dividing by 100 and placing *unity* before the result, the number 1·05. Then, if the given number of years be 4, and £162 the principal, we have, according to rule, amount $= £162 \times (1{\cdot}05)^4$.

The first rule requires no explanation; the second rule may be explained thus:—

Ex.—Find the compound interest of £360 for 3 years at 4 per cent.

Now, interest for £100 for 1 year = £4,
∴ „ £1 „ = £·04;
hence, amount of £1 „ = £1 + £·04 = £1·04.

We thus see that the amount of £1 for 1 year at 4 per cent. is 1·04 times the original sum. It therefore follows that—

Amount of the £1·04 for 1 year = 1·04 times £1·04,
∴ amount of £1 „ 2 years = £(1·04 × 1·04)
= £$(1·04)^2$;
and so, amount of £1 „ 3 years = 1·04 times £$(1·04)^2$
= £$(1·04)^3$,
hence, amount of £360 „ „ = 360 times £$(1·04)^3$
= £360 × $(1·04)^3$.

The compound interest is then found by subtracting from this the original principal.

Ex. 1.—Find the compound interest of £570 for 3 years at 5 per cent.

(We shall work this by Rule 1, and for convenience shall keep our quantities in a decimal form.)

£
570
5
£28·50 = interest for first year.

570
£598·5 = principal for second year.
5
£29·925 = interest „ „
598·5
£628·425 = principal for third year.
5
£31·42125 = interest „ „
628·425
£659·84625 = amount at end of third year, } therefore
570 = original principal; } subtracting,
£89·84625 = compound interest for 3 years.

Ex. 2.—Find the compound interest of £327. 12s. 6d. for 4 years at 3 per cent.

Now, £327. 12s. 6d. = £327·625.

Hence, by Rule 2—

Amount = £327·625 × $(1·03)^4$ = £368. 14s. $10\frac{3}{4}$d. nearly.

Ex. 3.—What sum of money, if put out for 2 years at 4 per cent., will amount to £324. 9s. $7\frac{1}{5}$d., compound interest being reckoned?

By Rule 2, the principal may be found by dividing the amount by $(1·04)^2$.

Now, given amount = £324. 9s. $7\frac{1}{5}$d. = £324·48.

Hence principal required = £324·48 ÷ $(1·04)^2$ = £300.

Ex. XVIII.

Find the compound interest of

1. £284 for 2 years at 4 per cent.
2. £312. 12s. $7\frac{1}{2}$d. for 3 years at 5 per cent.
3. £283. 10s. for 2 years at $3\frac{1}{2}$ per cent.
4. £605. 12s. 6d. for 4 years at 4 per cent.
5. What is the difference between the simple and compound interest of £150 for 2 years at 6 per cent.?
6. Find the amount of £381. 1 florin 3 cents 5 mils for 3 years at 5 per cent.

(£1 = 10 florins, 1 florin = 10 cents, 1 cent = 10 mils.)

7. Find the amount of £250 for 2 years at 4 per cent. per annum, interest being payable half-yearly.
8. What sum will amount in 3 years at $4\frac{1}{2}$ per cent. compound interest to £200?
9. A town has 200,000 inhabitants, and it increases at the rate of 5 per cent. per annum; find the number of inhabitants at the end of 3 years.
10. Find the difference in amount of £350 for 3 years at 4 per cent. simple interest, and £420 for 2 years at 5 per cent. compound interest.
11. How much would a person who lays by £50 a year at 5 per cent. compound interest, draw out at the end of 4 years?
12. A person expects to receive £450 in 3 years; what present sum is equivalent to this, reckoning compound interest at 4 per cent.?

Discount.

45. When money is paid before it is due, the payee may, of course, put out the money at interest for the rest of the term, and thereby increase it. It therefore follows that the amount which ought to be paid for the discharge of an account before its proper time should be such a sum that, if put out at interest for the remainder of the term, will just amount to the original sum in question.

Thus £102. 10s. (interest being reckoned at 5 per cent. per annum) payable 6 months hence, would be fully discharged by paying £100 at once. For £100 in 6 months at 5 per cent. per annum would amount to £102. 10s. Hence, the payee ought to remit £2. 10s. from the full account. The amount remitted is called **discount.**

It will be seen, therefore, that the discount on £102. 10s. due 6 months hence at 5 per cent. is £2. 10s.

Bankers, however, are in the habit of charging *interest* instead of *discount.* The banker's discount, therefore, on £100 due 6 months hence at 5 per cent. is £2. 10s.

Hence, the *true* discount on £102. 10s. due 6 months at 5 per cent. is the same as the *banker's* discount on £100 under the same circumstances; and bankers' discount on any given sum is in excess of the true discount.

Tradesmen's bills are legally due *three days* after the term for which they are drawn is completed. This extension of time is called *three days of grace.* When a bill falls due on a Sunday, it is usual in England to meet it on the previous Saturday.

Ex. 1.—Find the difference between the true discount and the banker's discount on £306 due 4 months hence at 6 per cent.

Now, £100 would in 4 months gain $\frac{1}{3}$ of £6, or £2. Hence, the true discount on £102 due four months hence at 6 per cent. is £2, and therefore we have—

£102 : £306 :: £2 : true discount required.

Hence, true discount $= £\frac{306 \times 2}{102} = £6.$

Again, proceeding according to the rule for simple interest—

$$\text{Banker's discount} = \frac{£306 \times 6 \times \frac{1}{3}}{100} = £\frac{612}{100} = £6.\ 2s.\ 4\tfrac{4}{5}d.$$

Hence, the excess of the banker's discount over the true discount is 2s. $4\frac{4}{5}$d.

Ex. 2.—A bill of £350 drawn on March 15, at 6 months, is cashed on May 20, 1872; what is the banker's discount at 6 per cent.?

The bill is legally due on Sept 18, and from May 20 to Sept. 18 are 121 days.

Now, the *interest* on £350 for 1 year at 6 per cent. is easily found to be £21. Hence,

365 days : 121 days :: £21 . banker's discount required;

and ∴ banker's discount = $£\frac{21 \times 121}{365}$ = £6. 19s. $2\frac{5}{7}\frac{8}{3}$d.

Percentages.

46. There are many questions which relate to ordinary commercial transactions which may be worked exactly as if we had to find the simple interest for one year—*e.g.*, questions in commission, brokerage, insurance, &c.

Commission is a sum of money charged, by an agent for buying or selling goods, at a certain rate per cent. upon the value of the goods.

Brokerage is similar to commission, but it is charged upon money transactions instead of upon the sale of goods.

Insurance is a sum charged per cent. upon the value of property, the said value being paid to the insured in case of loss from causes as per agreement.

Ex. 1.—Find the brokerage on £625. 5s. at $4\frac{1}{2}$ per cent.

£	s.	d.
625	5	0
		$4\frac{1}{2}$
2501	0	0
312	12	6
28·13	12	6
20		
2·72s.		
12		
8·70		

Ans.: £28. 2s. $8\frac{7}{10}$d.

Ex. 2.—The rate of insurance is $4\frac{1}{4}$ per cent., and the value of some property insured is worth £766. What will be the annual payment, so that, in case of fire, the owner may receive back his premium, as well as the value of his property?

If, instead of paying £4. 5s. as insurance on every £100, he pays £4. 5s. upon every (£100 – £4. 5s.) or upon every £95. 15s.; then, in case of fire, he will receive £100 for a damage of £95. 15s., and thus have the value of his property and the amount of his premium.

The problem is, therefore—

If £4. 5s. is the premium on £95. 15s., what is the premium on £766?

Hence—

£95. 15s. : £766 :: £4. 5s. : premium required.

And, therefore, premium = £34.

Ex. XIX.

1. Find the banker's discount on £412, due 6 months hence, at 6 per cent.
2. By how much does the banker's discount on £100, due 3 months hence, at 5 per cent., exceed the true?
3. What discount would be charged upon a bill drawn for £320, on April 15, at 4 months, and presented for payment on June 3 (discount at 7 per cent.)?
4. Find the discount on a bill drawn on Aug. 3 for £200, at 6 months, and cashed on Sept. 10, discount being reckoned at 6 per cent.
5. A man buys goods for £250, being allowed 6 months credit, and he immediately sells them for the same amount, allowing 3 months credit. What does he gain by the transaction, interest being reckoned at 5 per cent.?
6. Find the brokerage on £352. 17s. 6d. at 3 per cent.
7. What is the brokerage on £4500 at $\frac{1}{8}$ per cent.?
8. What is the commission on the sale of goods to the amount of £850 at 5 per cent.?
9. A person insures for £1050 at $3\frac{3}{4}$ per cent. What is his annual premium?

10. What will be the annual premium on property worth £965, so that the insured may obtain his premium back again with the value of his property, in case of loss—insurance being at $3\frac{1}{2}$ per cent. ?

11. The brokerage on a certain sum at $\frac{1}{4}$ per cent. is £5. 10s. $7\frac{1}{2}$d. Find the sum.

12. Together with a commission of 4 per cent., goods cost a person £339. 8s. 5d. Find the cost price to the agent.

Stocks and Shares.

47. When a large amount of capital is to be raised, a company is generally formed, which raises the money by the issue of shares. We will suppose a person to hold a £100 share; he will then be entitled to such a part of the profits of the company as £100 is of the whole capital. If there be a great demand for these shares, persons holding them may dispose of them for more than the nominal value, say for £106; whereas, if they are very little in demand, the seller may be glad to sell at, perhaps, £70. In the first instance, we should say that the shares were at 106, or at 6 *premium;* and in the second instance, that the shares were at 70, or at 30 *discount.* If the selling price of the shares is £100, they are said to be at *par.*

So, when we read that the Three per Cent. Consols are quoted at $96\frac{3}{8}$, it means that an acknowledgment of indebtedness on the part of the Government to the amount of £100, bearing interest at 3 per cent. per annum, may be bought for £$96\frac{3}{8}$.

The buying and selling of stocks and shares is carried on by brokers, who charge a percentage from $\frac{1}{8}$ to $\frac{1}{2}$, sometimes upon the *nominal* value of the stock, but mostly upon the *actual cash value.* When brokerage is to be taken into account in any example it will be mentioned, and it will be estimated by the first method, unless specified.

Ex. 1.—What is the value of £1000 stock at $89\frac{5}{16}$ per cent. ?

$$\text{No. of cents. stock} = \frac{1000}{100} = 10$$

$$\therefore \text{ value required} = £89\tfrac{5}{16} \times 10 = £893.\ 2s.\ 6d.$$

Ex. 2.—What would be the cost of £1180 stock at $153\frac{1}{4}$ per cent., including brokerage at $\frac{1}{4}$ per cent.?

No. of cents. stock $= \frac{1180}{100} = \frac{118}{10}$,

and total cost of each cent. $= £(153\frac{1}{4} + \frac{1}{4}) = £153\frac{1}{2}$.

Hence, required cost $= £153\frac{1}{2} \times \frac{118}{10} = £1811.\ 6s.$

Ex. 3.—What is the annual income arising from investing £6510 in the Four per Cents. at 93?

Here, price of £100 stock $= £93$,

and $\therefore$ No. of cents. stock for £6510 $= \frac{6510}{93} = 70$.

Hence, annual income $= £4 \times 70 = £280$.

Ex. XX.

1. Find the cost of £750 stock at $92\frac{1}{8}$ per cent.

2. I sell out £325 Three per Cent. Consols at 94. What do I get after allowing the broker $\frac{1}{4}$ per cent. upon the cash he receives for the sale?

3. Invest £5065. 9s. 9d. in the Three per Cent. Consols at $91\frac{5}{8}$.

4. What would be the cost of £413. 1s. 9d. reduced Three per Cents. at $92\frac{3}{8}$, including a brokerage of $\frac{1}{4}$ per cent. upon the cost to the broker?

5. What would be the proceeds of the sale of £6228 India Five per Cent. stock at $111\frac{1}{4}$, deducting $\frac{1}{4}$ per cent. upon the selling price for brokerage?

6. If I sell £8160 Spanish Three per Cent. Bonds at 31 per cent, and invest the proceeds, less $\frac{1}{8}$ per cent. brokerage, in Indian Railway Five per Cent. Stock at 107, what will be the difference in my annual income?

7. What is the value sterling of $4000 American Bonds at $93\frac{5}{8}$ per cent. ($ = 4s. 6d.)?

8. What is the value of $37,000 United States Bonds at $93\frac{5}{16}$ per cent.?

9. Invest £954. 0s. 7d. in India Five per Cent. Stock at $110\frac{3}{8}$, allowing $\frac{1}{8}$ per cent. brokerage.

10. What is the sterling value of 5000 francs Italian Bonds at 67 per cent. (Exchange 25 fr.)?

11. Find the sterling value of 6000 guilders Dutch $2\frac{1}{2}$ per Cent. Bonds at $56\frac{1}{4}$ (Exchange 12 guilders).

12. Invest £1025 in French Rentes at $51\frac{1}{8}$, allowing $\frac{1}{8}$ per cent. brokerage (Exchange 25 fr.).

Annuities.

43. Annuities are annual payments, the first payment being due at the end of a year. When an annuity is left untouched for a number of years, its *amount* is properly obtained by allowing compound interest.

49. To find the amount of an annuity of a given sum for a given time, at a given rate per cent.

Let us suppose the annuity to be £1, the time 4 years, and 5 the rate per cent. The first payment will not be due till the end of a year; so that at the end of 4 years it will have been accumulating for 3 years at compound interest; so the next payment, not being due till the end of the second year, will have been accumulating for 2 years at compound interest; and so on, the last payment being made when due.

Hence, the amount of an annuity of £1, for 4 years at 5 per cent., will be (reversing the above order) as follows:—

£1 + amount of £1 for 1 year + amount of £1 for 2 years + amount of £1 for 3 years (compound interest being reckoned).

And it is evident that for any other annuity we may follow the same method, and at the end multiply the sum by the number of £ in the annuity.

Ex.—Find the amount of an annuity of £300 for 4 years at 5 per cent.

We shall find the respective *amounts* by Rule 2, Art. 44. Thus—

£1·05 = amount for 1 year of £1 at 5 per cent.
1·05
525
105
£1·1025 = ,, ,, 2 years ,, ,,
1·05
55125
11025
1·157625 = ,, ,, 3 years ,, ,,

Hence, amount of annuity of £1 for 4 years—

= £1 + £1·05 + £1·1025 + £1·157625 = £4·310125.

∴ amount of given annuity = £4·310125 × 300 = £1293. 0s. 9d.

50. To find the present value of an annuity to continue for a given time.

By the present value of an annuity to continue a given number of years is meant such a lump sum which, paid down at once, would, by accumulating at compound interest for the same time, amount to just the same sum as the annuity itself if it were allowed to accumulate. In the absence of algebraical symbols, we shall best illustrate the method of finding such a lump sum by an example.

Ex.—Find the present value of an annuity of £50 for 3 years at 4 per cent.

As in Art. 49, we find the amount of the annuity—

$$= \{£1 + £1·04 + £(1·04)^2\} \times 50 = £156·08.$$

Now, whatever be the present value required, we know that its amount in 3 years at 4 per cent. compound interest is found (Art. 44) by multiplying it by $(1·04)^3$ or 1·124864.

It, therefore, follows that if we know this amount beforehand we can find the present value by dividing it by 1·124864.

Hence, present value = £156·08 ÷ 1·124864 = £138·755 nearly.

51. To find the present value of an annuity to continue for ever.

It is evident that the sum we require is one which, put out at interest, will annually produce a sum equal to that of the annuity itself. The problem then is simply this—

Having given the interest for 1 year of a certain sum, and the rate per cent., to find the principal.

We have therefore the following rule:—

RULE.—Divide the given annuity by the rate per cent.; and multiply by 100, and the result is the present value.

EX.—How much must a gentleman invest at 5 per cent. in order to endow a charity with £60 a year.

Present value of the annuity of £60 to continue for ever—

$$= £\frac{60 \times 100}{5} = £1200.$$

There are many other questions connected with annuities which are, however, best left till the student has a knowledge of Logarithms.

EX. XXI.

Find the amount of an annuity of—

1. £120 for 3 years at 4 per cent.
2. £250 for 4 years at $4\frac{1}{2}$ per cent.
3. £321 for 5 years at 5 per cent.
4. What is the present value of an annuity of £80, to continue for six years, at 6 per cent.?
5. A person who, according to the tables of mortality, is likely to live 10 years, wishes to insure an annual payment of £40 during life. What sum must he pay down, reckoning interest at 5 per cent.? (Give the result to four places of decimals.)
6. A house produces a clear rental of £30. How many years' purchase is it worth, interest being reckoned at 5 per cent.?
7. A gentleman invested a sum of money in the Three per Cent. Consols, in order that an annual payment of 7s. 6d. a year might be made in bread for ever. What sum did he invest?
8. Find the present value of a pension of £120 a-year, *payable half-yearly* for 5 years, interest being at the rate of 5 per cent. per annum.

9. A house, which ordinarily lets for £80 a-year, is leased for a term of four years, at a rent of £20, a certain sum being paid in addition at the time of letting. Find this latter amount.

10. What is the present value of a freehold which produces a clear rental of £50, but which cannot be entered upon for two years, reckoning interest at 5 per cent.?

11. Find the annuity which in four years, at 4 per cent., will amount to £100.

12. A corporation borrows a sum of £3000 at 4 per cent. What annual payment will clear off the debt in ten years? (Give the result correct to four places of decimals.)

Profit and Loss.

52. All questions involving the loss or gain per cent. by any transaction belong to this rule, and may be generally worked by Proportion.

Ex. 1.—A man buys goods at 5s. and sells them at 5s. 8d. Find his gain per cent.

The actual gain upon 5s. is 8d., and we are required to find the gain upon £100.

Now 5s. : £100 :: 8d. : gain upon £100,

$\therefore$ gain upon £100 = $£\frac{100 \times 8}{5 \times 12} = £13\frac{1}{3}$,

or, required gain per cent = $13\frac{1}{3}$.

Ex. 2.—By selling goods at 6s. 3d. there is a gain of 25 per cent. What will be the selling price to gain 10 per cent.?

Now, selling price of goods which cost £100, so as to gain 25 per cent, is £125, and that to gain 10 per cent. is £110.

Hence £125 : £110 :: 6s. 3d. : selling price required;

from which, selling price required = 5s. 6d.

Ex. 3.—Find the cost price when articles sold at 1s. 9d. entail a loss of $12\frac{1}{2}$ per cent.

Now, articles which cost £100 when sold at a loss of $12\frac{1}{2}$ per cent. must sell for $£87\frac{1}{2}$.

Hence $£87\frac{1}{2}$: 1s. 9d. :: £100 : cost price required;

from which, cost price = 2s.

Ex. XXII.

1. Find the cost price of goods which are sold at a loss of 10 per cent. for 4s. 10½d.

2. Goods which are sold for 7s. 11d. entail a loss of 5 per cent. What should be the price to gain 30 per cent.?

3. A tradesman reduces his goods 7½ per cent. What was the original price of an article which now fetches £1. 7s. 9d.?

4. In what proportion must tea at 4s. 2d. be mixed with tea at 6s. a pound, so that a grocer may sell the mixture at 5s. 6d. and gain by the sale 10 per cent.?

5. A quantity of silk, after paying a duty of 12½ per cent., cost £54. Find the original cost price.

6. An innkeeper buys 37½ gallons of brandy at 14s. a gallon, and adds to it sufficient water to enable him to sell it at the same price and gain 12 per cent. How much water does he add?

7. By selling goods at 8s. 2d. a tradesman gains 16⅔ per cent. What will be the gain or loss per cent. by selling at 6s. 1½d.

8. A company has a capital of £750,000, and the working expenses for the year have been £42,123. 12s. 6d. What must have been the gross receipts in order that the shareholders may receive a dividend of 4 per cent.?

9. If stock which is bought at 91½ is immediately sold at 91⅝, what is the gain per cent.?

10. A person buys goods at 6 months' credit and sells them for cash at the nominal cost price immediately. What is his gain per cent.? (Interest 5 per cent.)

11. Goods are marked at a ready-money price and a credit price allowing 12 months. The credit price is £4. 9s. 3d., what is the ready-money price?

12. Goods are now being sold at 10 per cent. loss. How much per cent. must be put upon the selling price in order that they may be sold at 20 per cent. gain?

Square Root and Cube Root.

53. To avoid unnecessary repetition, the student is referred to the articles on Involution, Algebra, stage I., where the arithmetical principles and methods are explained.

Estimates.

54. The following specimens will give the student an idea of what he may expect to meet with under the head of Estimates. It is usual, in ordinary transactions, to use certain abbreviations; as *cub.* for cubic measure, *sup.* for superficial measure, *run.* for running or lineal measure. Builders, too, are in the habit of calling twelfths of a foot—whether it be cubical, superficial, or lineal measure—by the name of inches. The names yards, feet, inches, are often written thus: yds., ′, ″.

Ex. 1.—Digger, Bricklayer, and Mason.

Yds.	Ft.	In.			£	s.	d.
25	0	0	Cub.	Digging in trenches, filling, wheeling, and carting away, ... 1/			
232	0	0	Sup.	9″ reduced common brickwork in mortar, pointed on both sides, ... 4/2			
5	0	0	,,	4½″ trimmer arches to hearths, ... 2/3			
16	4	0	,,	Best blue brick on edge, paving in cement, 4/6			
6	0	0	,,	6″ blue and red quarries do., 3/10			
14	0	0	Run.	12″ round blue coping bricks in cement, ... 2/4			
9	0	0	,,	Best red brick flat steps do., ... 1/2			
33	2	0	,,	Extra only to splayed brick angles to doors and windows, ... 0/2½			
12	1	0	,,	Do. to plinth in cement, ... 0/3			
5	0	0	,,	Do. to sailing course to chimney, ... 0/2			
12	0	0	,,	Do. to cornice to eaves of best red bricks, three courses, ... 0/5½			
			No.	2 core chimney flues, ... 1/6			
			,,	1 set sink, ... 4/			
			,,	4 do. stoves, ... 5/3			
			,,	1 do. copper, ... 5/			
			,,	12 do. ornamental air gratings, ... 0/6			
	19	6	Run.	Gable coping of hard stone 13″ × 4½″, twice splayed, ... 1/9			
	15	0	,,	2½ rubbed hearth and back hearth, ... 1/2			
	7	9	,,	Solid York step 14″ × 6″, tooled, ... 2/6			
			No.	2 knee stones to gables 18″ × 14″ × 10″, ea. 7/6			
			,,	1 apex stone do., 16″ × 14″ × 12″, ... 8/			
			,,	1 York stone sink 4′ 0″ × 1′ 9″, with rounded corners and hole cut for waste-pipe, ... 10/			
			,,	2 stone chimney-pieces with 7″ × 1½″, chamfered jambs, mantel, and shelf, ... ea. 24/			
				Total, ...			

Ex. 2.—Carpenter and Joiner. (A square = 100 sq. ft.)

Sq.	Ft.	In.				£	s.	d.
	178	6	Cub.	Fir framed-in roof timbers,.....	2/9			
	97	0	,,	Do. do., floors,.....	2/8			
	45	0	Sup.	Centreing to trimmer hearths,.....	0/4			
				Do. to 3 openings 5 0″ wide, with segment heads in $1\frac{1}{2}$ brick wall,.....ea.	2/			
				Do. to 4 do. 3′ 0″ wide do.,.....ea.	1/6			
	326	0	,,	Labour in planing roof and floor-timbers,..	$0/0\frac{3}{4}$			
	139	0	Run.	Do. in stop chamfering edges of do.,.....	$0/0\frac{1}{2}$			
	36	0	,,	$7'' \times 1\frac{1}{2}''$ ridge board,.....	0/3			
	21	0	,,	$3'' \times 1\frac{1}{2}''$ chamfered fillet to eaves,.....	$0/1\frac{1}{2}$			
3	45	0	Sup.	Inch clean red deal batten for boarded floors, wrought,.....	26/			
	61	0	Run.	$7'' \times 1''$ torus skirting, plugged,.....	0/3			
			No.	2 labour to mitred margins to hearths,....	$0/9\frac{1}{2}$			
			,,	16 mitres to skirting,.....ea.	0/2			
	114	3	Sup.	Inch deal treads and risers, ploughed, tongued, and screwed on 3 — $7'' \times 2\frac{1}{2}''$ carriages,.....	0/9			
	21	9	Run.	$1\frac{1}{4}''$ wall-string housed, for treads and risers,.....	0/8			
	14	6	,,	$1\frac{1}{2}''$ close-string do., and sunk and beaded,..	0/9			
	14	6	,,	$3'' \times 2\frac{1}{2}''$ rounded oak hand-rail, French-polished,.....	2/			
	12	8		$9'' \times 1''$ beaded fascia,.....	0/4			
	36	6	Sup.	2″ — 6 panel doors, bead, flush, and square,	1/3			
	23	7	,,	$1\frac{1}{2}''$ do. do.,	1/1			
	30	0	Run.	$4\frac{1}{2}'' \times 3''$ rebated and beaded frame,.....	0/6			
	26	0	,,	$4\frac{1}{2}'' \times 2''$ do. do., ...	$0/4\frac{1}{2}$			
	60	0	,,	$3'' \times 1\frac{1}{2}''$ moulding, mitred,.....	0/3			
	57	0	Sup.	2″ ovalo-moulded sashes, double hung, in deal-cased frames, with oak sunk and weathered sills, ...	1/6			
	37	0	,,	$1\frac{3}{4}''$ do. do.,	1/4			
	22	6	Run.	$\frac{7}{8}''$ mitred bead,.....	0/1			
	35	0	,,	$5\frac{1}{2}'' \times \frac{3}{4}''$ bead lining,.....	$0/3\frac{1}{2}$			
			No.	11 inch-beaded centre boards,.....	0/6			
				Total,.....				

MISCELLANEOUS EXAMPLES.

(Selected from University and other Examination Papers.)

1. Show by an easy example that the division of one whole number by another is equivalent to a series of subtractions.

Divide 1·02 by $\frac{1}{2}\frac{7}{4}$ of ·144.

2. If the Three per Cents. are at $91\frac{1}{8}$, what interest does this give on £100? (Omit brokerage and fractions of a penny.)

3. How many lbs. in ·321875 of a ton weight? Convert it into kilograms (omitting fractions), assuming that a cubic decimetre of distilled water weighs 15432·35 grains.

4. Reduce to their simplest forms $\frac{\sqrt{1\cdot75} - \sqrt{\cdot63}}{\sqrt{3\cdot5} - \sqrt{2\cdot1}}$ and

$$\frac{\sqrt[3]{5\cdot12} + \sqrt[3]{\cdot03375}}{\sqrt[3]{80} - \sqrt[3]{\cdot01}}.$$

5. Convert $_{1}\frac{1}{2}\frac{}{8}$ into a decimal fraction, and find the vulgar fraction corresponding to the recurring decimal $\cdot2\dot{2}9\dot{7}$.

6. Show, by proper attention to the value of the figures, in multiplying one number by another, that the order in which the figures of the multiplier are taken is of no importance. Multiply 61·143 by 47·982 correctly to three places of decimals, beginning with the left hand figure of the multiplier, and use as few figures as possible.

7. Extract the square root of 1095·61, and find to three places of decimals the value of $\frac{4}{\sqrt{5} - 1}$.

8. Find the compound interest of £55 for one year, payable quarterly, at 5 per cent. per annum.

A person bought into the Three per Cents. at 98, and after receiving three years' interest he sold at 90. How much per cent. on the sum invested did he gain or lose?

9. Three gardeners working all day can plant a field in 10 days; but one of them having other employment can only work half time. How long will it take them to complete the work?

10. What fraction of a crown is $\frac{3}{5}$ of 6s. 8d.? What is the value of $\frac{2}{7}$ of a guinea? Reduce $11\frac{3}{4}$d. to a decimal of a pound, correct to five places of decimals.

11. Reduce the expressions—

$$\frac{3}{20} + \frac{5}{12} + \frac{4}{15} - \frac{5}{6}, \text{ and } \sqrt{\left\{\frac{\frac{1}{4} + \frac{1}{5}}{\frac{1}{4} - \frac{1}{5}} + \frac{\frac{1}{3} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}\right\}}.$$

Multiply $49\frac{15}{16}$ by $50\frac{1}{16}$, and add $\frac{1}{2}\frac{1}{5}\frac{1}{6}$ to the result.

Divide $(2\frac{1}{16})^3 - 1$ by $(2\frac{1}{16})^2 + 3\frac{1}{16}$.

12. A bankrupt's estate amounts to £910. 3s. $1\frac{1}{2}$d., and his debts to £1875. What can he pay in the pound? and what will a creditor lose on a debt of £57?

13. A person having invested a sum of money in the Three per Cent. Consols received annually therefrom £233, after deducting the income-tax of 7d. in the pound. What is the sum of money? What can the stock be sold for when Consols are at $94\frac{1}{8}$.

14. Find the value of $\frac{\cdot003 \times \cdot004}{\cdot006}$, and of $\cdot\frac{1}{0}\frac{0}{0}\frac{7}{3}\frac{2}{1}\frac{4}{2}\frac{}{5}$.

15. Prove the rule for finding the value of a circulating decimal, and divide 4·367 by the circulating decimal $\cdot\dot{0}5\dot{2}$.

Reduce to its simplest form the quantity $\frac{3 + \sqrt{5}}{3 - \sqrt{5}} \times \frac{8 - \sqrt{20}}{8 + \sqrt{20}}$

16. Three persons, A, B, C, hold a pasture in common, for which they are to pay £30 per annum. A put in 7 oxen for 3 months; B, 9 oxen for 5 months; and C, 4 oxen for 12 months. How much rent ought each to pay?

17. Calculate to four places of decimals the value of the expression $\frac{\frac{5}{8} \text{ of } \cdot31416}{\sqrt{\cdot93}}$

18. Find the least common multiple of 16, 24, and 30, and explain the method.

19. What should be the price of English standard silver, 37-40ths fine, in order that the par of exchange between England and France should be 25 fr. 22 c.—200 francs being coined from 1 kilogram of silver, 9-10ths fine? (1 kilog. = 15·434 grains).

20. A person buys 100 shares in a company for £3,500; after receiving four half-yearly dividends of 15s. 4d., 20s. 10d., 30s. 4d., and 38s. 9d. per share, he sells at a profit of 43 per cent.; reckoning the simple interest of money at 4 per cent., how much above that interest has he gained?

21. The price of Three per Cent. Consols is $90\frac{3}{8}$; what sum must be invested in order to purchase £24 per annum; and what is the rate of interest on the money invested?

22. Three partners in trade contribute respectively the sums of £438, £292, £730, with the agreement that each was to receive 5 per cent. on their respective investments, and that the remainder of the gains of the firm, if any, was to be divided between them in the proportion of the sums originally advanced. The whole gain of the firm was £200. What was each man's share?

23. If 25 tons of goods are purchased for £37. 10s. and sold at 35s. a ton, what is the gain per ton?

At what rate per ton should the goods have been sold in order to obtain a profit of £9. 7s. 6d.?

24. Find the value of $\frac{5}{1\,7}$ of £3. 12s. $11\frac{1}{2}$d.; and find the fraction that 3 miles, 2 fur. 100 yards is of 12 leagues, 2 fur. 20 yards.

25. The sum of £9040. 16s. is placed in the Three and a Half per Cents. at 94; find the income obtained, allowing on the stock purchased $\frac{1}{8}$th per cent. to the broker, and $\frac{1}{2\,0}$ per cent. for other expenses.

26. Express as a fraction $\cdot 200\dot{1}\dot{2}\dot{3}$, and express as a recurring decimal $\cdot 0\dot{1}\dot{2} \div \cdot 00\dot{1}3\dot{2}$.

27. By the reduction of the income-tax from 7d. in the pound to 5d. a person saves £28. 2s. 6d. a year; what is his income?

28. If 81 bushels of wheat are consumed by 56 men in 5 days, how long will 16 men take to consume 28 bushels?

29. Find the square root of $\cdot\dot{1}$, and prove that $\sqrt{\cdot 69\dot{4}} = 83\dot{3}$.

30. The periods of three planets which move uniformly in circular orbits round the sun are respectively 200, 250, and 300 days. Supposing that their positions relative to each other and to the sun to be given at any moment, determine how many days must elapse before they again have exactly the same relative positions.

SECTION II.

GEOMETRY.

EUCLID'S ELEMENTS, BOOK I.

Definitions.

1. A **point** is that which has position, but not magnitude.

2. A **line** is length without breadth.

3. The **extremities of a line** are points.

4. A **straight line** is that which lies evenly between its extreme points.

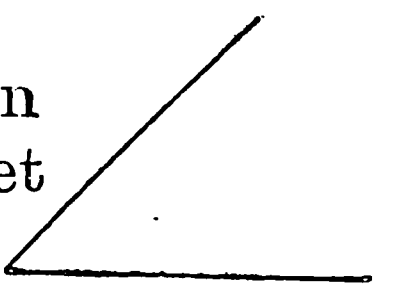
5. A **superficies** (or **surface**) is that which has only length and breadth.

6. The **extremities of a superficies** are lines.

7. A **plane superficies** is that in which any two points being taken, the straight line between them lies wholly in that superficies.

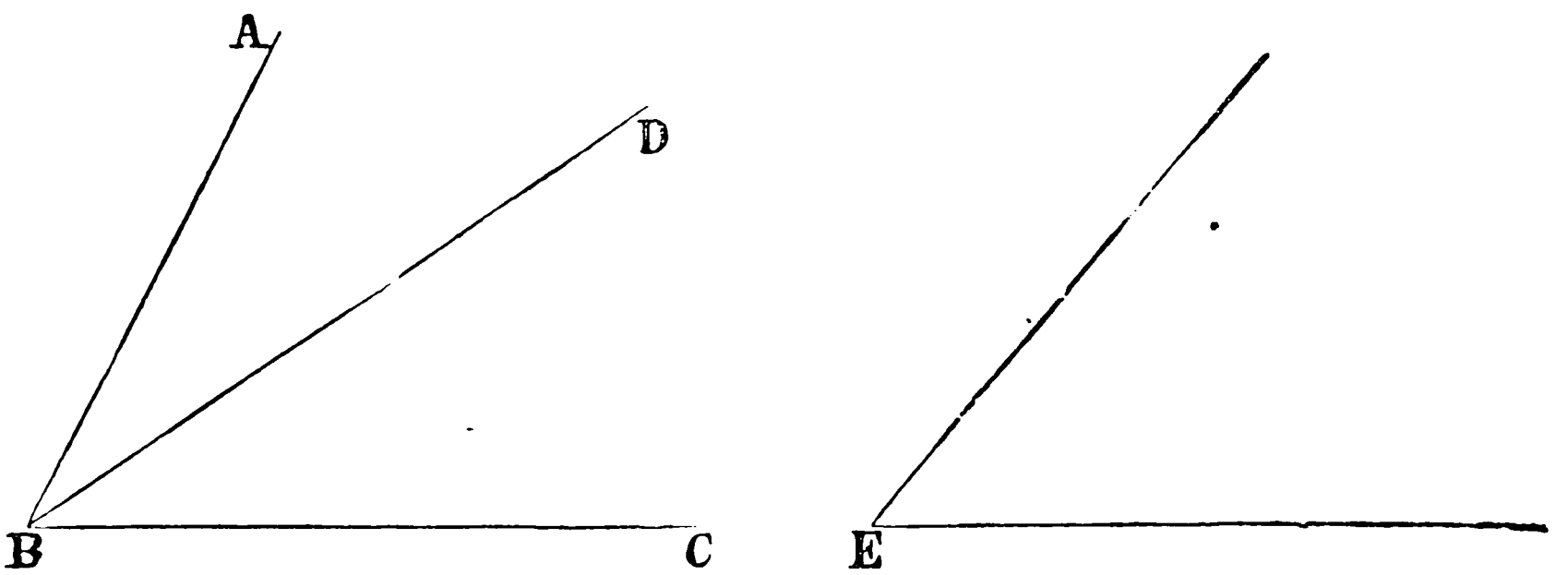
8. A **plane angle** is the inclination of two lines to one another in a plane, which meet together, but are not in the same direction.

9. A **plane rectilineal angle** is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.

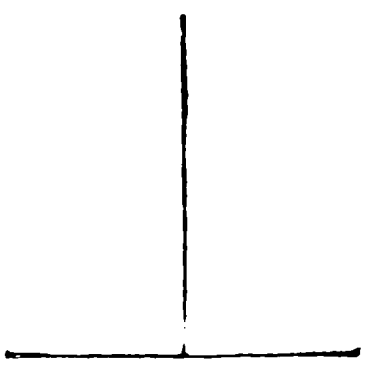


NOTE.—When several angles are at one point B, any one of them is expressed by three letters, of which the middle letter is B, and the first letter is on one of the straight lines which contain the angle, and the last letter on the other line.

Thus, the angle contained by the straight lines AB and BC is expressed either by ABC or CBA, and the angle contained by AB and

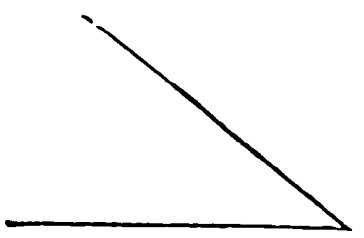


BD is expressed either by ABD or DBA. When there is only one angle at any given point, it may be expressed by the letter at that point, as the angle E.



10. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is called a **right angle**; and the straight line which stands on the other is called a **perpendicular** to it.

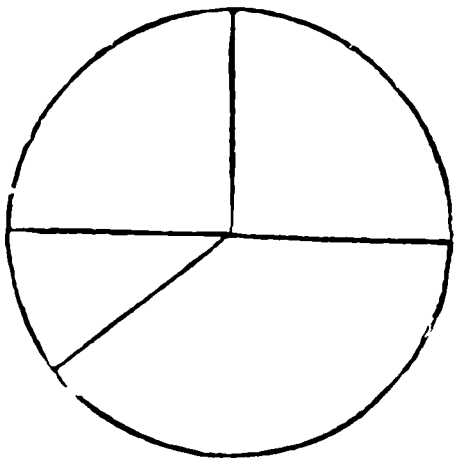
11. An **obtuse angle** is that which is greater than a right angle.



12. An **acute angle** is that which is less than a right angle.

13. A **term** or **boundary** is the extremity of anything.

14. A **figure** is that which is enclosed by one or more boundaries.



15. A **circle** is a plane figure contained by one line, which is called the **circumference**, and is such, that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the **centre** of the circle, [and any straight line drawn from the centre to the circumference is called a **radius** of the circle].

17. A **diameter** of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

18. A **semicircle** is the figure contained by a diameter and the part of the circumference cut off by the diameter.

19. A **segment** of a circle is the figure contained by a straight line and the part of the circumference which it cuts off.

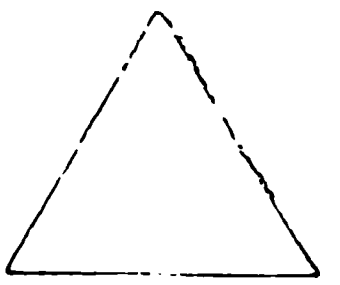
20. **Rectilineal figures** are those which are contained by straight lines.

21. **Trilateral figures,** or **triangles,** by three straight lines.

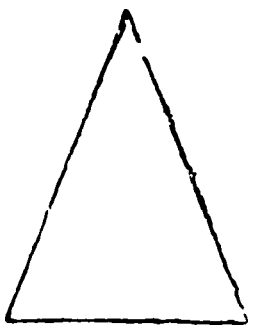
22. **Quadrilateral figures,** by four straight lines.

23. **Multilateral figures,** or **polygons,** by more than four straight lines.

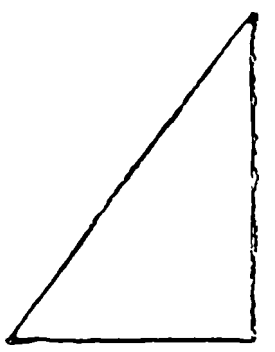
24. Of three-sided figures an **equilateral triangle** is that which has three equal sides.



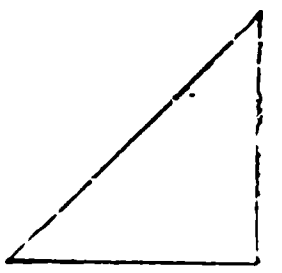
25. An **isosceles triangle** is that which has only two sides equal.



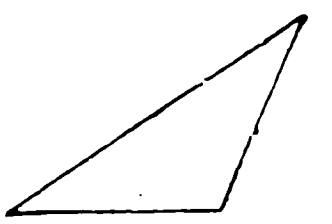
26. A **scalene triangle** is that which has three unequal sides.



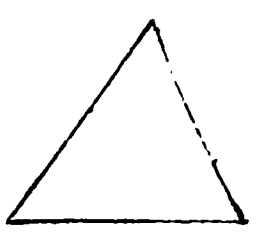
27. A **right-angled triangle** is that which has a right angle.

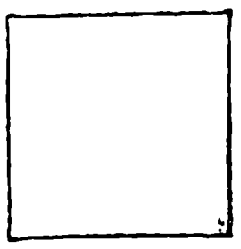


28. An **obtuse-angled triangle** is that which has an obtuse angle.

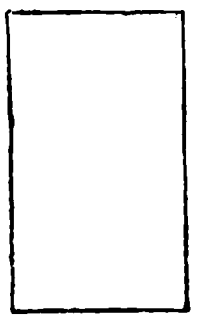


29. An **acute-angled triangle** is that which has three acute angles.

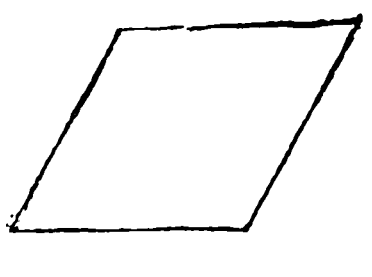




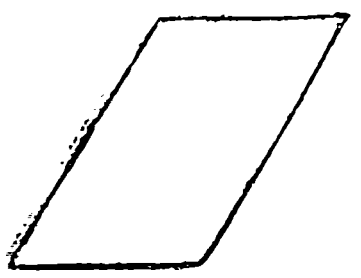
30. Of four-sided figures, a **square** is that which has all its sides equal, and all its angles right angles.



31. An **oblong** is that which has all its angles right angles, but not all its sides equal.



32. A **rhombus** is that which has all its sides equal, but its angles are not right angles.



33. A **rhomboid** is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.

34. **Parallel straight lines** are such as are in the same plane, and which being produced ever so far both ways do not meet.

35. A **parallelogram** is a four-sided figure of which the opposite sides are parallel; and the **diagonal** is the straight line joining two of its opposite angles. All other four-sided figures are called **trapeziums**.

Postulates.

1. Let it be granted that a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same are equal to one another.

7. Things which are halves of the same are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot inclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being continually produced shall at length meet on that side on which are the angles which are less than two right angles.

Explanation of Terms and Abbreviations.

An **Axiom** is a truth admitted without demonstration.

A **Theorem** is a truth which is capable of being demonstrated from previously demonstrated or admitted truths.

A **Postulate** states a geometrical process, the power of effecting which is required to be admitted.

A **Problem** proposes to effect something by means of admitted processes, or by means of processes or constructions, the power of effecting which has been previously demonstrated.

A **Corollary** to a proposition is an inference which may be easily deduced from that proposition.

The sign $=$ is used to express *equality*.

$\angle$ means *angle*, and $\triangle$ signifies *triangle*.

The sign $>$ signifies "is greater than," and $<$ "is less than."

$+$ expresses addition; thus AB + BC is the line whose length is the *sum* of the lengths of AB and BC.

$-$ expresses subtraction; thus AB − BC is the excess of the length of the line AB above that of BC.

AB^2 means the square described upon the straight line AB.

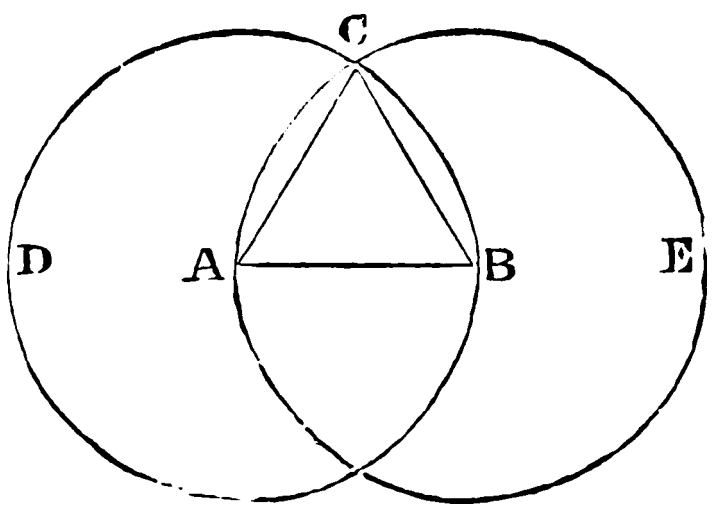
Proposition 1.—Problem.

To describe an equilateral triangle on a given finite straight line.

Let AB be the given straight line.

It is required to describe an equilateral triangle on AB.

From centres A and B, and radius = AB, describe circles.



CONSTRUCTION.—From the centre A, at the distance AB, describe the circle BCD (Post. 3).

From the centre B, at the distance BA, describe the circle ACE (Post. 3).

From the point C, in which the circles cut one another, draw the straight lines CA, CB to the points A and B (Post. 1).

Then ABC *shall be an equilateral triangle.*

AC=AB. PROOF.—Because the point A is the centre of the circle BCD, AC is equal to AB (Def. 15).

BC=AB. Because the point B is the centre of the circle ACE, BC is equal to BA (Def. 15).

AC and BC each=AB. Therefore AC and BC are each of them equal to AB.

AC=BC. But things which are equal to the same thing are equal to one another. Therefore AC is equal to BC (Ax. 1).

∴AB=BC=CA. Therefore AB, BC, and CA are equal to one another.

Therefore the triangle ABC is equilateral, and it is described on the given straight line AB. *Which was to be done.*

Proposition 2.—Problem.

From a given point to draw a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

It is required to draw from the point A a straight line equal to BC.

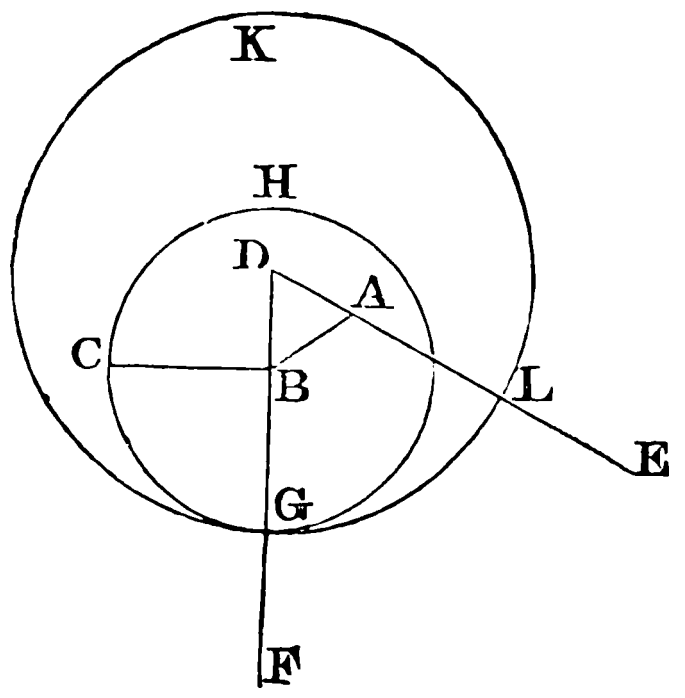
CONSTRUCTION.—From the point A to B draw the straight line AB (Post. 1). Draw AB.

Upon AB describe the equilateral triangle DAB (Book I., Prop. 1). Δ DAB equilateral.

Produce the straight lines DA, DB, to E and F (Post. 2).

From the centre B, at the distance BC, describe the circle CGH, meeting DF in G (Post. 3). B as centre.

From the centre D, at the distance DG, describe the circle GKL, meeting DE in L (Post. 3). D as centre.



Then AL *shall be equal to* BC.

PROOF.—Because the point B is the centre of the circle CGH, BC is equal to BG (Def. 15). BC=BG.

Because the point D is the centre of the circle GKL, DL is equal to DG (Def. 15). DL=DG.

But DA, DB, parts of them, are equal (Construction). DA=DB.

Therefore the remainder AL is equal to the remainder BG (Ax. 3). ∴ AL = BG.

But it has been shown that BC is equal to BG.

Therefore AL and BC are each of them equal to BG. ∴ AL and BC each = BG.

But things which are equal to the same thing are equal to one another, therefore AL is equal to BC (Ax. 1).

Therefore from the given point A a straight line AL has been drawn equal to the given straight line BC. *Which was to be done.* ∴ AL = BC.

Proposition 3.—Problem.

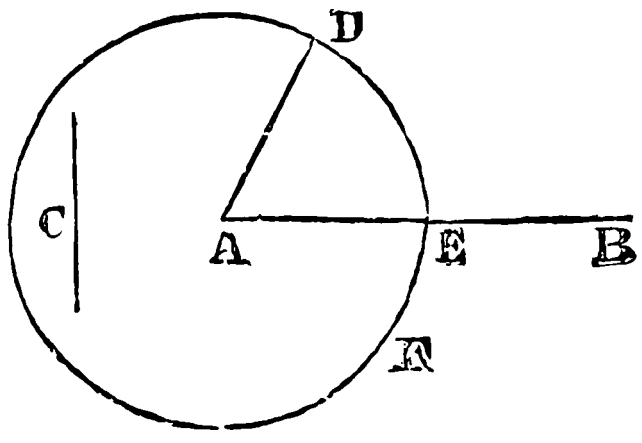
From the greater of two given straight lines to cut off a part equal to the less.

Let AB and C be the two given straight lines, of which AB is the greater.

It is required to cut off from AB, the greater, a part equal to C, the less.

Make AD = C.

A as centre and radius AD.



CONSTRUCTION.—From the point A draw the straight line AD equal to C (I. 2).

From the centre A, at the distance AD, describe the circle DEF, cutting AB in E (Post. 3).

Then AE *shall be equal to* C.

PROOF.—Because the point A is the centre of the circle DEF, AE is equal to AD (Def. 15).

AE=AD.

But C is also equal to AD (Construction).

AD=C.

Therefore AE and C are each of them equal to AD.

AE and C each=AD.

Therefore AE is equal to C (Ax. 1).

∴ AE=C.

Therefore, from AB, the greater of two given straight lines, a part AE has been cut off, equal to C, the less. *Q. E. F.**

Proposition 4.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have also the angles contained by those sides equal to one another: they shall have their bases, or third sides, equal; and the two triangles shall be equal, and their other angles shall be equal, each to each, viz., those to which the equal sides are opposite. Or,

If two sides and the contained angle of one triangle be respectively equal to those of another, the triangles are equal in every respect.

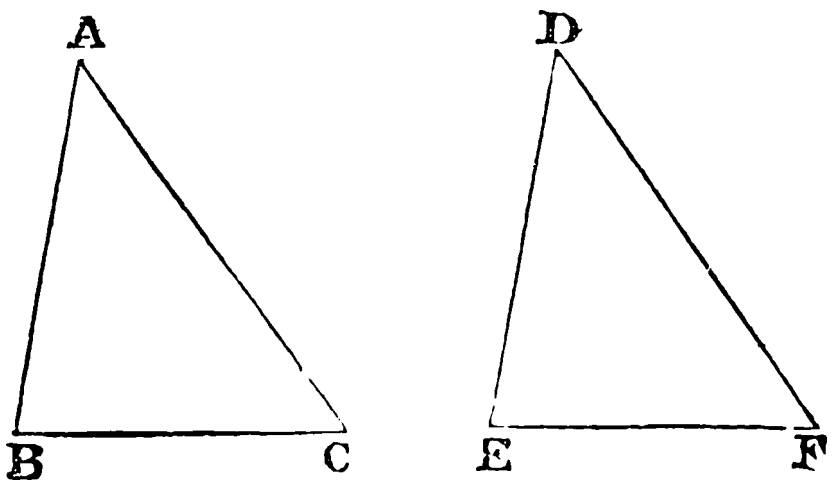
Let ABC, DEF be two triangles which have

AB=DE.

The two sides AB, AC, equal to the two sides DE, DF, each to each, viz., AB equal to DE, and AC equal to DF.

AC=DF.

∠ BAC = ∠ EDF.



And the angle BAC equal to the angle EDF:—then—

The base BC shall be equal to the base EF;

The triangle ABC shall be equal to the triangle DEF;

* Q. E. F. is an abbreviation for *quod erat faciendum*, that is "*which was to be done.*"

And the other angles to which the equal sides are opposite, shall be equal, each to each, viz., the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

PROOF.—For if the triangle ABC be applied to (*or placed upon*) the triangle DEF, Suppose △ ABC put upon △ DEF.

So that the point A may be on the point D, and the straight line AB on the straight line DE,

The point B shall coincide with the point E, because AB is equal to DE (Hypothesis).

And AB coinciding with DE, AC shall coincide with DF, because the angle BAC is equal to the angle EDF (Hyp.).

Therefore also the point C shall coincide with the point F, because the straight line AC is equal to DF (Hyp.).

But the point B was proved to coincide with the point E.

Therefore the base BC shall coincide with the base EF.

Because the point B coinciding with E, and C with F, if the base BC do not coincide with the base EF, two straight lines would enclose a space, which is impossible (Ax. 10).

Therefore the base BC coincides with the base EF, and is therefore equal to it (Ax. 8). BC=EF.

Therefore the whole triangle ABC coincides with the whole triangle DEF, and is equal to it (Ax. 8). ∴ △ ABC = △ DEF.

And the other angles of the one coincide with the remaining angles of the other, and are equal to them, viz., the angle ABC to DEF, and the angle ACB to DFE. ∠ ABC = ∠ DEF. ∠ ACB = ∠ DFE.

Therefore, if two triangles have, &c. (see Enunciation).

Which was to be shown.

Proposition 5.—Theorem.

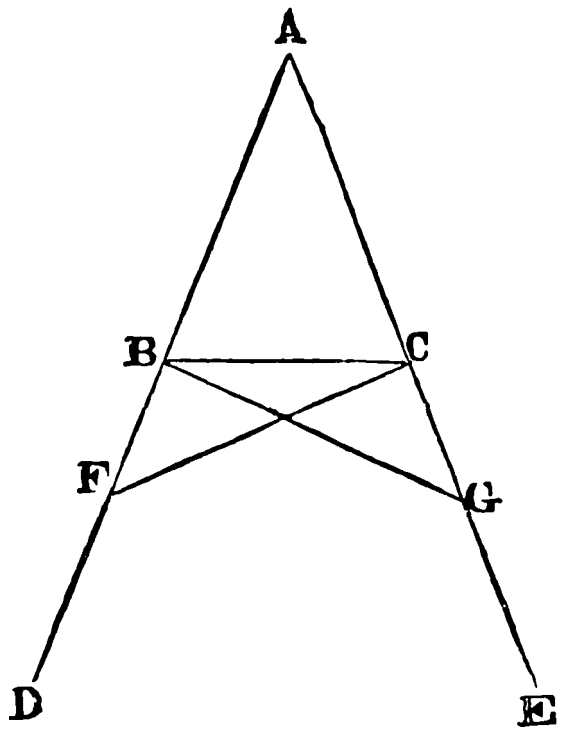
The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall also be equal.

Let ABC be an isosceles triangle, of which the side AB is equal to the side AC. AB = AC.

Let the straight lines AB, AC (*the equal sides of the triangle*), be produced to D and E.

The angle ABC shall be equal to the angle ACB (*angles at the base*),

And the angle CBD shall be equal to the angle BCE (*angles upon the other side of the base*).



CONSTRUCTION.—In BD take any point F.

$AG = AF$. From AE, the greater, cut off AG, equal to AF, the less (I. 3).

Join FC, GB.

PROOF.—Because AF is equal to AG (Construction), and AB is equal to AC (Hyp.),

FA, AC respectively $=$ GA, AB. Therefore the two sides FA, AC are equal to the two sides GA, AB, each to each;

And they contain the angle FAG, common to the two triangles AFC, AGB.

$\therefore FC = GB$ and $\triangle AFC = \triangle AGB$. Therefore the base FC is equal to the base GB (I. 4);

And the triangle AFC to the triangle AGB (I. 4);

$\angle ACF = \angle ABG$. $\angle AFC = \angle AGB$. And the remaining angles of the one are equal to the remaining angles of the other, each to each, to which the equal sides are opposite, viz., the angle ACF to the angle ABG, and the angle AFC to the angle AGB (I. 4).

And because the whole AF is equal to the whole AG, of which the parts AB, AC, are equal (Hyp.),

$BF = CG$. The remainder BF is equal to the remainder CG (Ax. 3).

And FC was proved to be equal to GB;

Therefore the two sides BF, FC are equal to the two sides CG, GB, each to each.

And the angle BFC was proved equal to the angle CGB;

Therefore the triangles BFC, CGB are equal; and their other angles are equal, each to each, to which the equal sides are opposite (I. 4).

$\therefore \angle FBC = \angle GCB$. $\angle BCF = \angle CBG$. Therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG.

And since it has been demonstrated that the whole angle ABG is equal to the whole angle ACF, and that the parts of these, the angles CBG, BCF, are also equal,

$\therefore \angle ABC = \angle ACB$. Therefore the remaining angle ABC is equal to the remaining angle ACB (Ax. 3),

Which are the angles at the base of the triangle ABC.

And it has been proved that the angle FBC is equal to the angle GCB (Dem. 11),

Which are the angles upon the other side of the base,

Therefore the angles at the base, &c. (see Enunciation). *Which was to be shown.*

COROLLARY.—Hence every equilateral triangle is also equiangular.

Proposition 6.—Theorem.

If two angles of a triangle be equal to one another, the sides also which subtend, or are opposite to, the equal angles, shall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB.

The side AB shall be equal to the side AC.

For if AB be not equal to AC, one of them is greater than the other. Let AB be the greater. Suppose AB > AC.

CONSTRUCTION.—From AB, the greater, cut off a part DB, equal to AC, the less (I. 3). Make DB = AC.

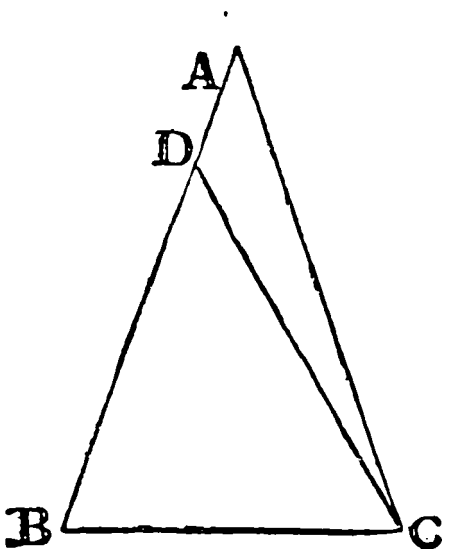
Join DC.

PROOF.—Because in the triangles DBC, ACB, DB is equal to AC, and BC is common to both,

Therefore the two sides DB, BC are equal to the two sides AC, CB, each to each;

And the angle DBC is equal to the angle ACB (Hyp.)

Therefore the base DC is equal to the base AB (I. 4).



And the triangle DBC is equal to the triangle ACB (I. 4), the less to the greater, which is absurd. Δ DBC = Δ ACB.

Therefore AB is not unequal to AC, that is, it is equal to it.

Wherefore, if two angles, &c. *Q. E. D.* *

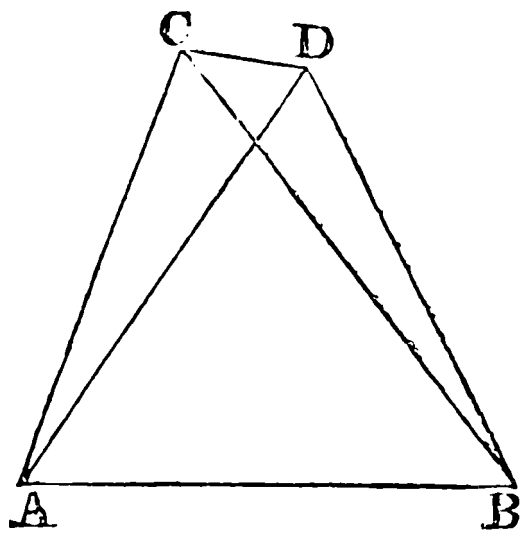
COROLLARY.—Hence every equiangular triangle is also equilateral.

* Q. E. D. is an abbreviation for *quod erat demonstrandum*, that is, "*which was to be shown or proved.*"

Proposition 7.—Theorem.

Upon the same base, and on the same side of it, there cannot be two triangles that have their sides, which are terminated in one extremity of the base, equal to one another, and likewise those which are terminated in the other extremity.

Let the triangles ACB, ADB, upon the same base AB, and on the same side of it, have, if possible,



Suppose CA = DA.

CB = DB.

Their sides CA, DA, terminated in the extremity A of the base, equal to one another;

And their sides CB, DB, terminated in the extremity B of the base, likewise equal to one another.

Case I.—Let the vertex of each triangle be without the other triangle.

Construction.—Join CD.

Proof.—Because AC is equal to AD (Hyp.),

The triangle ADC is an isosceles triangle, and the angle ACD is therefore equal to the angle ADC (I. 5).

∠ ACD = ∠ ADC.

But the angle ACD is greater than the angle BCD (Ax. 9).

Therefore the angle ADC is also greater than BCD.

Much more then is the angle BDC greater than BCD.

BDC > BCD.

Again, because BC is equal to BD (Hyp.),

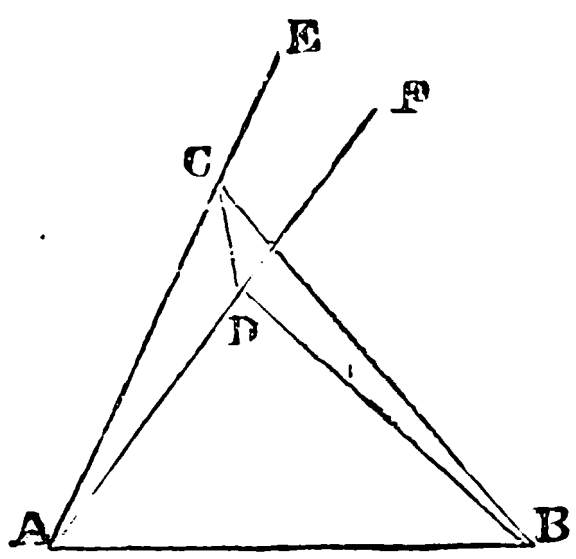
The triangle BCD is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5).

∠ BDC = ∠ BCD.

But the angle BDC has been shown to be greater than the angle BCD (Dem. 5).

Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible.

∠ BDC = and > ∠ BCD.



Case II.—Let the vertex of one of the triangles fall within the other.

Construction.—Produce AC, AD to E and F, and join CD.

Proof.—Because AC is equal to AD (Hyp.),

The triangle ADC is an isosceles triangle, and the angles ECD, FDC, upon the other side of its base CD, are equal to one another (I. 5).

Again ∠ ECD = ∠ FDC.

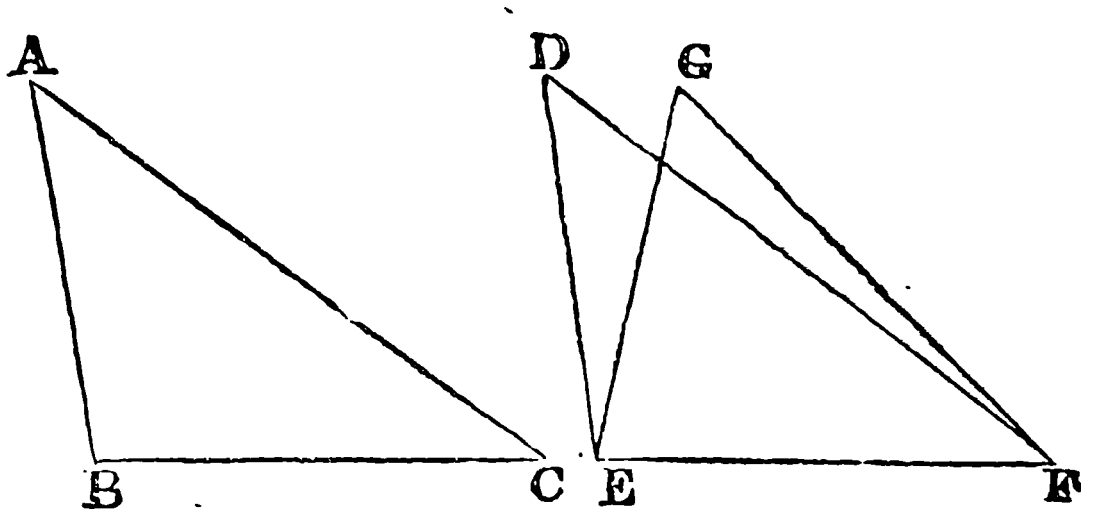
But the angle ECD is greater than the angle BCD (Ax. 9).
Therefore the angle FDC is likewise greater than BCD.
Much more then is the angle BDC greater than BCD. ∠ BDC > ∠ BCD.
Again, because BC is equal to BD (Hyp.),
The triangle BDC is an isosceles triangle, and the angle BDC is equal to the angle BCD (I. 5). ∠ BDC = ∠ BCD.
But the angle BDC has been shown to be greater than the angle BCD.
Therefore the angle BDC is both equal to, and greater than the same angle BCD, which is impossible. ∴ ∠ BDC = and > ∠ BCD.
Therefore, upon the same base, &c. *Q. E. D.*

Proposition 8.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one shall be equal to the angle contained by the two sides, equal to them, of the other. Or,

If two triangles have three sides of the one respectively equal to the three sides of the other, they are equal in every respect, those angles being equal which are opposite to the equal sides.

Let ABC, DEF be two triangles which have
The two sides AB, AC equal to the two sides DE, DF, each to each, viz., AB to DE, and AC to DF, Given AB = DE, AC = DF, and BC = EF.
And the base BC equal to the base EF.
The angle BAC shall be equal to the angle EDF.
Proof.—For if the triangle ABC be applied to the triangle DEF,
So that the point B may be on E, and the straight line BC on EF,
The point C shall coincide with the point F, because BC is equal to EF (Hyp.).
Therefore, BC coinciding with EF, BA and AC shall coincide with ED and DF.



Make BC coincide with EF.

For if the base BC coincides with the base EF,

But the sides BA, AC, do not coincide with the sides ED, DF, but have a different situation, as EG, GF,

Then upon the same base, and on the same side of it, there will be two triangles, which have their sides terminated in one extremity of the base equal to one another, and likewise their sides, which are terminated in the other extremity. But this is impossible (I. 7).

∴ BA, AC respectively coincide with ED, DF.

Therefore, if the base BC coincides with the base EF, the sides BA, AC must coincide with the sides ED, DF.

Therefore the angle BAC coincides with the angle EDF, and is equal to it (Ax. 8).

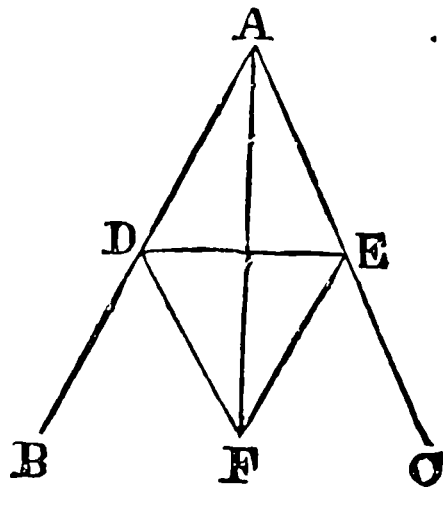
Also the triangle ABC coincides with the triangle DEF and is therefore equal to it in every respect (Ax. 8).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 9.—Problem.

To bisect a given rectilineal angle, that is, to divide it into two equal parts.

Let BAC be the given rectilineal angle.



It is required to bisect it.

Make AE = AD.

CONSTRUCTION.—Take any point D in AB.

From AC cut off AE equal to AD (I. 3).

Join DE.

△ DEF equilateral.

Upon DE, on the side remote from A, describe an equilateral triangle DEF (I. 1).

Join AF.

Then the straight line AF *shall bisect the angle* BAC.

PROOF.—Because AD is equal to AE (Const.), and AF is common to the two triangles DAF, EAF;

The two sides DA, AF are equal to the two sides EA, AF, each to each;

And the base DF is equal to the base EF (Const.);

∴ ∠ DAF = ∠ EAF.

Therefore the angle DAF is equal to the angle EAF (I. 8).

Therefore the given rectilineal angle BAC is bisected by the straight line AF. *Q. E. F.*

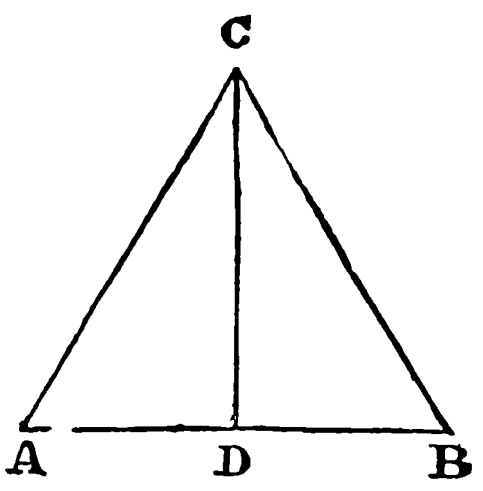
Proposition 10.—Problem.

To bisect a given finite straight line, that is, to divide it into two equal parts.

Let AB be the given straight line.

It is required to divide it into two equal parts.

CONSTRUCTION.—Upon AB describe the equilateral triangle ABC (I. 1).



Make Δ ABC equilateral and ∠ ACD = ∠ BCD.

Bisect the angle ACB by the straight line CD (I. 9).

Then AB *shall be cut into two equal parts in the point* D.

PROOF.—Because AC is equal to CB (Const.), and CD common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD (Const.);

Therefore the base AD is equal to the base DB (I. 4). ∴ AD = DB.

Therefore the straight line AB is divided into two equal parts in the point D. *Q. E. F.*

Proposition 11.—Problem.

To draw a straight line at right angles to a given straight line from a given point in the same.

Let AB be the given straight line, and C a given point in it.

It is required to draw a straight line from the point C at right angles to AB.

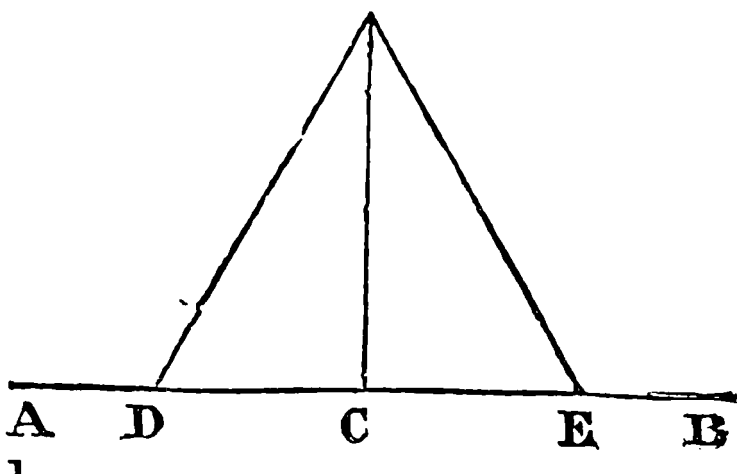
CONSTRUCTION.—Take any point D in AC.

Make CE equal to CD (I. 3).

Upon DE describe the equilateral triangle DFE (I. 1).

Join FC.

Make CE = CD and Δ DEF equilateral.



Then FC *shall be at right angles to* AB.

PROOF.—Because DC is equal to CE (Const.), and FC common to the two triangles DCF, ECF;

The two sides DC, CF, are equal to the two sides EC, CF, each to each;

And the base DF is equal to the base EF (Const.);

Therefore the angle DCF is equal to the angle ECF (I. 8); ∠ DCF = ∠ ECF.

And they are adjacent angles.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (Def. 10);

∴ ∠ DCF, ∠ ECF are right angles.

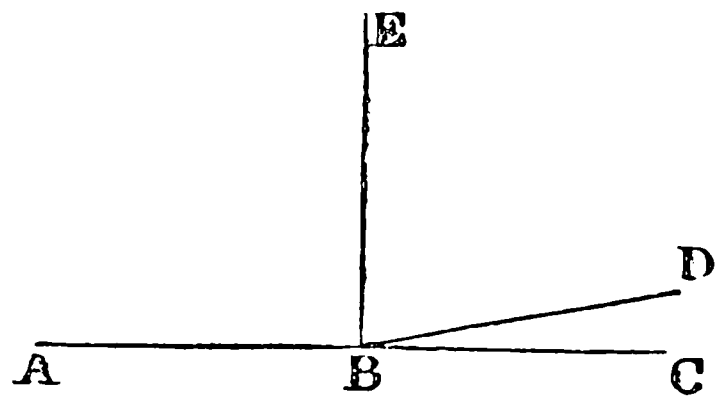
Therefore each of the angles DCF, ECF is a right angle.

Therefore from the given point C in the given straight line AB, a straight line FC has been drawn at right angles to AB. *Q. E. F.*

COROLLARY.—By help of this problem, it may be demonstrated that

Two straight lines cannot have a common segment.

If it be possible, let the two straight lines ABC, ABD, have the segment AB common to both of them.



Make ∠ ABE a right ∠

CONSTRUCTION.—From the point B, draw BE at right angles to AB (I. 11).

∠ CBE = ∠ EBA.

and

∠ DBE = ∠ EBA.

PROOF.—Because ABC is a straight line, the angle CBE is equal to the angle EBA (Def. 10).

Also, because ABD is a straight line, the angle DBE is equal to the angle EBA (Def. 10).

∴ ∠ DBE = ∠ CBE.

Therefore the angle DBE is equal to the angle CBE. The less to the greater; which is impossible.

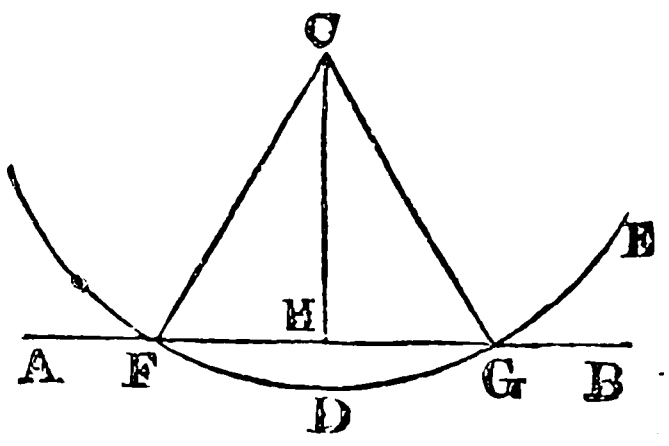
Therefore two straight lines cannot have a common segment.

Proposition 12.—Problem.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.

Let AB be the given straight line, which may be produced to any length both ways, and let C be a point without it.

It is required to draw from the point C, a straight line perpendicular to AB.



CD as radius.

Bisect FG in H.

CONSTRUCTION.—Take any point D upon the other side of AB.

From the centre C, at the distance CD, describe the circle EGF, meeting AB in F and G (Post. 3).

Bisect FG in H (I. 10).

Join CF, CH, CG.

Then CH *shall be perpendicular to* AB.

PROOF.—Because FH is equal to HG (Const.), and HC common to the two triangles FHC, GHC;

The two sides FH, HC are equal to the two sides GH, HC, each to each;

And the base CF is equal to the base CG (Def. 15);

Therefore the angle CHF is equal to the angle CHG (I. 8), and they are adjacent angles. ∴ adjacent angles CHF, CHG are equal.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle, and the straight line which stands on the other is called a perpendicular to it (Def. 10).

Therefore, from the given point C, a perpendicular has been drawn to the given straight line AB. *Q. E. F.*

Proposition 13.—Theorem.

The angles which one straight line makes with another upon one side of it, are either two right angles, or are together equal to two right angles.

Let the straight line AB make with CD, upon one side of it, the angles CBA, ABD.

These angles shall either be two right angles, or shall together be equal to two right angles.

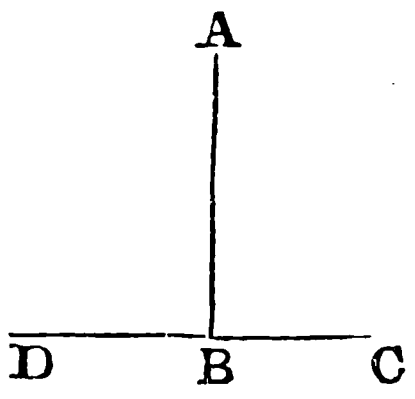
PROOF.—If the angle CBA be equal to the angle ABD, each of them is a right angle (Def. 10).

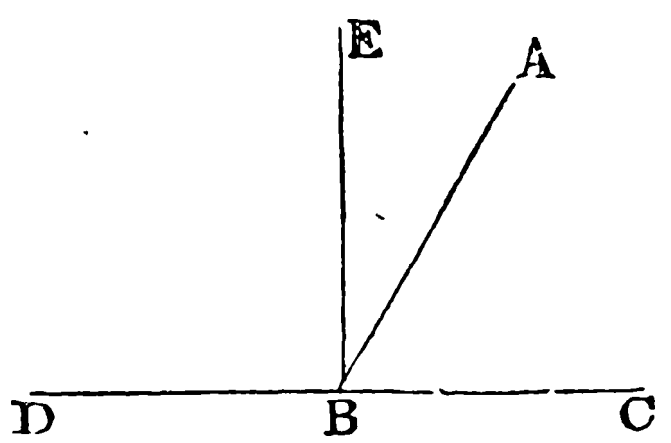
But if the angle CBA be not equal to the angle ABD, from the point B, draw BE at right angles to CD (I. 11).

Therefore the angles CBE, EBD, are two right angles.

Now the angle CBE is equal to the two angles CBA, ABE; to each of these equals add the angle EBD.

Make ∠ CBE = ∠ EBD = a right ∠.





Therefore the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD (Ax. 2).

∴ ∠ CBE + ∠ EBD = ∠ CBA + ∠ ABE + ∠ EBD, also

Again, the angle DBA is equal to the two angles DBE, EBA; to each of these equals add the angle ABC.

Therefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC (Ax. 2).

so ∠ DBA + ∠ ABC = ∠ DBE + ∠ EBA + ∠ ABC.

But the angles CBE, EBD have been shown to be equal to the same three angles ;

And things which are equal to the same thing are equal to one another ;

∴ ∠ CBE + ∠ EBD = ∠ DBA + ∠ ABC.

Therefore the angles CBE, EBD, are equal to the angles DBA, ABC (Ax. 1).

But the angles CBE, EBD are two right angles.

Therefore the angles DBA, ABC, are together equal to two right angles (Ax. 1).

Therefore, the angles which one straight line, &c. *Q. E. D.*

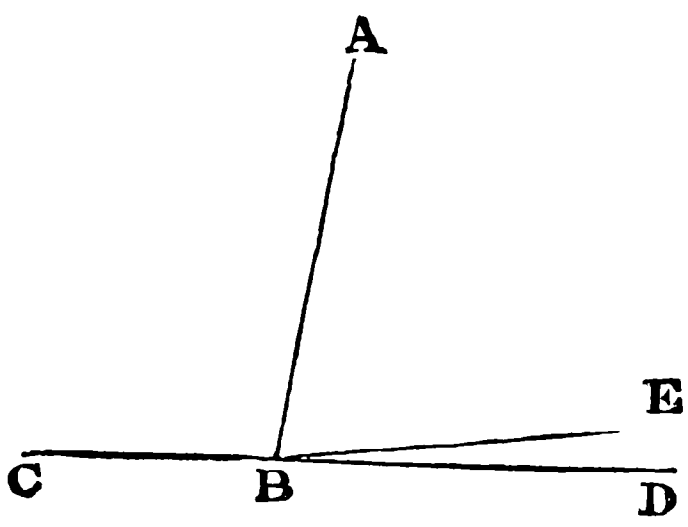
Proposition 14.—Theorem.

If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line.

Given ∠ ABC + ∠ ABD = two right angles.

At the point B in the straight line AB, let the two straight lines BC, BD, upon the opposite sides of AB, make the adjacent angles ABC, ABD together equal to two right angles.

BD shall be in the same straight line with BC.



If possible, let CBE be a straight line.

For if BD be not in the same straight line with BC, let BE be in the same straight line with it.

Proof.—Because CBE is a straight line, and AB meets it in B.

Therefore the adjacent angles ABC, ABE are together equal to two right angles (I. 13).

But the angles ABC, ABD, are also together equal to two right angles (Hyp.) ;

Therefore the angles ABC, ABE, are equal to the angles ABC, ABD (Ax. 1).

Take away the common angle ABC.

∴ ∠ ABE = ∠ ABD.

The remaining angle ABE is equal to the remaining angle ABD (Ax. 3), the less to the greater, which is impossible ;

Therefore BE is not in the same straight line with BC.

And, in like manner, it may be demonstrated that no other can be in the same straight line with it but BD.

Therefore BD is in the same straight line with BC.

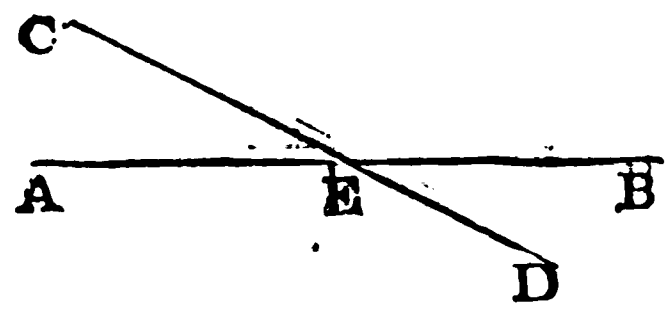
Therefore, if at a point, &c. *Q. E. D.*

Proposition 15.—Theorem.

If two straight lines cut one another, the vertical, or opposite angles shall be equal.

Let the two straight lines AB, CD cut one another in the point E.

The angle AEC shall be equal to angle DEB, and the angle CEB to the angle AED.



PROOF.—Because the straight line AE makes with CD, the angles CEA, AED, these angles are together equal to two right angles (I. 13). ∠ CEA + ∠ AED = 2 right angles.

Again, because the straight line DE makes with AB the angles AED, DEB, these also are together equal to two right angles (I. 13). ∠ AED + ∠ DEB = 2 right angles.

But the angles CEA, AED have been shown to be together equal to two right angles,

Therefore the angles CEA, AED are equal to the angles AED, DEB (Ax. 1).

Take away the common angle AED.

The remaining angle CEA is equal to the remaining angle DEB (Ax. 3). ∴ ∠ CEA = ∠ DEB.

In the same manner it can be shown that the angles CEB, AED are equal.

Therefore, if two straight lines, &c. *Q. E. D.*

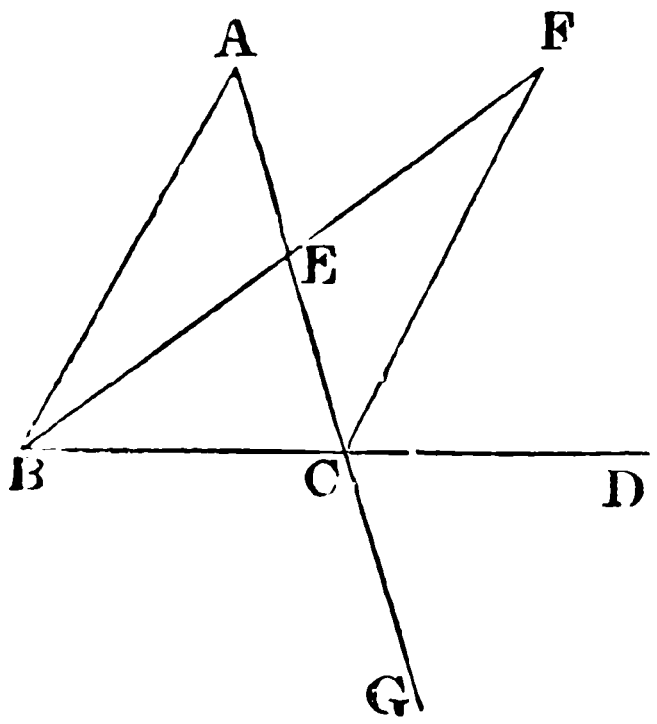
COROLLARY 1.—From this it is manifest that if two straight lines cut one another, the angles which they make at the point where they cut, are together equal to four right angles.

COROLLARY 2.—And, consequently, that all the angles made by any number of lines meeting in one point are together equal to four right angles, provided that no one of the angles be included in any other angle.

Proposition 16.—Theorem.

If one side of a triangle be produced, the exterior angle shall be greater than either of the interior opposite angles.

Let ABC be a triangle, and let its side BC be produced to D. The exterior angle ACD shall be greater than either of the interior opposite angles CBA, BAC.



Make AE = EC. and EF = BE.

Construction.—Bisect AC in E (I. 10).

Join BE, and produce it to F, making EF equal to BE (I. 3), and join FC.

Proof.—Because AE is equal to EC, and BE equal to EF (Const.),

AE, EB are equal to CE, EF, each to each ;

And the angle AEB is equal to the angle CEF, because they are opposite vertical angles (I. 15).

Therefore the base AB is equal to the base CF (I. 4) ;

And the triangle AEB to the triangle CEF (I. 4) ;

And the remaining angles to the remaining angles, each to each, to which the equal sides are opposite.

∴ ∠BAE = ∠ ECF.

Therefore the angle BAE is equal to the angle ECF (I. 4).

But the angle ECD is greater than the angle ECF (Ax. 9) ;

∴ ∠ACD > ∠BAE.

Therefore the angle ACD is greater than the angle BAE.

In the same manner, if BC be bisected, and the side AC be produced to G, it may be proved that the angle BCG (or its equal ACD), is greater than the angle ABC.

Therefore, if one side, &c. *Q. E. D.*

Proposition 17.—Theorem.

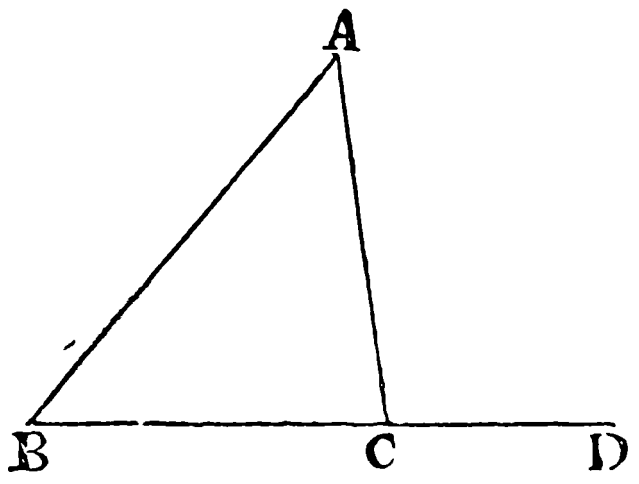
Any two angles of a triangle are together less than two right angles.

Let ABC be any triangle.

Any two of its angles together shall be less than two right angles.

CONSTRUCTION.—Produce BC to D.

PROOF.—Because ACD is the exterior angle of the triangle ABC, it is greater than the interior and opposite angle ABC (I. 16).



∠ ACD > ∠ ABC.

To each of these add the angle ACB.

Add to each ∠ ACB.

Therefore the angles ACD, ACB are greater than the angles ABC, ACB (Ax. 4).

∠ ABC + ∠ ACB < 2 right angles.

But the angles ACD, ACB are together equal to two right angles (I. 13);

Therefore the angles ABC, ACB are together less than two right angles.

In like manner, it may be proved that the angles BAC, ACB, as also the angles CAB, ABC are together less than two right angles.

Therefore, any two angles, &c. *Q. E. D.*

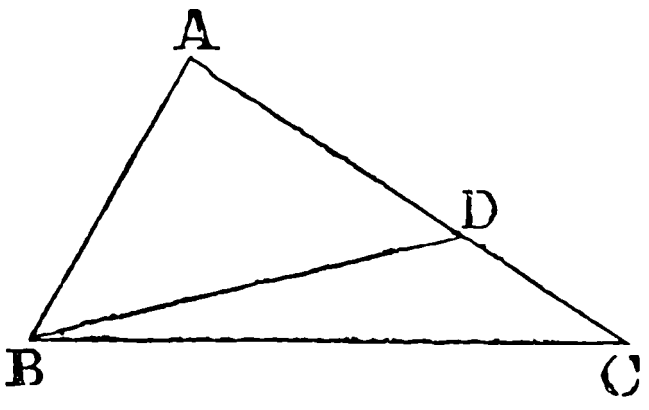
Proposition 18.—Theorem.

The greater side of every triangle is opposite the greater angle.

Let ABC be a triangle, of which the side AC is greater than the side AB.

AC > AB.

The angle ABC shall be greater than the angle BCA.



CONSTRUCTION.—Because AC is greater than AB, make AD equal to AB (I. 3), and join BD.

Make AD = AB.

PROOF.—Because ADB is the exterior angle of the triangle BDC, it is greater than the interior and opposite angle BCD (I. 16).

∠ ADB > ∠ BCD, and

But the angle ADB is equal to the angle ABD; the triangle BAD being isosceles (I. 5),

∠ ADB = ∠ ABD and

Therefore the angle ABD is greater than the angle BCD (*or* ACB).

∴ ∠ ABD > ∠ BCD.

Much more then is the angle ABC greater than the angle ACB.

Therefore, the greater side, &c. *Q. E. D.*

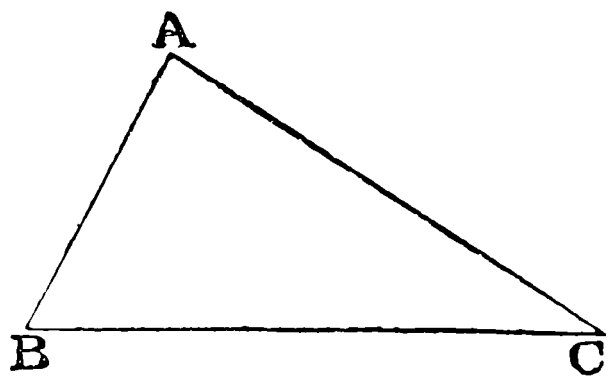
Proposition 19.—Theorem.

The greater angle of every triangle is subtended by the greater side, or has the greater side opposite to it.

Given ∠ ABC > ∠ BCA.

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA;

The side AC shall be greater than the side AB.



Proof.—If AC be not greater than AB, it must either be equal to or less than AB.

It is not equal, for then the angle ABC would be equal to the angle BCA (I. 5); but it is not (Hyp.);

AC not = AB.

Therefore AC is not equal to AB.

Neither is AC less than AB, for then the angle ABC would be less than the angle BCA (I. 18); but it is not (Hyp.);

AC not < AB.

Therefore AC is not less than AB.

And it has been proved that AC is not equal to AB;

Therefore AC is greater than AB.

Therefore, the greater angle, &c. *Q. E. D.*

Proposition 20.—Theorem.

Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle;

Any two sides of it are together greater than the third side.

Make AD = AC.

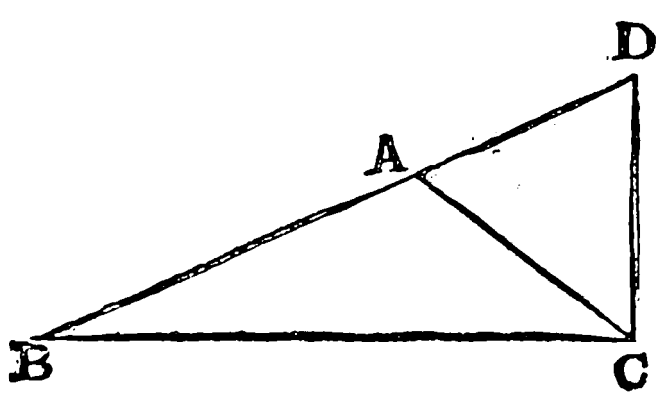
Construction.—Produce BA to the point D, making AD equal to AC (I. 3), and join DC.

Proof.—Because DA is equal to AC, the angle ADC is equal to the angle ACD (I. 5).

But the angle BCD is greater than the angle ACD (Ax. 9);

∠ BCD > ∠ BDC.

Therefore the angle BCD is greater than the angle ADC (*or* BDC).



And because the angle BCD of the triangle DCB is greater than its angle BDC, and that the greater angle is subtended by the greater side;

∴ DB > BC.

Therefore the side DB is greater than the side BC (I. 19).

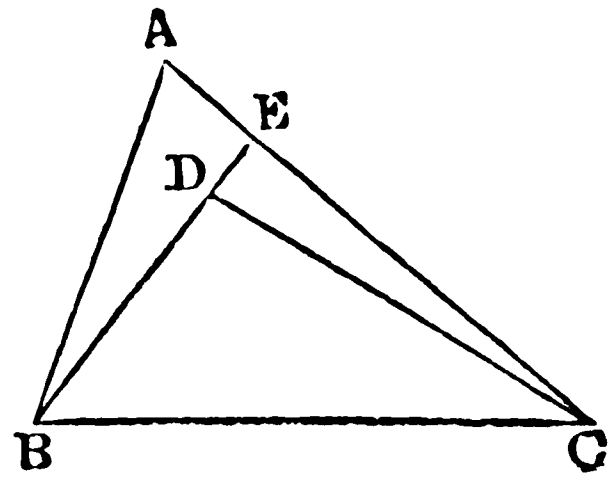
But BD is equal to BA and AC;
Therefore BA, AC are greater than BC. ∴ BA + AC > BC.
In the same manner it may be proved that AB, BC are greater than AC; and BC, CA greater than AB.
Therefore any two sides, &c. *Q. E. D.*

Proposition 21.—Theorem.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these shall be less than the other two sides of the triangle, but shall contain a greater angle.

Let ABC be a triangle, and from the points B, C, the ends of the side BC, let the two straight lines BD, CD be drawn to the point D within the triangle;
BD, DC shall be less than the sides BA, AC;
But BD, DC shall contain an angle BDC greater than the angle BAC.

CONSTRUCTION.—Produce BD to E.



PROOF.—1. Because two sides of a triangle are greater than the third side (I. 20), the two sides BA, AE, of the triangle BAE are greater than BE.
To each of these add EC.
Therefore the sides BA, AC, are greater than BE, EC (Ax. 4). BA + AC > BE + EC.
Again, because the two sides CE, ED, of the triangle CED are greater than CD (I. 20),
To each of these add DB.
Therefore CE, EB are greater than CD, DB (Ax. 4).
But it has been shown that BA, AC are greater than BE, EC; ∴ EC + EB > CD + DB.
Much more then are BA, AC greater than BD, DC.
PROOF. — 2. Again, because the exterior angle of a triangle is greater than the interior and opposite angle (I. 16), therefore BDC, the exterior angle of the triangle CDE, is greater than CED or CEB. Again ∠ BDC > ∠ CEB,
For the same reason, CEB, the exterior angle of the triangle ABE, is greater than the angle BAE or BAC, and ∠ CEB > ∠ BAE.

And it has been shown that the angle BDC is greater than CEB ;

$\therefore \angle$ BDC $> \angle$ BAC. Much more then is the angle BDC greater than the angle BAC.

Therefore, if from the ends, &c. *Q. E. D.*

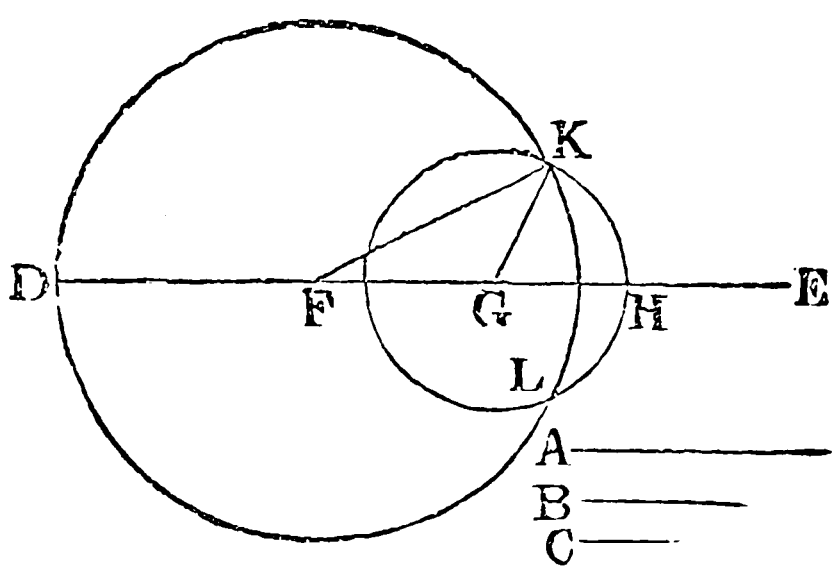
Proposition 22.—Problem.

To make a triangle of which the sides shall be equal to three given straight lines, but any two whatever of these lines must be greater than the third (I. 20).

Let A, B, C be the three given straight lines, of which any two whatever are greater than the third—namely, A and B greater than C, A and C greater than B, and B and C greater than A ;

It is required to make a triangle of which the sides shall be equal to A, B, and C, each to each.

CONSTRUCTION.—Take a straight line DE terminated at the point D, but unlimited towards E.



DF, FG, GH respectively = A, B, C.

Make DF equal to A, FG equal to B, and GH equal to C (I. 3).

FD as radius.

From the centre F, at the distance FD, describe the circle DKL (Post. 3).

and GH as radius.

From the centre G, at the distance GH, describe the circle HLK (Post. 3).

Join KF, KG.

Then the triangle KFG *shall have its sides equal to the three straight lines* A, B, C.

PROOF.—Because the point F is the centre of the circle DKL, FD is equal to FK (Def. 15).

But FD is equal to A (Const.) ;

FK = A. Therefore FK is equal to A (Ax. 1).

Again, because the point G is the centre of the circle HLK, GH is equal to GK (Def. 15).

But GH is equal to C (Const.) ;

GK = C. Therefore GK is equal to C (Ax. 1),

And FG is equal to B (Const.); FG = B

Therefore the three straight lines KF, FG, GK are equal to the three A, B, C, each to each.

Therefore the triangle KFG has its three sides KF, FG, GK equal to the three given straight lines A, B, C. *Q. E. F.*

Proposition 23.—Problem.

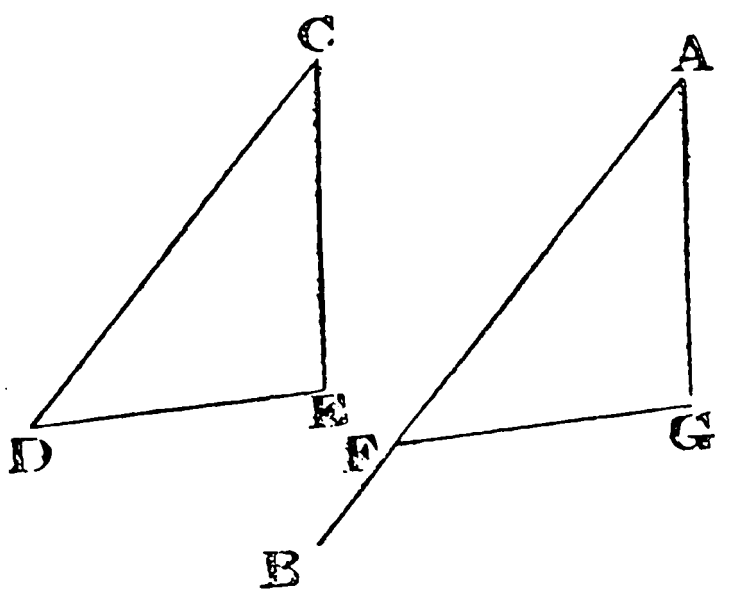
At a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given straight line, and A the given point in it, and DCE the given rectilineal angle.

It is required to take an angle at the point A, in the straight line AB, equal to the rectilineal angle DCE.

CONSTRUCTION. — In CD, CE, take any points D, E, and join DE.

On AB construct a triangle AFG, the sides of which shall be equal to the three straight lines CD, DE, EC — namely, AF equal to CD, FG to DE, and AG to EC (I. 22);



Make △ AFG so that AF = CD, FG = DE, AG = CE.

Then the angle FAG *shall be equal to the angle* DCE.

PROOF.—Because DC, CE are equal to FA, AG, each to each, and the base DE equal to the base FG (Const.),

The angle DCE is equal to the angle FAG (I. 8). Then ∠ DCE = ∠ FAG

Therefore, at the given point A, in the given straight line AB, the angle FAG has been made equal to the given rectilineal angle DCE. *Q. E. F.*

Proposition 24.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other.

Let ABC, DEF, be two triangles which have

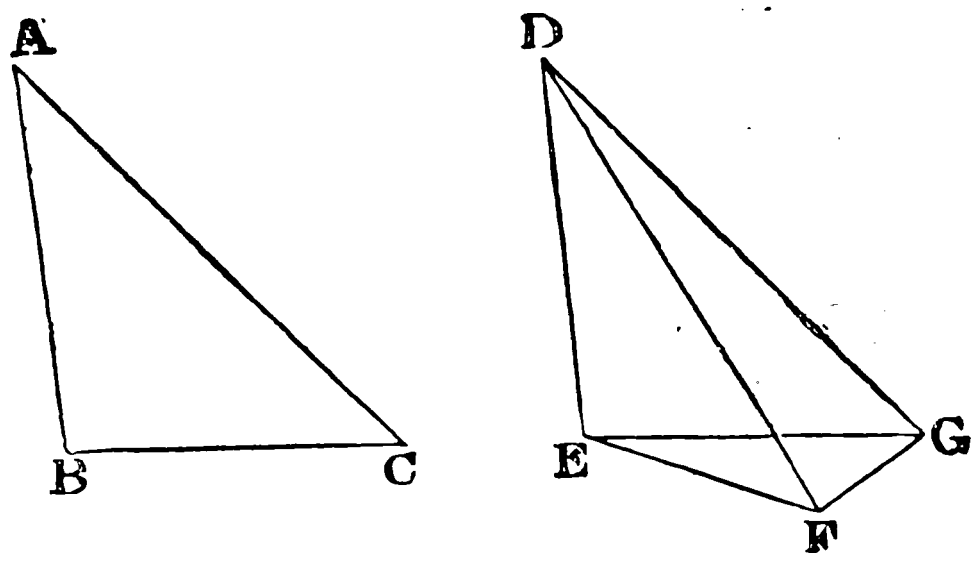
The two sides AB, AC equal to the two DE, DF, each to each—namely, AB to DE, and AC to DF,

But the angle BAC greater than the angle EDF;

The base BC shall be greater than the base EF.

Suppose DF > DE.

CONSTRUCTION.—Let the side DF of the triangle DEF be greater than its side DE.



Make ∠ EDG = ∠ BAC.

Then at the point D, in the straight line ED, make the angle EDG equal to the angle BAC (I. 23).

Make DG = AC, and = DF.

Make DG equal to AC or DF (I. 3).

Join EG, GF.

PROOF.—Because AB is equal to DE (Hyp.), and AC to DG (Const.), the two sides BA, AC are equal to the two ED, DG, each to each;

And the angle BAC is equal to the angle EDG (Const.);

∴ BC = EG.

Therefore the base BC is equal to the base EG (I. 4).

And because DG is equal to DF (Const.), the angle DFG is equal to the angle DGF (I. 5).

But the angle DGF is greater than the angle EGF (Ax. 9);

and ∠ EFG > ∠ EGF.

Therefore the angle DFG is greater than the angle EGF;

Much more then is the angle EFG greater than the angle EGF.

And because the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater angle is subtended by the greater side,

∴ EG > EF.

Therefore the side EG is greater than the side EF (I. 19).

But EG was proved equal to BC;

Therefore BC is greater than EF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 25.—Theorem.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, the angle contained by the sides of that which has the greater base shall be greater than the angle contained by the sides equal to them of the other.

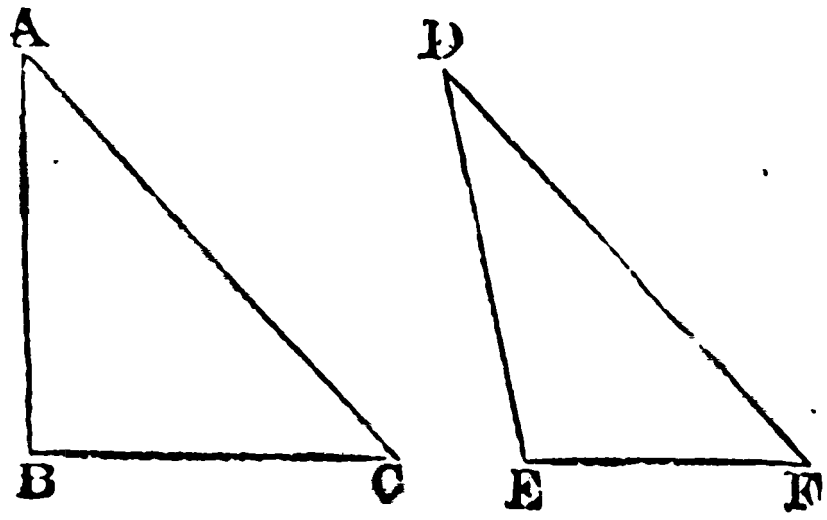
Let ABC, DEF, be two triangles, which have

The two sides AB, AC equal to the two sides DE, DF, each to each—namely, AB to DE, and AC to DF,

But the base BC greater than the base EF;

The angle BAC shall be greater than the angle EDF.

PROOF.—For if the angle BAC be not greater than the angle EDF, it must either be equal to it or less.



But the angle BAC is not equal to the angle EDF, for then the base BC would be equal to the base EF (I. 4), but it is not (Hyp.);

Therefore the angle BAC is not equal to the angle EDF; ∠ BAC not = ∠ EDF.

Neither is the angle BAC less than the angle EDF, for then the base BC would be less than the base EF (I. 24), but it is not (Hyp.),

Therefore the angle BAC is not less than the angle EDF. ∠ BAC not < ∠ EDF.

And it has been proved that the angle BAC is not equal to the angle EDF;

Therefore the angle BAC is greater than the angle EDF.

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 26.—Theorem.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side—namely, either the side adjacent to the equal angles in each, or the side opposite to them; then shall the other sides be equal, each to each; and also the third angle of the one equal to the third angle of the other. Or,

If two angles and a side in one triangle be respectively equal to two angles and a corresponding side in another triangle, the triangles shall be equal in every respect.

Let ABC, DEF be two triangles, which have

The angles ABC, BCA equal to the angles DEF, EFD, each to each—namely, ABC to DEF, and BCA to EFD;

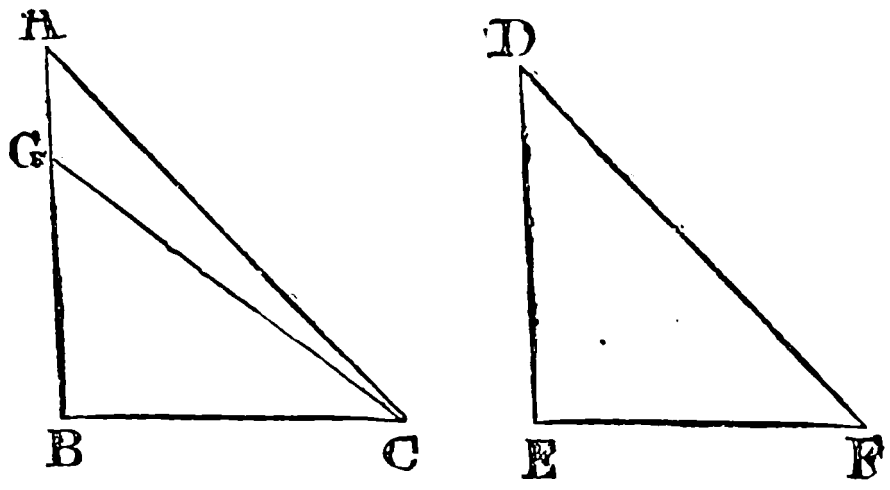
Also one side equal to one side.

CASE 1.—First, let the sides adjacent to the equal angles in each be equal—namely, BC to EF; Given BC = EF.

Then shall the side AB be equal to DE, the side AC to DF, and the angle BAC to the angle EDF.

Suppose AB > DE.

For if AB be not equal to DE, one of them must be greater than the other. Let AB be the greater of the two.



Make BG = DE.

CONSTRUCTION.—Make BG equal to DE (I. 3), and join GC.

PROOF.—Because BG is equal to DE (Const.), and BC is equal to EF (Hyp.), the two sides GB, BC are equal to the two sides DE, EF, each to each.

And the angle GBC is equal to the angle DEF (Hyp.);

Therefore the base GC is equal to the base DF (I. 4),

And the triangle GBC to the triangle DEF (I. 4),

And the other angles to the other angles, each to each, to which the equal sides are opposite;

∴ ∠ GCB = ∠ DFE.

Therefore the angle GCB is equal to the angle DFE (I. 4).

But the angle DFE is equal to the angle BCA (Hyp.);

Therefore the angle GCB is equal to the angle BCA (Ax. 1), the less to the greater, which is impossible;

AB not unequal to DE.

Therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF (Hyp.);

Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,

And the angle ABC is equal to the angle DEF (Hyp.);

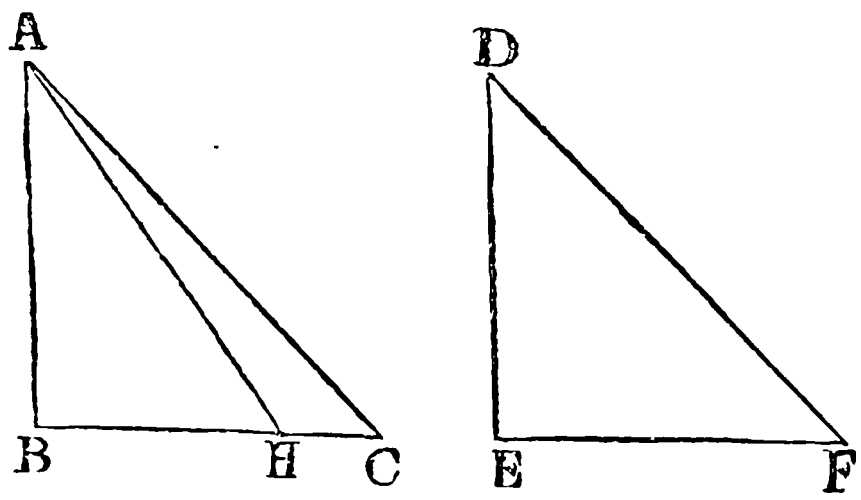
Therefore the base AC is equal to the base DF (I. 4),

And the third angle BAC to the third angle EDF (I. 4).

CASE 2.—Next, let the sides which are opposite to the equal angles in each triangle be equal to one another—namely, AB equal to DE.

Likewise in this case the other sides shall be equal, AC to DF, and BC to EF; and also the angle BAC to the angle EDF.

Suppose BC > EF



For if BC be not equal to EF, one of them must be greater than the other. Let BC be the greater of the two.

Make BH = EF.

CONSTRUCTION.—Make BH equal to EF (I. 3), and join AH.

PROOF.—Because BH is equal to EF (Const.), and AB is equal to DE (Hyp.), the two sides AB, BH are equal to the two sides DE, EF, each to each, BH = EF, AB = DE.

And the angle ABH is equal to the angle DEF (Hyp.); ∠ ABH = ∠ DEF.

Therefore the base AH is equal to the base DF (I. 4),

And the triangle ABH to the triangle DEF (I. 4),

And the other angles to the other angles, each to each, to which the sides are opposite;

Therefore the angle BHA is equal to the angle EFD (I. 4). ∴ ∠ BHA

But the angle EFD is equal to the angle BCA (Hyp.); = ∠ EFD

Therefore the angle BHA is also equal to the angle BCA (Ax. 1); = ∠ BCA.

That is, the exterior angle BHA of the triangle AHC, is equal to its interior and opposite angle BCA, which is impossible (I. 16);

Therefore BC is not unequal to EF—that is, it is equal to it; and AB is equal to DE (Hyp.); BC not unequal to EF.

Therefore the two sides AB, BC are equal to the two sides DE, EF, each to each,

And the angle ABC is equal to the angle DEF (Hyp.);

Therefore the base AC is equal to the base DF (I. 4),

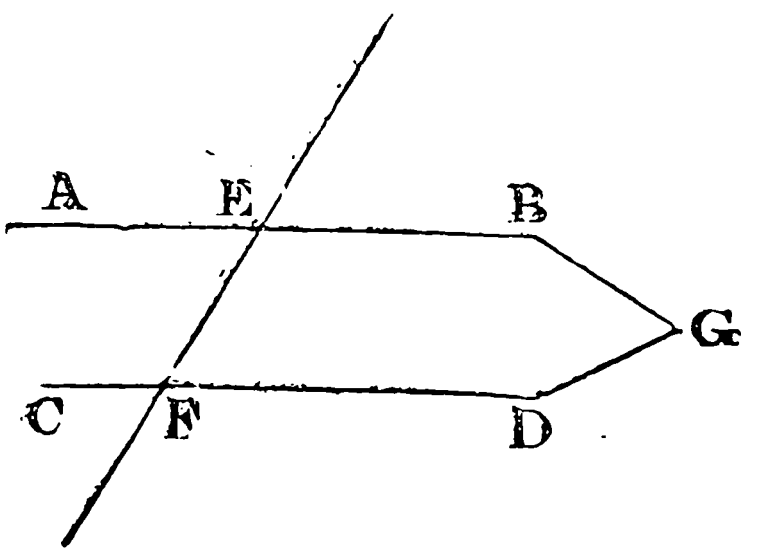
And the third angle BAC is equal to the third angle EDF (I. 4).

Therefore, if two triangles, &c. *Q. E. D.*

Proposition 27.—Theorem.

If a straight line falling upon two other straight lines make the alternate angles equal to one another, these two straight lines shall be parallel.

Let the straight line EF, which falls upon the two straight lines AB, CD, make the alternate angles AEF, EFD, equal to one another.



Given ∠ AEF = ∠ EFD.

AB shall be parallel to CD.

For if AB and CD be not parallel, they will meet if produced, either towards B, D; or towards A, C. Let them be produced, and meet towards B, D, in the point G.

∠ AEF > ∠ EFG, and also = ∠ EFG.

PROOF.—Then GEF is a triangle, and its exterior angle AEF is greater than the interior and opposite angle EFG (I. 16).

But the angle AEF is also equal to EFG (Hyp.), which is impossible;

Therefore AB and CD, being produced, do not meet towards B, D.

In like manner it may be shown that they do not meet towards A, C.

But those straight lines in the same plane which being produced ever so far both ways do not meet are parallel (Def. 34);

Therefore AB is parallel to CB.

Therefore, if a straight line, &c. *Q. E. D.*

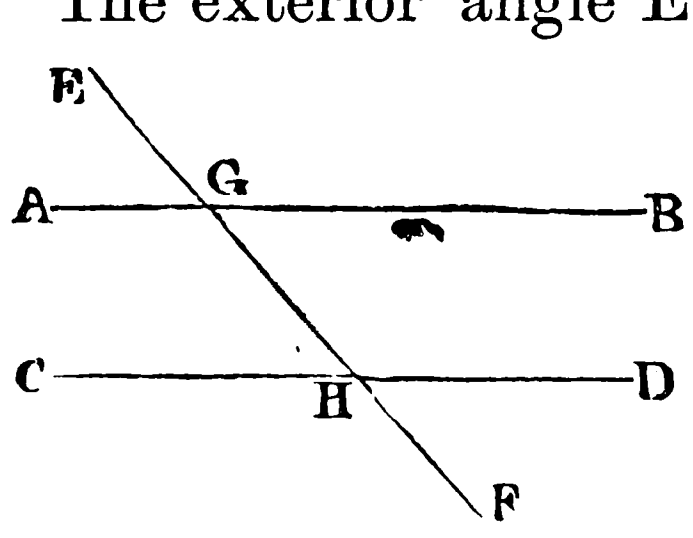
Proposition 28.—Theorem.

If a straight line falling upon two other straight lines make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles, the two straight lines shall be parallel to one another.

Let the straight line EF, which falls upon the two straight lines AB, CD, make—

The exterior angle EGB equal to the interior and opposite angle GHD, upon the same side;

Or make the interior angles on the same side, the angles BGH, GHD, together equal to two right angles;



AB shall be parallel to CD.

PROOF 1.—Because the angle EGB is equal to the angle GHD (Hyp.),

And the angle EGB is equal to the angle AGH (I. 15);

∠ AGH = ∠ GHD.

Therefore the angle AGH is equal to the angle GHD (Ax. 1), and these angles are alternate;

Therefore AB is parallel to CD (I. 27).

PROOF 2.—Again, because the angles BGH, GHD are equal to two right angles (Hyp.),

And the angles BGH, AGH are also equal to two right angles (I. 13).

Therefore the angles BGH, AGH are equal to the angles BGH, GHD (Ax. 1).

$\therefore \angle BGH + \angle AGH = \angle BGH + \angle GHD.$

Take away the common angle BGH.

Therefore the remaining angle AGH is equal to the remaining angle GHD (Ax. 3), and they are alternate angles.

$\therefore \angle AGH = \angle GHD$

Therefore AB is parallel to CD (I. 27).

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 29.—Theorem.

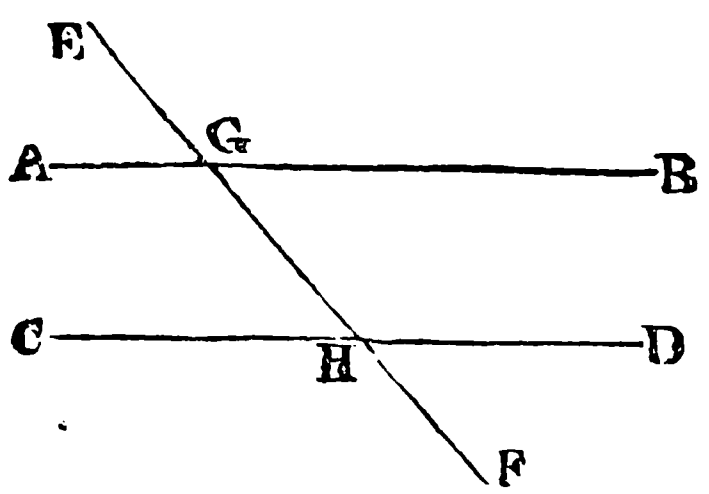
If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side; and also the two interior angles upon the same side together equal to two right angles.

Let the straight line EF fall upon the parallel straight lines AB, CD;

The alternate angles AGH, GHD shall be equal to one another.

The exterior angle EGB shall be equal to GHD, the interior and opposite angle upon the same side;

And the two interior angles on the same side BGH, GHD shall be together equal to two right angles.



For if AGH be not equal to GHD, one of them must be greater than the other. Let AGH be the greater.

$\angle AGH >$ GHD. (suppose.)

Proof.—Then the angle AGH is greater than the angle GHD; to each of them add the angle BGH.

Therefore the angles BGH, AGH are greater than the angles BGH, GHD (Ax. 4).

But the angles BGH, AGH are together equal to two right angles (1. 3).

$\therefore \angle BGH + \angle GHD <$ two right angles.

Therefore the angles BGH, GHD are less than two right angles.

But if a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines being

continually produced, shall at length meet on that side on which are the angles which are less than two right angles (Ax. 12);

Hence AB and CD meet, and are parallel.

Therefore the straight lines AB, CD will meet if produced far enough.

∴ ∠ AGH not unequal to ∠ GHD.

But they cannot meet, because they are parallel straight lines (Hyp.);

Therefore the angle AGH is not unequal to the angle GHD—that is, it is equal to it.

and ∠ EGB = ∠ GHD,

But the angle AGH is equal to the angle EGB (I. 15);

Therefore the angle EGB is equal to the angle GHD (Ax. 1).

Add to each of these the angle BGH.

Therefore the angles EGB, BGH, are equal to the angles BGH, GHD (Ax. 2).

But the angles EGB, BGH, are equal to two right angles (I. 13).

also ∠ BGH + ∠ GHD = two right angles.

Therefore also BGH, GHD, are equal to two right angles (Ax. 1).

Therefore, if a straight line, &c. *Q. E. D.*

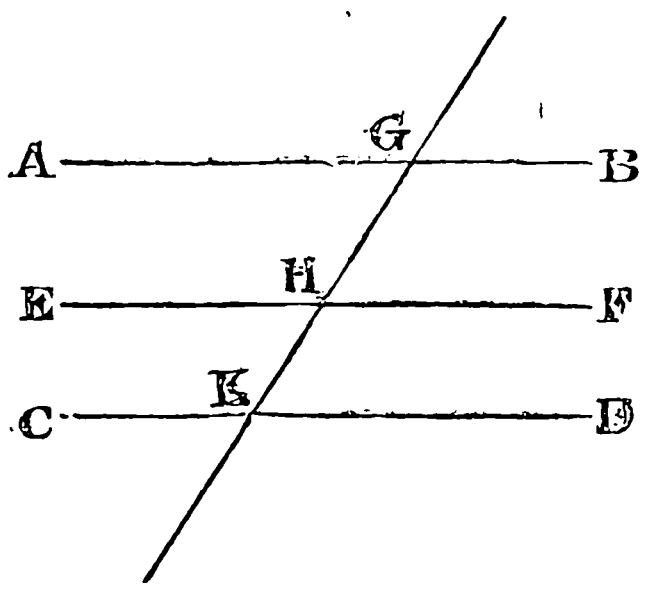
Proposition 30.—Theorem.

Straight lines which are parallel to the same straight lines are parallel to one another.

Let AB, CD be each of them parallel to EF;

AB shall be parallel to CD.

CONSTRUCTION.—Let the straight line GHK cut AB, EF, CD.



∠ AGH or ∠ AGK = ∠ GHF,

and ∠ GHF = ∠ GKD.

PROOF.—Because GHK cuts the parallel straight lines AB, EF, the angle AGH is equal to the angle GHF (I. 29).

Again, because GK cuts the parallel straight lines EF, CD, the angle GHF is equal to the angle GKD (I. 29).

And it was shown that the angle AGK is equal to the angle GHF;

∴ ∠ AGK = ∠ GKD.

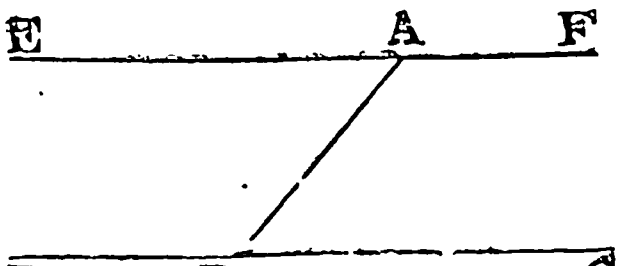
Therefore the angle AGK is equal to the angle GKD (Ax. 1), and they are alternate angles;

Therefore AB is parallel to CD (I. 27).
Therefore, straight lines, &c. *Q. E. D.*

Proposition 31.—Problem.

To draw a straight line through a given point, parallel to a given straight line.

Let A be the given point, and BC the given straight line.
It is required to draw a straight line through the point A, parallel to BC.



Make ∠ DAE = ∠ ADC.

CONSTRUCTION.—In BC take any point D, and join AD.
At the point A, in the straight line AD, make the angle DAE equal to the angle ADC (I. 23).
Produce the straight line EA to F.
Then EF *shall be parallel to* BC.
PROOF.—Because the straight line AD, which meets the two straight lines BC, EF, makes the alternate angles EAD, ADC equal to one another;

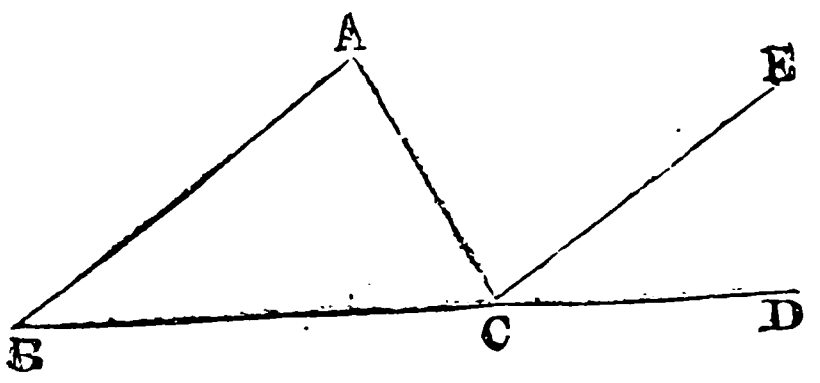
They are alternate angles.

Therefore EF is parallel to BC (I. 27).
Therefore, the straight line EAF is drawn through the given point A, parallel to the given straight line BC. *Q. E. F.*

Proposition 32.—Theorem.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its sides BC be produced to D;
The exterior angle ACD shall be equal to the two interior and opposite angles CAB, ABC;
And the three interior angles of the triangle—namely, ABC, BCA, CAB, shall be equal to two right angles.



Make CE parallel to AB.

CONSTRUCTION.—Through the point C, draw CE parallel to AB (I. 31).

Then ∠ BAC = ∠ ACE, and ∠ ECD = ∠ ABC.

PROOF.—Because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal (I. 29).

Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC (I. 29).

But the angle ACE was shown to be equal to the angle BAC;

∴ ∠ ACD = ∠ BAC + ∠ ABC. Add ∠ ACB.

Therefore the whole exterior angle ACD is equal to the two interior and opposite angles BAC, ABC (Ax. 2).

To each of these equals add the angle ACB.

Therefore the angles ACD, ACB are equal to the three angles CBA, BAC, ACB (Ax. 2).

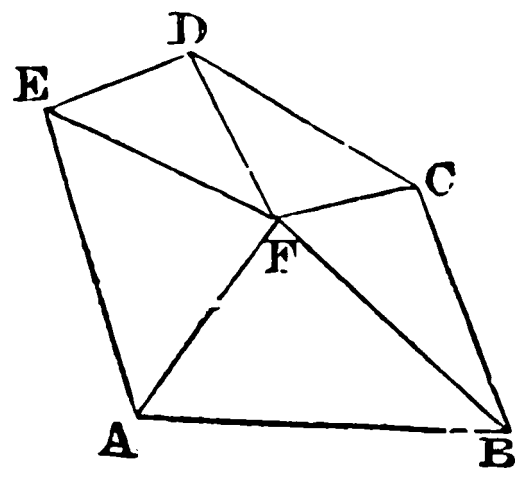
But the angles ACD, ACB are equal to two right angles (I. 13);

Therefore also the angles CBA, BAC, ACB are equal to two right angles (Ax. 1).

Therefore, if a side of any triangle, &c. *Q. E. D.*

COROLLARY 1.—*All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*

For any rectilineal figure ABCDE can, by drawing straight lines from a point F within the figure to each angle, be divided into as many triangles as the figure has sides.



And, by the preceding proposition, the angles of each triangle are equal to two right angles.

Therefore all the angles of the triangles are equal to twice as many right angles as there are triangles; that is, as there are sides of the figure.

But the same angles are equal to the angles of the figure, together with the angles at the point F;

And the angles at the point F, which is the common vertex of all the triangles, are equal to four right angles (I. 15, Cor. 2);

Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

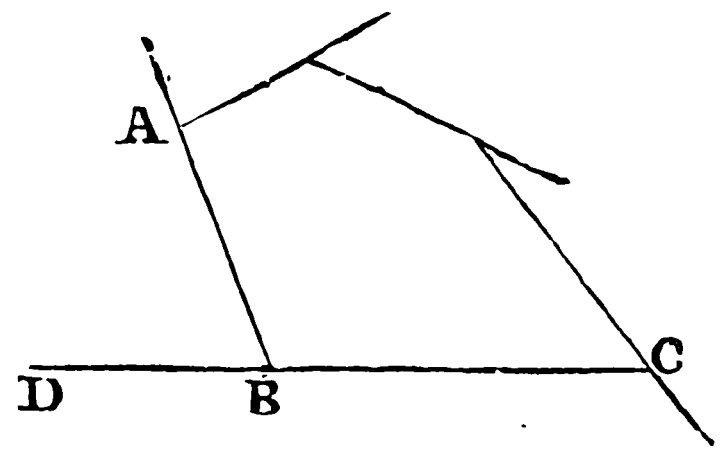
COROLLARY 2.—*All the exterior angles of any rectilineal figure are together equal to four right angles.*

The interior angle ABC, with its adjacent exterior angle ABD, is equal to two right angles (I. 13);

Therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as the figure has sides.

But all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides (I. 32, Cor. 1);

Therefore all the interior angles, together with all the exterior angles, are equal to all the interior angles and four right angles (Ax. 1).



Take away the interior angles which are common;

Therefore all the exterior angles are equal to four right angles (Ax. 3).

Proposition 33.—Theorem.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.

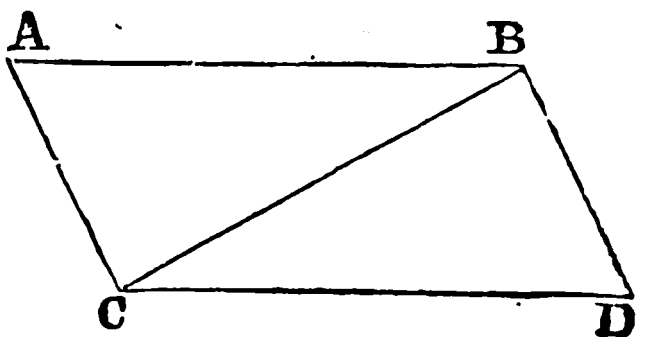
Let AB and CD be equal and parallel straight lines joined towards the same parts by the straight lines AC and BD;

AC and BD shall be equal and parallel.

CONSTRUCTION.—Join BC.

PROOF.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal (I. 29). ∠ ABC = ∠ BCD.

Because AB is equal to CD, and BC common to the two triangles ABC, DCB, the two sides AB, BC are equal to the two sides DC, CB, each to each;



And the angle ABC was proved to be equal to the angle BCD;

Therefore the base AC is equal to the base BD (I. 4), ∴ AC = BD,

And the triangle ABC is equal to the triangle BCD (I. 4),

And the other angles are equal to the other angles, each to each, to which the equal sides are opposite;

and ∠ ACB = ∠ CBD.

Therefore the angle ACB is equal to the angle CBD.

And because the straight line BC meets the two straight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another;

Therefore AC is parallel to BD (I. 27); and it was shown to be equal to it.

Therefore, the straight lines, &c. *Q. E. D.*

Proposition 34.—Theorem.

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram—that is, divides it into two equal parts.

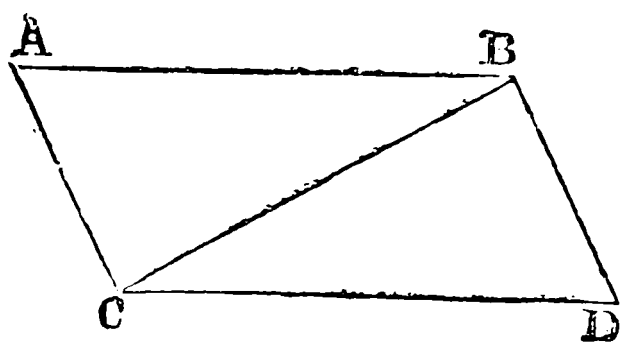
Let ACDB be a parallelogram, of which BC is a diagonal;

The opposite sides and angles of the figure shall be equal to one another,

And the diagonal BC shall bisect it.

PROOF.—Because AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal to one another (I. 29);

∠ ABC = ∠ BCD,



Because AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal to one another (I. 29);

and ∠ ACB = ∠ CBD.

Therefore the two triangles ABC, BCD have two angles, ABC, BCA in the one, equal to two angles, BCD, CBD in the other, each to each; and the side BC, adjacent to the equal angles in each, is common to both triangles.

Therefore the other sides are equal, each to each, and the third angle of the one to the third angle of the other—namely, AB equal to CD, AC to BD, and the angle BAC to the angle CDB (I. 26).

∴ AB = CD, AC = BD, ∠ BAC = ∠ CDB,

And because the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB,

Therefore the whole angle ABD is equal to the whole angle ACD (Ax. 2).

and ∴ ∠ ABD = ∠ ACD,

And the angle BAC has been shown to be equal to the angle BDC; therefore the opposite sides and angles of a parallelogram are equal to one another.

Also the diagonal bisects it.

For AB being equal to CD, and BC common,

The two sides AB, BC are equal to the two sides CD and CB, each to each.

And the angle ABC has been shown to be equal to the angle BCD;

Therefore the triangle ABC is equal to the triangle BCD (I. 4), also $\triangle ABC = \triangle BCD$.

And the diagonal BC divides the parallelogram ABCD into two equal parts.

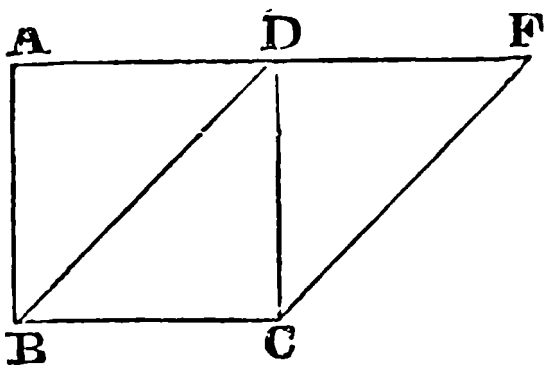
Therefore, the opposite sides, &c. *Q. E. D.*

Proposition 35.—Theorem.

Parallelograms upon the same base, and between the same parallels, are equal to one another.

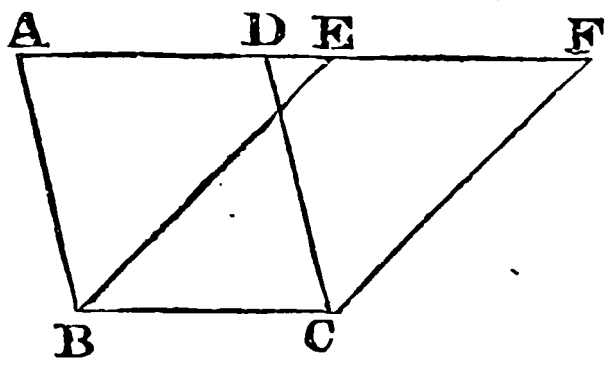
Let the parallelograms ABCD, EBCF be on the same base BC, and between the same parallels AF, BC;

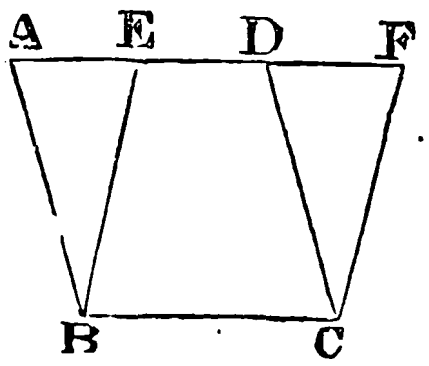
The parallelogram ABCD shall be equal to the parallelogram EBCF.



Case 1.—If the sides AD, DF of the parallelograms ABCD, DBCF, opposite to the base BC, be terminated in the same point D, it is plain that each of the parallelograms is double of the triangle DBC (I. 34), and that they are therefore equal to one another (Ax. 6).

Case 2.—But if the sides AD, EF, opposite to the base BC, of the parallelograms ABCD, EBCF, be not terminated in the same point, then—





Proof.—Because ABCD is a parallelogram, AD is equal to BC (I. 34). $AD = BC$.

For the same reason EF is equal to BC; $EF = BC$.

Therefore AD is equal to EF (Ax. 1), and DE is common;

Therefore the whole, or the remainder, AE, is equal to the whole, or the remainder, DF (Ax. 2, or 3), $\therefore AE = DF$.

And AB is equal to DC (I. 34).

Therefore the two EA, AB are equal to the two FD, DC, each to each;

And the exterior angle FDC is equal to the interior EAB (I. 29);

Therefore the base EB is equal to the base FC (I. 4),

Hence △ EAB = △ FDC.

And the triangle EAB equal to the triangle FDC (I. 4).

Take the triangle FDC from the trapezium ABCF, and from the same trapezium ABCF, take the triangle EAB, and the remainders are equal (Ax. 3),

That is, the parallelogram ABCD is equal to the parallelogram EBCF.

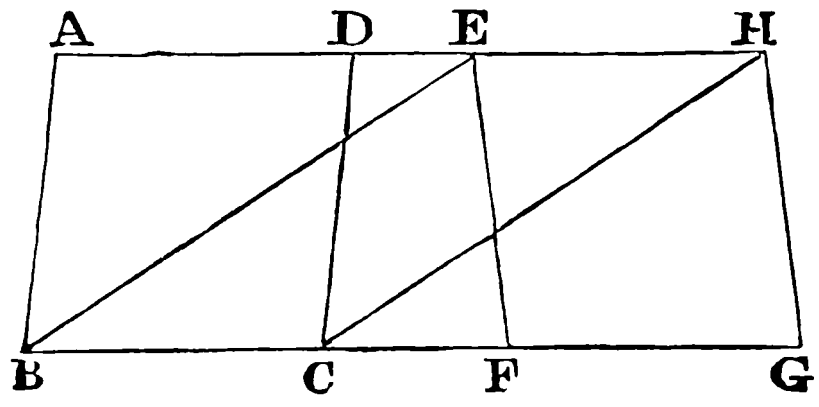
Therefore, parallelograms, &c. *Q. E. D.*

Proposition 36.—Theorem.

Parallelograms upon equal bases, and between the same parallels, are equal to one another.

Let ABCD, EFGH be parallelograms on equal bases BC, FG, and between the same parallels AH, BG;

The parallelogram ABCD shall be equal to the parallelogram EFGH.



Construction.—Join BE, CH.

Proof.—Because BC is equal to FG (Hyp.), and FG to EH (I. 34),

BC = EH,

Therefore BC is equal to EH (Ax. 1); and they are parallels, and joined towards the same parts by the straight lines BE, CH.

But straight lines which join the extremities of equal and parallel straight lines towards the same parts, are themselves equal and parallel (I. 33);

and BE = CH.

Therefore BE, CH are both equal and parallel;

EBCH a parallelogram, equal each of the given ones.

Therefore EBCH is a parallelogram (Def. 35),

And it is equal to the parallelogram ABCD, because they are on the same base BC, and between the same parallels BC, AH (I. 35).

For the like reason, the parallelogram EFGH is equal to the same parallelogram EBCH;

Therefore the parallelogram ABCD is equal to the parallelogram EFGH (Ax. 1).

Therefore, parallelograms, &c. *Q. E. D.*

Proposition 37.—Theorem.

Triangles upon the same base, and between the same parallels, are equal to one another.

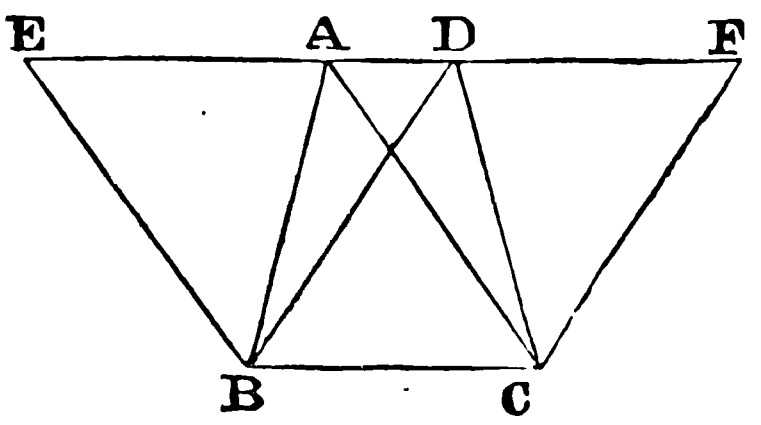
Let the triangles ABC, DBC be on the same base BC, and between the same parallels AD, BC;

The triangle ABC shall be equal to the triangle DBC.

CONSTRUCTION.—Produce AD both ways, to the points E, F.

Through B draw BE parallel to CA, and through C draw CF parallel to BD (I. 31).

PROOF. — Then each of the figures EBCA, DBCF, is a parallelogram (Def. 35), and they are equal to one another, because they are on the same base BC, and between the same parallels BC, EF (I. 35.);



Figures EBCA and DBCF are equal;

And the triangle ABC is half of the parallelogram EBCA, because the diagonal AB bisects it (I. 34);

And the triangle DBC is half of the parallelogram DBCF, because the diagonal DC bisects it (I. 34).

and the triangles are respectively half of these.

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DBC.

Therefore, triangles, &c. *Q. E. D.*

Proposition 38.—Theorem.

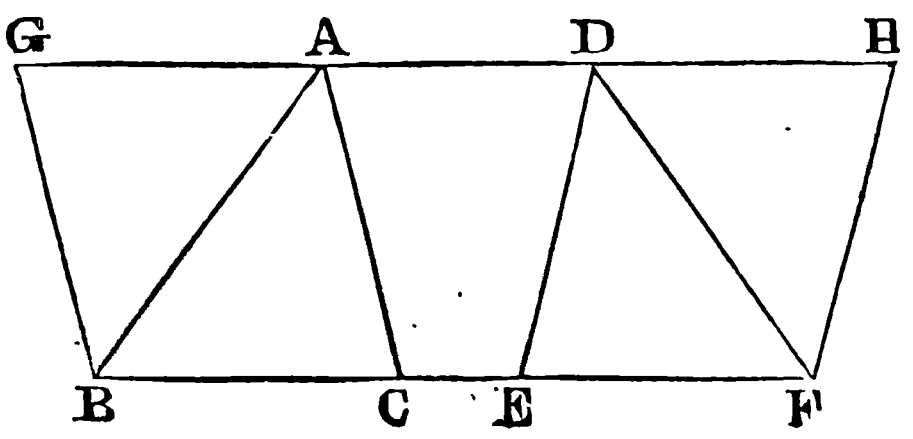
Triangles upon equal bases, and between the same parallels, are equal to one another.

Let the triangles ABC, DEF, be on equal bases BC, EF, and between the same parallels BF, AD.

The triangle ABC shall be equal to the triangle DEF.

CONSTRUCTION.—Produce AD both ways to the points G, H.

Through B draw BG parallel to CA, and through F draw FH parallel to ED (I. 31).



Figures GBCA and DEFH are equal;

PROOF.—Then each of the figures GBCA. DEFH, is a

parallelogram (Def. 35), and they are equal to one another, because they are on equal bases BC, EF, and between the same parallels BF, GH (I. 36);

and the triangles are half of these respectively.

And the triangle ABC is half of the parallelogram GBCA, because the diagonal AB bisects it (I. 34);

And the triangle DEF is half of the parallelogram DEFH, because the diagonal DF bisects it (I. 34).

But the halves of equal things are equal (Ax. 7);

Therefore the triangle ABC is equal to the triangle DEF.

Therefore, triangles, &c. *Q. E. D.*

Proposition 39.—Theorem.

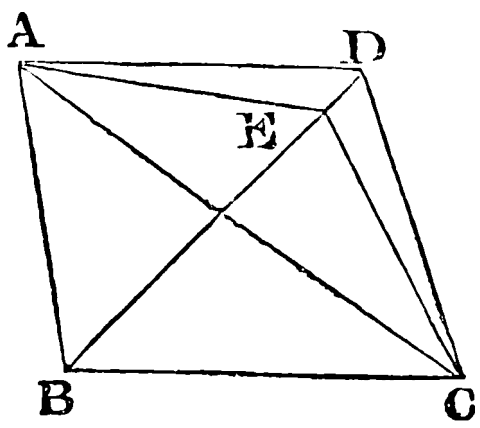
Equal triangles upon the same base, and on the same side of it, are between the same parallels.

Let the equal triangles ABC, DBC be upon the same base BC, and on the same side of it;

They shall be between the same parallels.

CONSTRUCTION.—Join AD; AD shall be parallel to BC.

AE parallel to EC suppose.



For if it is not, through A draw AE parallel to BC (I. 31), and join EC.

PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

But the triangle ABC is equal to the triangle DBC (Hyp.);

Then △ DBC = △ EBC, an absurdity.

Therefore the triangle DBC is equal to the triangle EBC (Ax. 1), the greater equal to the less, which is impossible;

Therefore AE is not parallel to BC.

In the same manner, it can be demonstrated that no line passing through A can be parallel to BC, except AD;

Therefore AD is parallel to BC.

Therefore, equal triangles, &c. *Q. E. D.*

Proposition 40.—Theorem.

Equal triangles upon the same side of equal bases, that are in the same straight line, are between the same parallels.

Let the equal triangles ABC, DEF, be upon the same side of equal bases BC, EF, in the same straight line BF.

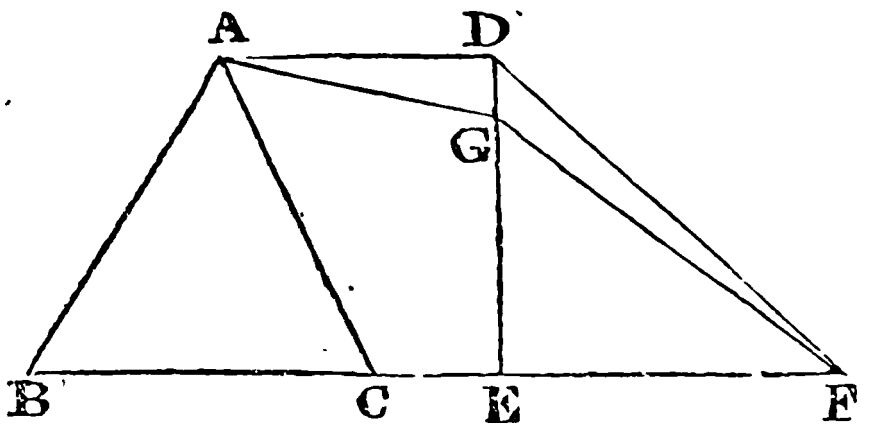
The triangles ABC, DEF shall be between the same parallels.

CONSTRUCTION.—Join AD ; AD shall be parallel to BF.

For if it is not, through A draw AG parallel to BF (I. 31), and join GF.

AG parallel to BF suppose.

PROOF.—The triangle ABC is equal to the triangle GEF, because they are upon equal bases BC, EF, and are between the same parallels BF, AG (I. 38).



△ DEF = △ GEF, an absurdity.

But the triangle ABC is equal to the triangle DEF ;

Therefore the triangle DEF is equal to the triangle GEF (Ax. 1), the greater equal to the less, which is impossible ;

Therefore AG is not parallel to BF.

In the same manner, it can be demonstrated that no line, passing through A, can be parallel to BF, except AD ;

Therefore AD is parallel to BF.

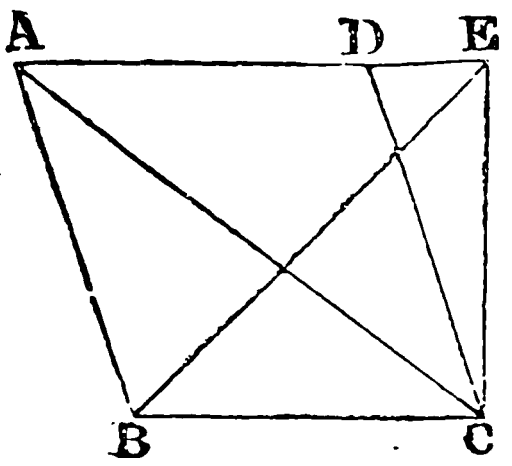
Therefore, equal triangles, &c.

Proposition 41.—Theorem.

If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram shall be double of the triangle.

Let the parallelogram ABCD, and the triangle EBC be upon the same base BC, and between the same parallels BC, AE ;

The parallelogram ABCD shall be double of the triangle EBC.



△ ABC = △ EBC.

CONSTRUCTION.—Join AC.

PROOF.—The triangle ABC is equal to the triangle EBC, because they are upon the same base BC, and between the same parallels BC, AE (I. 37).

But the parallelogram ABCD is double of the triangle ABC, because the diagonal AC bisects the parallelogram (I. 34).

And parallelogram = 2 △ ABC.

Therefore the parallelogram ABCD is also double of the triangle EBC (Ax. 1).

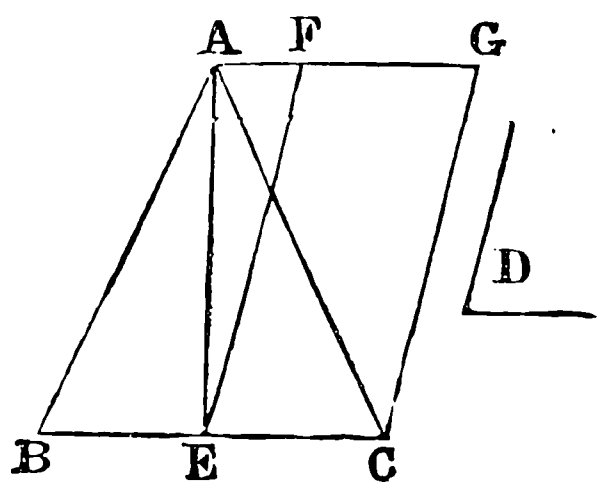
Therefore, if a parallelogram, &c. *Q. E. D.*

Proposition 42.—Problem.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle;

It is required to describe a parallelogram that shall be equal to the given triangle ABC, and have one of its angles equal to D.



Make BE = EC

and ∠ CEF = D

Construction.—Bisect BC in E (I. 10), and join AE.

At the point E, in the straight line CE, make the angle CEF equal to D (I. 23).

Through A draw AFG parallel to EC (I. 31).

Through C draw CG parallel to EF (I. 31).

Then FECG *is the parallelogram required.*

Proof.—Because BE is equal to EC (Const.), the triangle ABE is equal to the triangle AEC, since they are upon equal bases and between the same parallels (I. 38);

△ ABC = 2 △ AEC, and also figure FECG = 2 △ AEC.

Therefore the triangle ABC is double of the triangle AEC.

But the parallelogram FECG is also double of the triangle AEC, because they are upon the same base, and between the same parallels (I. 41);

Therefore the parallelogram FECG is equal to the triangle ABC (Ax. 6),

And it has one of its angles CEF equal to the given angle D (Const.).

Therefore a parallelogram FECG has been described equal to the given triangle ABC, and having one of its angles CEF equal to the given angle D. *Q. E. F.*

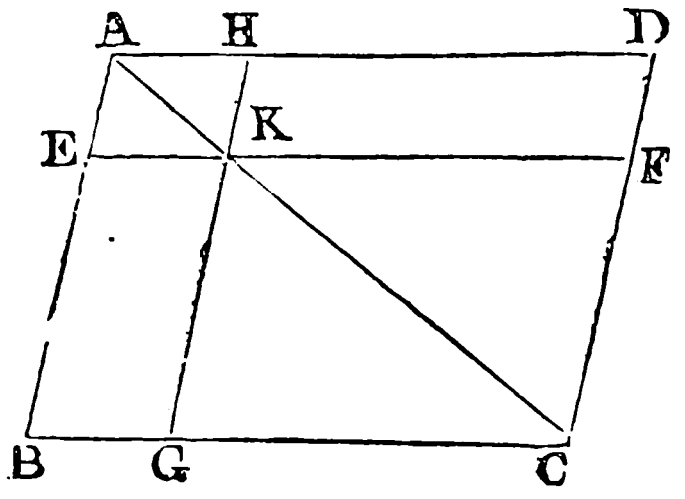
Proposition 43.—Theorem.

The complements of the parallelograms which are about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, of which the diagonal is AC; and EH, GF parallelograms about AC, that is, through which AC passes; and BK, KD the other parallelograms, which make up the whole figure ABCD, and are therefore called the complements.

The complement BK shall be equal to the complement KD.

Proof.—Because ABCD is a parallelogram, and AC its diagonal, the triangle ABC is equal to the triangle ADC (I. 34). △ ABC = △ ADC.



Again, because AEKH is a parallelogram, and AK its diagonal, the triangle AEK is equal to the triangle AHK (I. 34). Also △ AEK = △ AHK.

For the like reason the triangle KGC is equal to the triangle KFC. And △ KGC = △ KFC.

Therefore, because the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC,

The triangles AEK, KGC are equal to the triangles AHK, KFC (Ax. 2).

But the whole triangle ABC was proved equal to the whole triangle ADC;

Therefore the remaining complement BK is equal to the remaining complement KD (Ax. 3). ∴ BK = KD.

Therefore, the complements, &c. *Q. E. D.*

Proposition 44.—Problem.

To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given straight line, C the given triangle, and D the given angle.

It is required to apply to the straight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make parallelogram BEFG = △ C, and ∠ at B = ∠ D, and EBA a straight line.

CONSTRUCTION 1.—Make the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D (I. 42);

And let the parallelogram BEFG be made so that BE may be in the same straight line with AB.

Produce FG to H.

Through A draw AH parallel to BG or EF (I. 31).

Join HB.

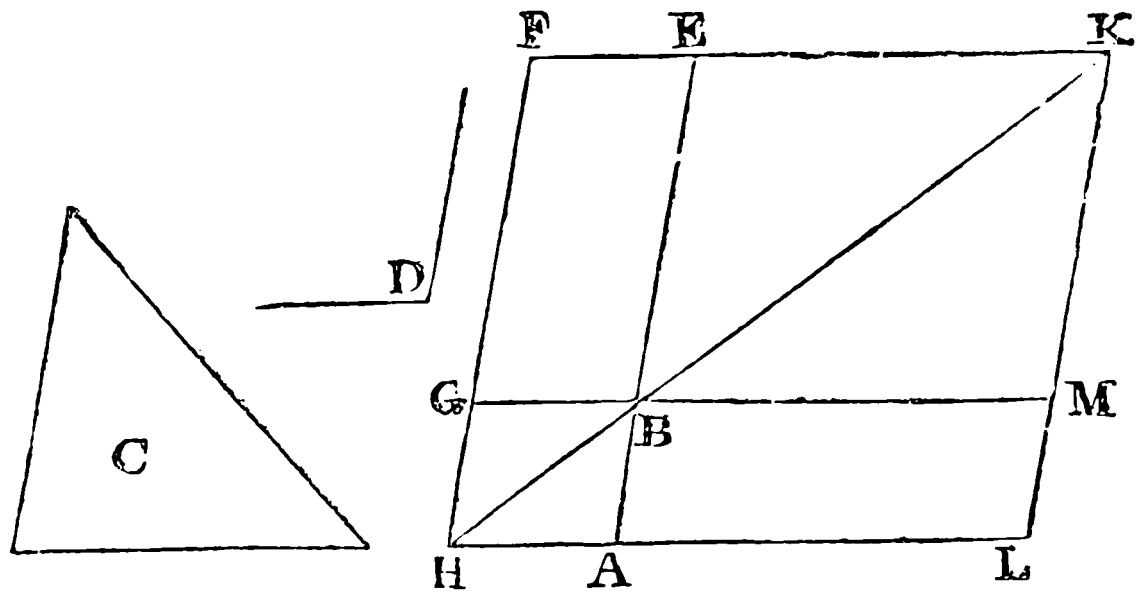
PROOF 1.—Because the straight line HF falls on the parallels AH, EF, the angles AHF, HFE are together equal to two right angles (I. 29).

Therefore the angles BHF, HFE are together less than two right angles (Ax. 9). But straight lines which with another straight line make the interior angles on the same side together less than two right angles, will meet on that side, if produced far enough (Ax. 12);

HB and FE meet.

Therefore HB and FE shall meet if produced.

CONSTRUCTION 2.—Produce HB and FE towards BE, and let them meet in K.



Through K draw KL parallel to EA or FH (I. 31).

Produce HA, GB to the points L, M.

Then LB *shall be the parallelogram required.*

PROOF 2.—Because HLKF is a parallelogram, of which the diagonal is HK; and AG, ME are the parallelograms about HK; and LB, BF are the complements;

Figures LB = BF.

Therefore the complement LB is equal to the complement BF (I. 43).

But BF is equal to the triangle C (Const.);
Therefore LB is equal to the triangle C (Ax. 1).
And because the angle GBE is equal to the angle ABM (I. 15), and likewise to the angle D (Const.);
Therefore the angle ABM is equal to the angle D (Ax. 1).
Therefore, the parallelogram LB is applied to the straight line AB, and is equal to the triangle C, and has the angle ABM equal to the angle D. *Q. E. F.*

But $BF = \triangle C$, $\therefore LB = \triangle C$;

Also $\angle ABM = \angle GBE = \angle D$.

Proposition 45.—Problem.

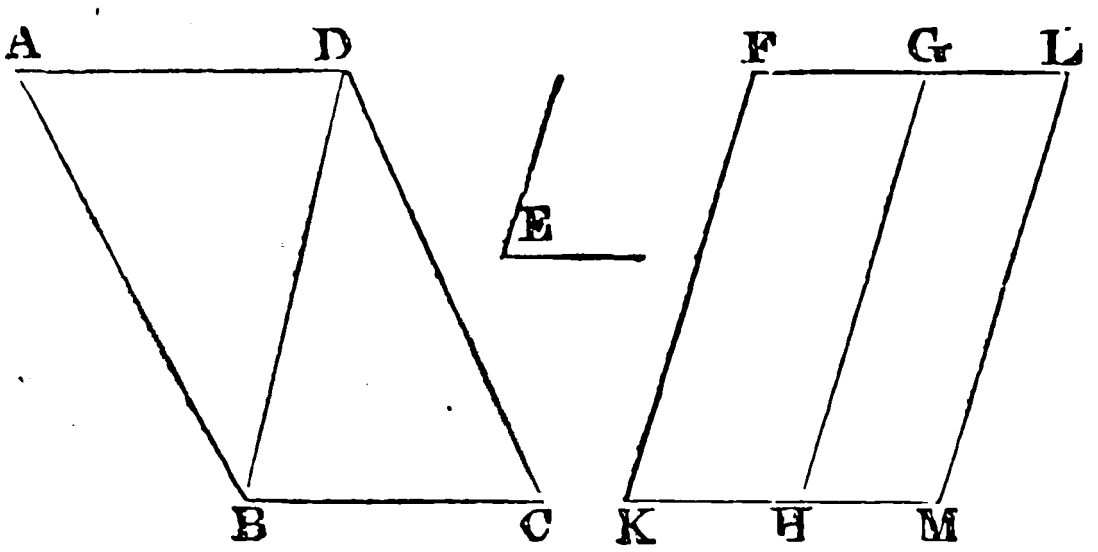
To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle.

It is required to describe a parallelogram equal to ABCD, and having an angle equal to E.

CONSTRUCTION.—Join DB.

Describe the parallelogram FH equal to the triangle ADB, and having the angle FKH equal to the angle E (I. 42).



Make FH $= \triangle ADB$. Apply to GH, $GM = \triangle DBC$, with $\angle GHM = \angle E$.

To the straight line GH apply the parallelogram GM equal to the triangle DBC, and having the angle GHM equal to the angle E (I. 44).

Then the figure FKML *shall be the parallelogram required.*

PROOF.—Because the angle E is equal to each of the angles FKH, GHM (Const.),

Therefore the angle FKH is equal to the angle GHM (Ax. 1).

Add to each of these equals the angle KHG;

Therefore the angles FKH, KHG are equal to the angles KHG, GHM (Ax. 2).

But FKH, KHG are equal to two right angles (I. 29);

Therefore also KHG, GHM are equal to two right angles (Ax. 1).

And because at the point H, in the straight line GH, the two straight lines KH, HM, on the opposite sides of it, make the adjacent angles together equal to two right angles,

Then KHM is a straight line;

Therefore KH is in the same straight line with HM (I. 14).

And because the straight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal (I. 29).

Add to each of these equals the angle HGL;

Therefore the angles MHG, HGL are equal to the angles HGF, HGL (Ax. 2).

But the angles MHG, HGL are equal to two right angles (I. 29);

Therefore also the angles HGF, HGL are equal to two right angles,

and FGL is a straight line.

And therefore FG is in the same straight line with GL (I. 14).

And because KF is parallel to HG, and HG parallel to ML (Const.);

Therefore KF is parallel to ML (I. 30).

And KM, FL are parallels (Const);

∴ KFLM a parallelogram.

Therefore KFLM is a parallelogram (Def. 35).

And because the triangle ABD is equal to the parallelogram HF, and the triangle DBC equal to the parallelogram GM (Const.),

And the figure ABCD is equal to it.

Therefore the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM (Ax. 2).

Therefore, the parallelogram KFLM has been described equal to the given rectilineal figure ABCD, and having the angle FKM equal to the given angle E. *Q. E. F.*

COROLLARY.—From this it is manifest how to apply to a given straight line a parallelogram, which shall have an angle equal to a given rectilineal angle, and shall be equal to a given rectilineal figure—namely, by applying to the given straight line a parallelogram equal to the first triangle ABD, and having an angle equal to the given angle; and so on (I. 44).

Proposition 46.—Problem.

To describe a square upon a given straight line.

Let AB be the given straight line;

It is required to describe a square upon AB.

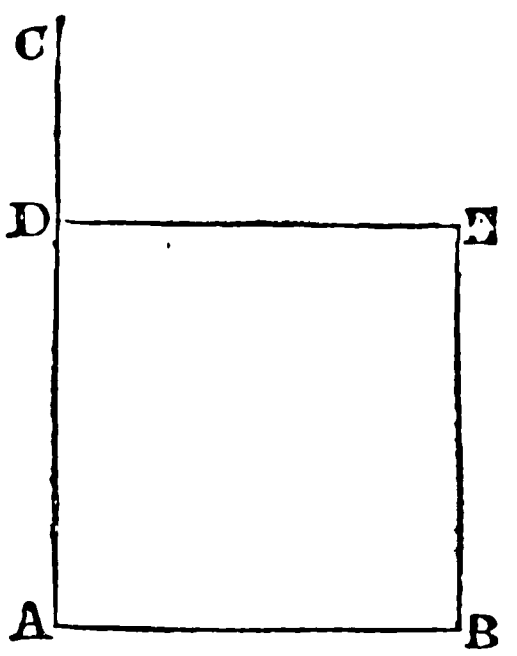
CONSTRUCTION.—From the point A draw AC at right angles to AB (I. 11),

And make AD equal to AB (I. 3).

Through the point D draw DE parallel to AB (I. 31).

Through the point B draw BE parallel to AD (I. 31).

Then ADEB *shall be the square required.*



ADEB a parallelogram.

PROOF.—Because DE is parallel to AB, and BE parallel to AD (Const.), therefore ADEB is a parallelogram;

Therefore AB is equal to DE, and AD to BE (I. 34).

But AB is equal to AD (Const.);

Therefore the four straight lines BA, AD, DE, EB are equal to one another (Ax. 1),

And the parallelogram ADEB is therefore equilateral. It is equilateral.

Likewise all its angles are right angles.

For since the straight line AD meets the parallels AB, DE, the angles BAD, ADE are together equal to two right angles (I. 29).

But BAD is a right angle (Const.);

Therefore also ADE is a right angle (Ax. 3).

But the opposite angles of parallelograms are equal (I. 34);

Therefore each of the opposite angles ABE, BED is a right angle (Ax. 1);

Therefore the figure ADEB is rectangular; and it has been proved to be equilateral; therefore it is a square (Def. 30). It is rectangular.

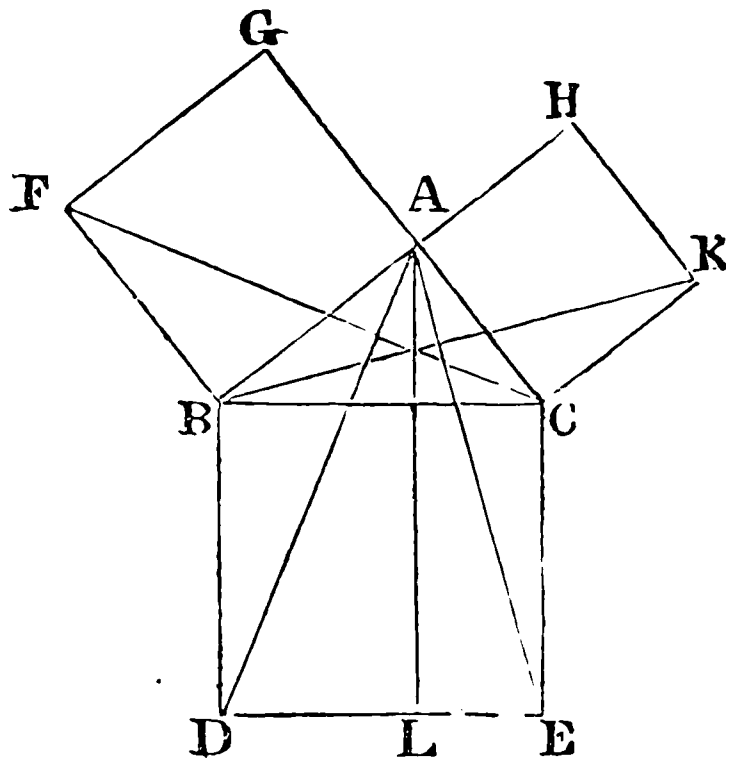
Therefore, the figure ADEB is a square, and it is described upon the given straight line AB. *Q. E. F.* ∴a square.

COROLLARY.—Hence every parallelogram that has one right angle has all its angles right angles.

Proposition 47.—Theorem.

In any right-angled triangle, the square which is described upon the side opposite to the right angle is equal to the squares described upon the sides which contain the right angle.

Let ABC be a right-angled triangle, having the right angle BAC;



The square described upon the side BC shall be equal to the squares described upon BA, AC.

CONSTRUCTION. — On BC describe the square BDEC (I. 46).

On BA, AC describe the squares GB, HC (I. 46).

Through A draw AL parallel to BD or CE (I. 31).

Join AD, FC.

PROOF.—Because the angle BAC is a right angle (Hyp.), and that the angle BAG is also a right angle (Def. 30),

The two straight lines AC, AG, upon opposite sides of AB, make with it at the point A the adjacent angles equal to two right angles;

CAG is a straight line.

Therefore CA is in the same straight line with AG (I. 14).

BAH is a straight line.

For the same reason, AB and AH are in the same straight line.

Now the angle DBC is equal to the angle FBA, for each of them is a right angle (Ax. 11); add to each the angle ABC.

Therefore the whole angle DBA is equal to the whole angle FBC (Ax. 2).

And because the two sides AB, BD are equal to the two sides FB, BC, each to each (Def. 30), and the angle DBA equal to the angle FBC;

△ ABD = △ FBC.

Therefore the base AD is equal to the base FC, and the triangle ABD to the triangle FBC (I. 4).

Now the parallelogram BL is double of the triangle ABD, because they are on the same base BD, and between the same parallels BD, AL (I. 41).

And the square GB is double of the triangle FBC, because they are on the same base FB, and between the same parallels FB, GC (I. 41).

Hence parallelogram BL = square GB, and parallelogram CL = square HC.

But the doubles of equals are equal (Ax. 6), therefore the parallelogram BL is equal to the square GB.

In the same manner, by joining AE, BK, it can be shown that the parallelogram CL is equal to the square HC.

Therefore the whole square BDEC is equal to the two squares GB, HC (Ax. 2);

And the square BDEC is described on the straight line BC, and the squares GB, HC upon BA, AC.

$\therefore BC^2 = BA^2 + AC^2$

Therefore the square described upon the side BC is equal to the squares described upon the sides BA, AC.

Therefore, in any right-angled triangle, &c. *Q. E. D.*

Proposition 48.—Theorem.

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.

Let the square described upon BC, one of the sides of the triangle ABC, be equal to the squares described upon the other sides BA, AC;

The angle BAC shall be a right angle.

CONSTRUCTION.—From the point A draw AD at right angles to AC (I. 11).

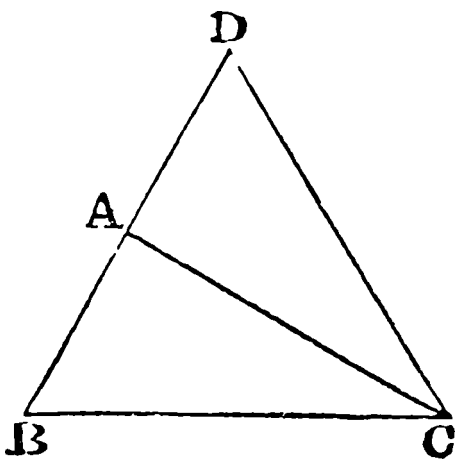
Draw AD at right angles to AC. *(Do not produce BA.)*

Make AD equal to BA (I. 3), and join DC.

PROOF.—Because DA is equal to AB, the square on DA is equal to the square on BA.

To each of these add the square on AC.

Therefore the squares on DA, AC are equal to the squares on BA, AC (Ax. 2).



But because the angle DAC is a right angle (Const.), the square on DC is equal to the squares on DA, AC (I. 47),

And the square on BC is equal to the squares on BA, AC (Hyp.);

Therefore the square on DC is equal to the square on BC (Ax. 1);

Then $DC^2 = BC^2$, and $DC = BC$.

And therefore the side DC is equal to the side BC.

And because the side DA is equal to AB (Const.), and AC common to the two triangles DAC, BAC, the two sides DA, AC are equal to the two sides BA, AC, each to each.

And the base DC has been proved equal to the base BC;

Hence ∠DAC = ∠BAC.

Therefore the angle DAC is equal to BAC (I. 8).
But DAC is a right angle (Const.);
Therefore also BAC is a right angle (Ax. 1).
Therefore, if the square, &c. *Q. E. D.*

EXERCISES.

Prop. 1—15.

1. From the greater of two given straight lines to cut off a portion which is three times as long as the less.

2. The line bisecting the vertical angle of an isosceles triangle also bisects the base.

3. Prove Euc. I. 5, by the method of *super-position.*

4. In the figure to Euc. I. 5, show that the line joining A with the point of intersection of BG and FC, makes equal angles with AB and AC.

5. ABC is an isosceles triangle, whose base is BC, and AD is perpendicular to BC; every point in AD is equally distant from B and C.

6. Show that the sum of the sum and difference of two given straight lines is twice the greater, and that the difference of the sum and difference is twice the less.

7. Prove the same property with regard to angles.

8. Make an angle which shall be three-fourths of a right angle.

9. If, with the extremities of a given line as centres, circles be drawn intersecting in two points, the line joining the points of intersection will be perpendicular to the given line, and will also bisect it.

10. Find a point which is at a given distance from a given point and from a given line.

11. Show that the sum of the angles round a given point are together equal to four right angles.

12. If the exterior angle of a triangle and its adjacent interior angle be bisected, the bisecting lines will be at right angles.

13. If three points, A, B, C, be taken not in the same straight line, and AB and AC be joined and bisected by perpendiculars which meet in D, show that DA, DB, DC are equal to each other.

Prop. 16—32.

14. The perpendiculars from the angular points upon the opposite sides of a triangle meet in a point.

15. To construct an isosceles triangle on a given base, the sides being each of them double the given base.

16. Describe an isosceles triangle having a given base, and whose vertical angle is half a right angle.

17. AB is a straight line, C and D are points on the same side of it; find a point E in AB such that the sum of CE and ED shall be a minimum.

18. Having given two sides of a triangle and an angle, construct the triangle. Examine the cases when there will be (1.) one solution; (2.) two solutions; (3.) none.

19. Given an angle of a triangle and the sum and difference of the two sides including the angle, to construct the triangle.

20. Show that each of the angles of an equilateral triangle is two-thirds of a right angle, and hence show how to trisect a right angle.

21. If two angles of a triangle be bisected by lines drawn from the angular points to a given point within, then the line bisecting the third angle will pass through the same point.

22. The difference of any two sides of a triangle is less than the third side.

23. If the angles at the base of a right-angled isosceles triangle be bisected, the bisecting line includes an angle which is three halves of a right angle.

24. The sum of the lines drawn from any point within a polygon to the angular points is greater than half the sum of the sides of the polygon.

PROP. 33—48.

25. Show that the diagonals of a square bisect each other at right angles, and that the square described upon a semi-diagonal is half the given square.

26. Divide a given line into any number of equal parts, and hence show how to divide a line similarly to a given line.

27. If D and E be respectively the middle points of the sides BC and AC of the triangle ABC, and AD and BE be joined, and intersect in G, show that GD and GE are respectively one-third of AD and BE.

28. The lines drawn to the bisections of the sides of a triangle from the opposite angles meet in a point.

29. Describe a square which is five times a given square.

30. Show that a square, hexagon, and dodecagon will fill up the space round a point.

31. Divide a square into three equal areas, by lines drawn parallel to one of the diagonals.

32. Upon a given straight line construct a regular octagon.

33. Divide a given triangle into equal triangles by lines drawn from one of the angles.

34. If any two angles of a quadrilateral are together equal to two right angles, show that the sum of the other two is two right angles.

35. The area of a trapezium having two parallel sides is equal to half the rectangle contained by the perpendicular distance between the parallel sides of the trapezium, and the sum of the parallel sides.

36. The area of any trapezium is half the rectangle contained by one of the diagonals of the trapezium, and the sum of the perpendiculars let fall upon it from the opposite angles.

37. If the middle points of the sides of a triangle be joined, the lines form a triangle whose area is one-fourth that of the given triangle.

38. If the sides of a triangle be such that they are respectively the sum of two given lines, the difference of the same two lines, and twice the side of a square equal to the rectangle contained by these lines, the triangle shall be right-angled, having the right angle opposite to the first-named side.

39. If a point be taken within a triangle such that the lengths of the perpendiculars upon the sides are equal, show that the area of the rectangle contained by one of the perpendiculars and the perimeter of the triangle is double the area of the triangle.

40. In the last problem, if O be the given point, and OD, OE, OF the respective perpendiculars upon the sides BC, AC, and AB, show that the sum of the squares upon AD, OB, and DC exceeds the sum of the squares upon AF, BD, and CD by three times the square upon either of the perpendiculars.

41. Having given the lengths of the segments AF, BD, CE, in Problem 40, construct the triangle.

42. Draw a line, the square upon which shall be seven times the square upon a given line.

43. Draw a line, the square upon which shall be equal to the sum or difference of two given squares.

44. Reduce a given polygon to an equivalent triangle.

45. Divide a triangle into equal areas by drawing a line from a given point in a side.

46. Do the same with a given parallelogram.

47. If in the fig., Euc. I. 47, the square on the hypothenuse be on the other side, show how the other two squares may be made to cover exactly the square on the hypothenuse.

48. The area of a quadrilateral whose diagonals are at right angles is half the rectangle contained by the diagonals.

SECTION III.

ALGEBRA.

CHAPTER I.

ELEMENTARY PRINCIPLES.

1. **Algebra** treats of numbers, the numbers being represented by letters (symbols of *quantity*), affected with certain symbols of *quality*, and connected by symbols of *operation*.

It is easy to see that these symbols of quantity may be dealt with very much as we deal with concrete quantities in arithmetic. Thus, allowing the letter a to stand for the number of units in any quantity, and allowing also $2\,a$, $3\,a$, $4\,a$, &c., to stand respectively for *twice*, *thrice*, *four times*, &c., as large a quantity as the letter a, it at once follows that we may perform the operations of addition, subtraction, multiplication, and division upon these symbols exactly as we do in ordinary arithmetic upon concrete quantities. For instance, $4\,a$ and $6\,a$ make $10\,a$, $9\,a$ exceeds $5\,a$ by $4\,a$, $15\,a$ is 5 times $3\,a$, and $7\,a$ is contained 8 times in $56\,a$.

Neither is it necessary in these operations to *state*, or even to *know* the exact number of units for which any symbol of quantity stands, nor indeed the nature of these units; it is simply sufficient that it is a symbol of *quantity*. Thus, in the science of chemistry, we use a weight called a *crith*; and a person unacquainted with chemistry might not know whether a crith were a measure of length, weight, or capacity, or indeed whether it were a measure at all, yet he would at once allow that 6 criths and 5 criths are 11 criths, that twice 4 criths are 8 criths, &c.

The Signs + and − as Symbols of Operation.

2. In purely arithmetical operations, the signs + and − are respectively the signs of addition and subtraction. In this sense, too, they are used in algebra.

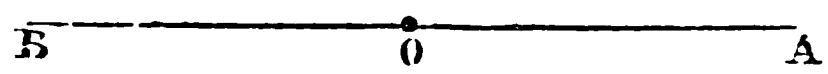
Thus, $a + b$ means that b is to be *added* to a, and $a - b$ means that b is to be *subtracted* from a.

Hence, as long as a and b represent ordinary arithmetical numbers, $a + b$ admits of easy interpretation, as also does $a - b$, when b is not greater than a. But when b is greater than a, the expression $a - b$ has no arithmetical meaning. By an extension, however, of the use of the signs + and −, we are able to give such expressions an intelligible signification, whatever may be the quantities represented by a and b.

Positive and Negative Quantities.—The Signs + and − as Symbols of Affection or Quality.

3. DEF.—A **positive** quantity is one which is affected with a + sign, and a **negative** quantity is one which is affected with a − sign.

Let BA be a straight line, and O a point in the line; and suppose a person, starting from O, to walk a miles in the direction OA. Suppose also another person, starting from the same or any other point in BA to walk a miles in the direction OB. These persons will thus walk a miles each in *exactly opposite* directions. Now, we call one of these directions *positive* (it matters not which) and the other *negative*. Let us take the direction OA as *positive*. We then have the first person walking a miles in a *positive* direction, and the second walking a miles in a *negative* direction. We represent these distances algebraically by $+a$ and $-a$ respectively.

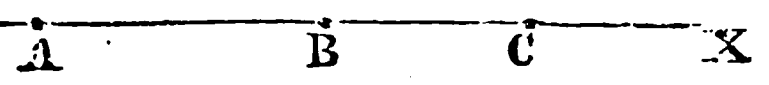


It will therefore be seen that the signs + and − have no effect upon the magnitudes of quantities, but that they express the *quality* or *affection* of the quantities before which they stand.

Again, suppose a person in business to get a profit of £6, while another suffers a loss of £6. We may express these facts algebraically in two ways. We may consider *gain* as positive, and *loss* as negative gain, and say that the former has gained + 6 pounds, while the latter has gained − 6 pounds. Or we may consider *loss* as positive, and *gain* as negative loss, and say that the former has lost − 6 pounds, while the latter has lost + 6 pounds. We hence see that the gain of + 6 is equal to a loss of − 6, and that a gain of − 6 is equal to a loss of + 6.

The Sum of Algebraical Quantities.

4. Let a distance AB be measured to the *right* along the line AX. And let a further distance BC be measured from B in the *same* direction. By the *sum* of these lines we mean the resulting distance of the point C from the original point A, that is to say, the distance AC.



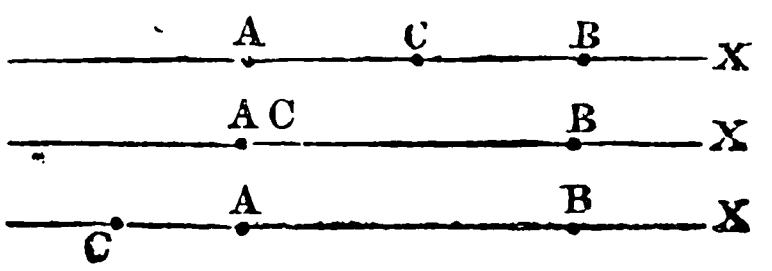
(It may be remarked that we *add* the line BC to the line AB by measuring BC in its *own proper direction* from the extremity B of AB. It is hardly necessary to remind the student that both lines are in the same straight line AX.)

Let us represent the distances AB and BC by + *a* and + *b* respectively; then the *algebraical sum* of the lines will be represented writing these quantities side by side, each with its *own proper sign* of affection.

Thus the sum of the distances AB and BC is expressed by + *a* + *b*, or, as it is usual to omit the + sign of a positive quantity when the quantity stands alone or at the head of an algebraical expression, the sum of AB and BC is expressed by *a* + *b*.

Hence, the interpretation of *a* + *b* is that it represents the distance AC.

Again, taking as above + *a* to represent the distance AB along the straight line AX, and measured to the *right*, let a distance BC be measured from B in the same straight line AX, but this time to the *left*.



Let the latter distance be represented by $-b$.

Then, on the principle above, AC is the *sum* of these distances, and this *sum* is represented algebraically by $+a-b$ or $a-b$.

(It will be seen that the distance BC is again measured from B in its *own proper direction*, and that the resultant distance AC of the point C is again the *sum* of the line AB and BC.)

There will evidently be three cases, viz. :—

1. When the distance BC is less in magnitude than AB, in which case the point C is on the right of A, and the distance AC is *positive*.

2. When the distance BC is *equal* in magnitude to AB, in which case the point C coincides with A, and the distance AC is *zero*.

3. When the distance BC is *greater* in magnitude than AB, the point C being then on the left of A, and the distance is *negative*.

Now, $a-b$ in all these cases represents the distance AC. It therefore admits of intelligible interpretation whether b be less than, equal to, or greater than a.

And, since the distance AC is obtained in the first two cases by subtracting the distance BC from that of AB, and in the second case by subtracting as far as AB will allow of subtraction, and measuring the remainder to be subtracted in an *opposite* direction, it follows that—

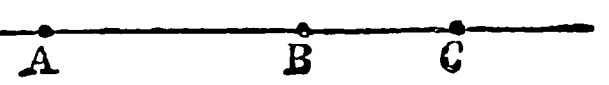
The sign $-$, which, standing before a letter, is a symbol of quality, becomes at once a symbol of subtraction in all cases when the quantity in question is placed immediately after any other given quantity with its proper sign of affection.

Hence also we may conclude that the addition of a *negative* quantity is equivalent to the subtraction of the corresponding *positive* quantity.

5. We may prove in a similar way that—

The subtraction of a *negative* quantity is equivalent to the addition of the corresponding *positive* quantity.

Let, as before, $+a$ represent the distance AB, measured from the point A to the *right*, and let it be required to *subtract* from $+a$ the distance represented by $-b$.



Now, in the last article, we *added* a distance to a given distance AB, by measuring the second distance in its *own direction* from the extremity B. We shall therefore be consistent if we *subtract* a given distance $-b$ by measuring this distance in a direction *exactly opposite* to its own direction, from the same extremity B.

Now, the direction of $-b$ is to the *left*. If, therefore, we measure a distance BC to the *right*, equal in magnitude to the distance to be subtracted, we obtain a distance AC which is correctly represented by $a-(-b)$. But AC is also correctly represented by $a+b$, and hence it follows that $a-(-b)=a+b$.

We may apply the above principle to all magnitudes which admit of continuous and indefinite extension; as, for instance, to forces which pull and push, attract and repel; to time past and time to come, to temperatures above zero and below zero, to money due and money owed, to distance up and distance down, &c., in all which cases, having represented one by a quantity affected with a + sign, we may represent the other by a quantity affected with a − sign.

6. *In expressing the sum of a number of quantities, the order of the terms is immaterial.*

We will take, as our illustration, a body subject to various alterations of temperature, and we will suppose the temperature of the body, before the changes in question, to be *zero* or 0°. Let the temperature now undergo the following changes—viz., a rise of $a°$, a fall of $b°$, a fall of $c°$, and, lastly, a rise of $d°$. Let us consider a *rise* as positive, and therefore a *fall* as negative. We may then represent these changes respectively by $+a$, $-b$, $-c$, $+d$.

And it is further evident that the resulting temperature will be represented by the *sum* of these quantities, which, as previously written, will be $a-b-c+d$.

But again, it is plain that the resulting temperature of the body will not be affected if these changes of temperature take place in the reverse order, or in any other order. Thus, suppose the temperature first falls $c°$, then rises $a°$, then rises $d°$, and, lastly, falls $b°$, it is evident that the final temperature will be the same as before. And the *sum* of the quantities $-c$, $+a$, $+d$, $-b$, represents this final temperature.

Now, expressing the sum of these quantities by writing them (Art. 4) side by side with their proper signs of affection, and in the order in which they stand, we have $-c+a+d-b$ for the *sum*.

It therefore follows, since we might have chosen any other order of these terms with a similar result, that the sum of any number of quantities, $+a$, $-b$, $-c$, $+d$, may be expressed by writing the terms side by side with their proper signs of affection *in any order whatever*.

Nevertheless, for convenience, and for other reasons, we write the terms *generally* in *alphabetical order*, or we arrange them according to the *power* (see Art. 16) of some particular letter.

7. We may sum up the results and remarks of the last four articles as follows :—

1. *Positive* and *negative* are used in exactly opposite senses.

2. The sign + before an algebraical quantity *affirms* the quality of the quantity as represented without the sign in question.

Thus, $+(+a) = +a$, and $+(-b) = -b$.

3. The sign − before an algebraical quantity *reverses* the quality of the quantity as represented without the sign in question.

Thus, $-(+a) = -a$, and $-(-b) = +b$.

4. The algebraical subtraction of a quantity is the same as the addition of the quantity with the sign of affection *reversed*.

5. The algebraical addition of quantities is expressed by writing the quantities down side by side with their proper signs of affection. And they *may* be written down in any order, though we generally write them in *alphabetical* order, or arranged according to the power (Art. 16) of some letter.

Brackets.

8. **Brackets**—( ), { }, [ ]—are used, for the most part, whenever we wish to consider an algebraical expression containing more than one term *as a whole*.

Thus, if we wish to express that the quantity $3a + 7b$ is to be added as a whole to $4a$, we write—

$$4a + (3a + 7b),$$

and, while inclosed within brackets, we think and speak of $3a + 7b$ as *one quantity*.

Again, if we wish to express that $b - c$ is to be subtracted from a, we write—

$$a - (b - c).$$

Let us consider what is the result of subtracting $(b - c)$ from a. We may evidently, if we please, subtract b first, then afterwards $- c$ from the quantity so obtained, without affecting this result.

Now, we know by Art. 7 (4.) that this is equivalent to *adding* the quantities $- b$ and $+ c$ successively.

Now, the sum of a, $- b$, $+ c$, is $a - b + c$.

We have therefore $a - (b - c) = a - b + c$.

We observe that the sign of b *within* the brackets is $+$, and that of c is $-$, whereas, in our final result, these signs are both reversed. And we hence arrive at the following important principle:—

When a *minus* sign stands before a bracket, its effect on removing the brackets is to *reverse* the sign of affection of every term *within*.

And it is evident that we may, by a similar course of reasoning, arrive at a principle equally important, viz.:—

When a *plus* sign stands before a bracket, its effect on removing the bracket is to *affirm* the sign of affection of every term within.

We shall show, in Art. 9, the use of brackets in expressing the product or quotient of quantities.

Though, as stated above, brackets are, for the most part, used to group together as a whole a number of quantities, they are sometimes used to inclose single terms. Thus, in Art. 5, we have the expression $a - (-b)$. Now, the brackets are used here to express that the *negative quantity* is to be subtracted as a *negative quantity*. And, in the same way, the expression $a + (-b)$ indicates that the *negative quantity* $(-b)$ is to be *added* to the quantity a. When one pair only is required we generally use the brackets ( ); if,

however, a quantity already in brackets is to be inclosed in a second pair, we use { }, as in the expression—

$$3\,a - \left\{6\,b + (4\,c - d)\right\}.$$

If a third pair be required we use the brackets [ ], and finally, we sometimes find it convenient to group a number of terms by means of a *vinculum*, thus—

$$4\,x - \left[6\,x - \left\{5\,y + (3\,z - \overline{7\,x - y})\right\} + 2\,y\right].$$

It must be remembered that the *vinculum* has in an expression exactly the same force as brackets.

9. We shall, in this Article, show how to find the value of a few algebraical expressions, as illustrations of the foregoing principles :—

Ex. 1.—If $a = 1$, $b = 2$, $c = 4$, find the value of $3\,a + 5\,b + 7\,c$.

We have only to substitute the value of the letters in the given expression, putting a sign of multiplication to avoid ambiguity.

Thus we have—

$$3\,a + 5\,b + 7\,c = 3 \times 1 + 5 \times 2 + 7 \times 4.$$
$$= 3 + 10 + 28 = 41.$$

Ex. 2.—If $x = 5$, $y = 2$, $z = 6$, find the value of $4\,x + 3\,y - 9\,z$.

Here, $4\,x + 3\,y - 9\,z = 4 \times 5 + 3 \times 2 - 9 \times 6$
$$= 20 + 6 - 54$$
$$= 26 - 54.$$

The negative quantity is here the larger, and exceeds the positive quantity by 28, and hence (Art. 4 (3),) the result will be *negative.*

We therefore have—

$$4\,x + 3\,y - 9\,z = -28.$$

Ex. 3.—If $x = 1$, $y = 3$, $z = 0$, find the value of—

$$3\,x - \left\{2\,y + (6\,z - \overline{5\,x - y})\right\}.$$

The given expression—

$$= 3 \times 1 - \left\{2 \times 3 + (6 \times 0 - \overline{5 \times 1 - 3})\right\}$$
$$= 3 - \Big\} \quad 6 + (0 - \overline{5 - 3})\Big\}$$

$$= 3 - \{ 6 + (0 - 2) \}.$$
$$= 3 - (6 - 2) = 3 - 4 = -1.$$

It may be advisable to simplify expressions of this kind before substituting the given value of the letters. The method of doing this, however, is deferred till we come to Chapter II.

Ex. I.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, $e = 4$, find the value of the following :—

1. $4a + 2b$, $3b + 7c$, $6a + 4d$, $4c - 7e$.
2. $a + b + c$, $a - b + c$, $b + c - a$, $a + b - c$.
3. $3a + 7b - 4c$, $2a + 7d + 3c$, $7a - 10b + 2c$.
4. $6a - (2b - 3c)$, $7b + (2a - 4d)$.
5. $3c + (6a - \overline{7b + c})$, $2b - \{ 7c + (4d - b) \}$.
6. $\{ 7b - (2d - c) \} - (4b - c + e)$.

If $x = 2$, $y = 3$, $z = 4$, find the sum and difference of the following expressions :—

7. $3x - 4y$ and $3y - 4x$.
8. $7(x - y)$ and $4(y - z)$.
9. $3x - (7y + 4z)$ and $8y + 5z - 3x$.
10. $12y - \overline{6y + 4z}$ and $12y + \overline{6y + 4z}$.
11. $x - y - z$ and $x - (y - z)$.
12. $x - (-3y)$ and $y - \{ 6x - (-3z) \}$.

Product of Two or more Quantities.

10. The product of two or more quantities may be expressed in several ways.

Thus, the product of a, b, c may be written as follows :—

1. abc, by placing the letters side by side without *any sign* between them.
2. $a \times b \times c$, by placing between them the sign $\times$.
3. $a \,.\, b \,.\, c$, by placing a dot between them.

4. $(a)\,(b)\,(c)$, by inclosing the quantities in brackets, writing them without any sign between the brackets.

When, however, the quantities are negative, or either of them consists of more than one term, it is *best* to inclose such quantities in brackets, and, in most cases, it is *necessary* to do so.

Thus, the product of a, $-b$, c would not be correctly expressed by $a - bc$ (for this means, that the product of b and c is to be subtracted from a), but must be written $a\,(-b)\,c$. Again, the product of $2a + 3b$ and $a + 5b$ cannot be written $2a + 3ba + 5b$, as this expression means that three times the product of $b \times a$ is to be added to $2a$, and then $5b$ to the result. The product is correctly expressed thus—

$$(2a + 3b)\,(a + 5b).$$

(The student cannot be too strongly cautioned against leaving out brackets in cases of this kind.)

The Order of the Letters.

11. It is evident that a times $b = b$ times a; for, if we arrange a rows of b things so as to have a horizontal rows and b vertical columns, we may either consider the number of *rows* or the number of columns. In the former case we have a times b things, and, in the latter, b times a things.

We may therefore write the letters whose product we wish to express in any order.

Thus, the product of a, b, c may be written in either of the following ways :—

$$abc,\ acb,\ bac,\ cba,\ bca,\ cab.$$

For convenience, however, and for reasons which the student will see as he proceeds with the subject, we write them in the order of the alphabet, unless there happen to be special reasons to the contrary.

Rule of Signs in Multiplication.

12. It was shown (Art. 3) that a *minus* sign does not affect the magnitude of a quantity, but simply its *affection* or

quality. It therefore follows that the product of $+ a$ and $- b$ will be the same in magnitude as that of $+ a$ and $+ b$, or of $a \times b$—that is, will be equal in magnitude to ab. But it is evident that the quality of the product will be the same as that of $- b$, for it is a times *a negative* quantity.

We therefore have $+ a (- b) = - ab$.

So also, the product of $- a$ and $+ b$, will have the same magnitude as that of $+ a$ and $+ b$—that is, its magnitude will be ab. But since, to take a geometrical illustration, a line of length, b, drawn *negatively once, negatively twice*, &c., must give a negative result, the quality of the product in question must be negative.

We therefore have $(- a) (+ b) = - ab$.

Again, the product of $- a$ and $- b$ will be equal in magnitude to ab, and its quality will be evidently the *reverse* of the quality of the product of $- a$ and $+ b$, and the quality of the latter product is above shown to be *negative.* The product of $- a \times - b$ will be therefore positive.

Hence we have $(- a) (- b) = + ab$.

Collecting these results, and remembering that $(+ a) (+ b) = + ab$, we have—

$$\begin{aligned} (+ a) (+ b) &= + ab. \\ (+ a) (- b) &= - ab. \\ (- a) (+ b) &= - ab. \\ (- a) (- b) &= + ab. \end{aligned}$$

We have then the following rule of signs in multiplication :—

RULE.—Like signs give $+$, and unlike signs give $-$.

Quotient of Two Quantities.

13. The quotient of two quantities is expressed in either of the two following ways :—

1. By placing the divisor under the dividend, separated by a line.

Thus, the quotient of a by $b = \frac{a}{b}$.

2. By placing the divisor after the dividend with a sign $\div$ between them; thus, $a \div b$.

When either of the quantities is negative, it is better, if the second be used, to inclose the negative quantity, if the divisor, in brackets.

Thus, the quotient of a by $-b$ may be written $a \div -b$, but it is better written $a \div (-b)$.

And when either of the quantities contains more than one term, it must always be inclosed in brackets when expressed by the second method.

Thus, the quotient of $a + 2b$ by $2c = (a + 2b) \div 2c$, not $a + 2b \div 2c$, for this would mean that the quotient of $2b$ by $2c$ is to be added to a.

Rule of Signs in Division.

It is evident from the last article that—

$$\begin{aligned}(+ab) \div (+a) &= +b.\\ (-ab) \div (+a) &= -b.\\ (-ab) \div (-a) &= +b.\\ (+ab) \div (-a) &= -b.\end{aligned}$$

We have therefore the following rule:—

RULE.—In division, as in multiplication, like signs give $+$, and unlike signs give $-$.

Coefficients.

14. When a quantity can be broken up into two factors, each of those factors is called a **coefficient** of the other.

Thus, taking the quantity $4abc$, we see that 4 is the coefficient of abc, $4b$ that of ac, $4c$ that of ab, $4bc$ that of a, &c. We call 4 a *numerical* coefficient; but when the coefficient contains a letter or letters we call it a *literal* coefficient. We often speak of the numerical coefficient of a quantity as *the coefficient* of the quantity. It is, moreover, unusual to write down unity as a numerical coefficient, and so, when an algebraical quantity has no expressed numerical coefficient, we may, conversely, consider *unity* as such.

15. *Like quantities* are generally defined as those which differ only in their *numerical* coefficients. Thus, $9\,x$, $3\,y$, $4\,xyz$, $12\,ab$ are respectively *like* to the quantities $10\,x$, $5\,y$, $6\,xyz$, $4\,ab$, while $3\,x^2$, $6\,xy$, $7\,b^2$ are called *unlike*.

It is sometimes convenient, however, to *consider as like quantities* those which may contain perhaps one *common* letter only, though containing other letters which are not common. Thus, for the purposes of addition and subtraction, we may consider $3\,ax$ and $4\,bx$ as *like quantities*, having $3\,a$ and $4\,b$ respectively as their coefficients. Now, their sum, since they are respectively equivalent to $3\,a$ and $4\,b$ times x, will be equivalent to $(3\,a + 4\,b)$ times x, and may then be written $(3\,a + 4\,b)\,x$.

Hence we learn that addition and subtraction of quantities having any common factor may, considering the uncommon factor as coefficient, be expressed according to the following rule :—

Add together the coefficients of the common factor, and take the sum as a new coefficient of the common factor.

And a similar rule will apply to the subtraction of such quantities.

Thus, we have the sum of $4\,axy$, $-\,2\,bcy$, $3\,dey$,

$$= (4\,ax - 2\,bc + 3\,de)\,y.$$

Powers.

16. We have seen (Art. 10) that abc means that a is to be multiplied by b, and the product by c. And so, if we wish to express that a is to be multiplied by a, and this product again by a, we might write the expression thus: aaa. Instead of this, however, we usually place a small figure at the top and on the right hand of the letter a, to indicate how many times the letter appears as a factor. In this case, therefore, we write a^3. We call quantities of this form **powers** of the letter in question; and so, remembering that a may be considered as a^1 on this principle, we have :—

a, the first power of a ; a^2, the *second* power of a ; a^3, the *third* power of a, &c. The figure written at the right hand at the top of the letter is called the **index** or **exponent**, and it is usual to call a^2, a^3 respectively the *square* and *cube* of a,

from the fact they express respectively the *area* of a square whose side is a, and the *volume* of cube whose edge is a.

17. The **square root** of a quantity is that quantity which, when raised to the *second power* or *squared*, will give the original quantity.

It is generally written $\sqrt{\ }$. Thus, $\sqrt{16} = 4$, $\sqrt{144} = 12$.

The **cube root** of a quantity is that quantity which, when raised to the *third power* or *cubed*, will give the original quantity.

It is generally written $\sqrt[3]{\ }$. Thus, $\sqrt[3]{8} = 2$, $\sqrt[3]{27} = 3$, $\sqrt[3]{1728} = 12$. And so the fourth, fifth, &c., roots are indicated by the symbols $\sqrt[4]{\ }$, $\sqrt[5]{\ }$, &c., respectively.

18. The **dimensions** of an algebraical quantity are the sum of the indices or exponents of the *literal* factors.

Thus, the dimensions of $3a^2b^3c^4 = 2 + 3 + 4 = 9$, and the dimensions of $abx^2 = 1 + 1 + 2 = 4$.

19. A **homogeneous** expression is one in which the dimensions of every term are the same.

Thus, $a^3 + 3a^2b + 3ab^2 + b^3$ is *homogeneous*,

Whereas, $5a + 3ab + 7a^2bc$ is *not homogeneous*.

Ex. II.

If $a = 2$, $b = 3$, $c = 0$, $d = 1$,

Find the values of—

1. $6a^2 + 3b^2 - 5c^2$; $ab + ac + bc$, $bc + bd + cd$.

2. $a^3 + 3a^2b + 3ab^2 + b^3$; $a^3 + b^3 + c^3 - 3abc$.

3. $(3a + 7b)(4a - 9b)$; $(a^2 + b^2)(a + b)(a - b)$.

4. $6\left\{2a^3 - 4\ (2b^3 - \overline{2c^3 - d^3})\right\}$; $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$.

If $a = 1$, $b = -2$, $c = -3$, $d = 0$, $e = 4$,

Find the values of—

5. $(a + b + c + d + e)^2$; $(a^2 + 2ab + b^2 - c^2) \div (a + b + c)$.

6. $\dfrac{a^4 - d^4}{a^3 + a^2d + ad^2 + d^3}$; $\dfrac{c^3 + 3c^2d + 3cd^2 + d^3}{b^3 - 3b^2c + 3bc^2 - c^3}$.

7. $\dfrac{\sqrt{c^2 + e^2} - \sqrt{5a^2 + b^2}}{ab + ac + bc}$; $\sqrt[3]{c^3 + 3c^2 + 3c + 1}$.

8. $3\,(4\,b + 5\,c)^2 + 4\,(c + e)^2$; *abcde.*

If $x = 3$, $y = 4$, $z = 0$,

Find the value of—

9. $(3\,x - \sqrt{x^2 + y^2})^2\,(2\,x + \sqrt[3]{x^2 + y^2 + 2})$.

10. $\{5\,x^2 + 2\,(y + 2)^2\}\ \{5\,x^2 - 2\,(y + z)^2\}$.

11. $x^y + y^x + z^x$.

12. $(x^3 - y^3) \div \sqrt[3]{\{3\,x^3 + 3\,(3\,x^2 + 3\,xy + y^2)\,y\}}$.

CHAPTER II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION.

Addition.

20. RULE.—Arrange the terms of the given quantities so that *like* quantities may be under each other; add separately the positive and negative coefficients of each column; take the difference and prefix the sign of the *greater*, and annex the common letter.

(When the coefficients are all positive or all negative, we, of course, simply add them together and prefix the common sign for the coefficient of the sum.)

Ex. 1.

$$\begin{array}{r} 3\,a + 5\,b - 3\,c \\ 2\,a - 7\,b + 4\,c \\ 5\,a + b - 2\,c \\ 4\,a - 3\,b + 8\,c \\ \hline \text{Ans. } 14\,a - 4\,b + 7\,c \end{array}$$

Ex. 2.

$$\begin{array}{r} -3\,a + 7\,b + c - 4\,d \\ 2\,a - 2\,b + 5\,c + 3\,d \\ -7\,a - 3\,b + 2\,c - 6\,d \\ 2\,a + 5\,b - 8\,c + 6\,d \\ \hline \text{Ans. } -6\,a + 7\,b - d \end{array}$$

Ex. 3. Add together $5\,x^2 - 3\,y^2 + 3\,y$, $6\,y^2 + 7\,xy - 4\,x$, $4\,xy + 6\,x - 5$, $-2\,x^2 - 3\,xy + 2$.

Arranging like quantities in each expression under each other, we have :—

$$\begin{array}{rrrrrr} 5\,x^2 & & -3\,y^2 & & +3\,y & \\ & 7\,xy & +6\,y^2 & -4\,x & & \\ & 4\,xy & & +6\,x & & -5 \\ -2\,x^2 & -3\,xy & & & & +2 \\ \hline \text{Ans. } 3\,x^2 & +8\,xy & +3\,y^2 & +2\,x & +3\,y & -3 \end{array}$$

Ex. III.

Add together-

1. $3a - 2b$, $4a + 7b$, $2a + 3b$, $a - 5b$.

2. $9a^2 + 7b^2$, $-3a^2 + 4b^2$, $a^2 + b^2$, $4a^2 - 12b^2$.

3. $a + b + c$, $3a + 2b + 3c$, $-4a + 7b - c$, $2b + 5c$.

4. $x - y - z$, $y - x - z$, $z - x - y$, $x + y + z$.

5. $3a^2 - 4ab + 6b^2$, $7ab - a^2 - b^2$, $2a^2 - 3ab - 4b^2$, $4a^2 + ab - b^2$.

6. $2x^4 - 7x^2 + 3$, $-4x^3 + 6x^2 - 2x + 7$, $x^4 - 2x^3 - 4x$, $6x^3 - 9x - 12$.

7. $2a^2 + 7ab + 3b^2 - 6a - 5b - 2$, $a^2 + 3a - 2b + 9$, $9ab - 2a - 3b + 4$, $-3a^2 - 12ab - 3b^2 + 5a + 10b - 15$.

8. $x^3 - x^2y - x^2z + xy^2 - xyz + xz^2$, $x^2y - xy^2 - xyz + y^3 - y^2z + yz^2$, $x^2z - xyz - xz^2 + y^2z - yz^2 + z^3$.

9. $x^4 + x^2y^2 + x^3y$, $-x^3y - x^2y^2 - xy^3$, $y^4 + xy^3 + x^2y^2$.

10. $a^3 + ab^2 + ac^2 + 2a^2b - 2a^2c - 2abc$, $a^2b + b^3 + bc^2 + 2ab^2 - 2abc - 2b^2c$, $a^2c + b^2c + c^3 + 2abc - 2ac^2 - 2bc^2$.

11. $x^4 - xy^3 + xz^3 - 3x^3y + 3x^3z$, $3x^2y^2 + 3x^2z^2 + 3xy^2z - 3xyz^2 - 6x^2yz$, $y^4 - x^3y - yz^3 + 3x^2y^2 - 3x^2yz$, $-3xy^3 - 3xyz^2 - 3y^3z + 3y^2z^2 + 6xy^2z$, $z^4 + x^3z - y^3z - 3x^2yz + 3x^2z^2$, $3xy^2z + 3xz^3 + 3y^2z^2 - 3yz^3 - 6xyz^2$.

12. $a^4 - a^3b + 3a^2c^2 + ab^2c - 3abc^2 - b^3c$, $a^3b - a^3c + 3abc^2 - 3ac^3 - b^2c^2 + b^3c$, $a^3c - a^3d + 3ac^3 - 3ac^2d + b^2c^2 - b^2cd$, $-a^4 + a^3d - 3a^2c^2 + 3ac^2d - ab^2c + b^2cd$.

Subtraction.

21. We have seen, Art. 7 (4.), that the subtraction of a quantity is equivalent to the addition of the same quantity with its sign of affection reversed. We therefore have the following rule :—

RULE.—Change the sign of each term of the subtrahend, and proceed as in addition.

Ex. 1.

$$\begin{array}{r} 5x + 6y - 3z \\ 2x - 2y + 2z \\ \hline \text{Ans. } 3x + 8y - 5z. \end{array}$$

Ex. 2.

$$\begin{array}{r} 6a^2 - 3ab + 4b^2 \\ -2a^2 - 3ab - 2b^2 \\ \hline \text{Ans. } 8a^2 \qquad + 6b^2. \end{array}$$

Ex. 3.

$$\begin{array}{r} 2\,x^3 \qquad\qquad + 6xy^2 + 3y^3 \\ x^3 + 3\,x^2y + 3\,xy^2 + 2\,y^3 \\ \hline x^3 - 3\,x^2y + 3\,xy^2 + \quad y^3. \end{array}$$

Ex. IV.

1. From $6\,a + 7\,b + 3\,c$ take $2\,a + 5\,b - 2\,c$.
2. From $2\,x - 3\,y - 8\,z$ take $6\,x - 5\,y - 2\,z$.
3. Take $5\,a^2 + 3\,ab + 4\,b^2 + 3\,a + 7\,b + 8$ from $6\,a^2 + 3\,b^2 - 2\,a$.
4. Take $6\,a^4 + 8\,a^2x^2 + x^4$ from $8\,a^4 + 6\,a^2x^2 + 2\,x^4$.
5. Subtract the sum of the quantities $a^4 + 2\,a^2b^2 + b^4$, $a^4 - 2\,a^2b^2 + b^4$ from $6\,a^4 + 8\,a^2b^2 + 6\,b^4$.
6. From $x^3 + y^3 + z^3 - 3\,xyz$ take $4\,x^3 + y^3 + 4\,z^3 + 3\,x^2z + 3\,xz^2 - 3\,xyz$.
7. From $3\,x^4 + 3\,ax^3 - 9\,a^2x^2 + a^3x - a^4$ take $2\,x^4 + 4\,ax^3 + 4\,a^3x + a^4$.
8. Take $a^3 - 5\,a^2b + 7\,ab^2 - 2\,b^3$ from the sum of $2\,a^3 - 9\,a^2b + 11\,ab^2 - 3\,b^3$ and $b^3 - 4\,ab^2 + 4\,a^2b - a^3$.
9. Subtract $a + b + c + d$ from $e + f + g + h$.
10. Take $x^4 - 4\,x^3y + 6\,x^2y^2 - 4\,xy^3 + y^4$ from $x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + y^4$, and subtract the result from their sum.
11. Add together the given quantities in the last example, and subtract the result from $3\,x^4 + 10\,x^2y^2 + 3\,y^4$.
12. Take $a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc$ from $2\,a^2 + 2\,b^2 + 4\,ab - c^2$.

Brackets—*continued.*

22. It was shown, in Art. 8, that, when a quantity inclosed in brackets is to be added, we may remove the sign (+) of addition and the brackets without changing the sign of the terms within the brackets. On the contrary, when the quantity in brackets is to be subtracted, or has the sign *minus* before it, we must change the sign of every term within the brackets on removing the brackets and the sign of subtraction.

We shall now see how to simplify expressions involving brackets connected by the signs of addition and subtraction:—

Ex. 1. Simplify $(3a + 5b) - (6b - 2c) + (-2a + b - 3c)$.

The given expression $= 3a + 5b - 6b + 2c - 2a + b - 3c$, or adding together the like quantities,

$$= a - c.$$

Ex. 2. Reduce to its simplest form—

$$a - (b - c) + \left\{b + (a - c)\right\} - \left\{(a - b) - c\right\}.$$

(When a pair of brackets is inclosed within another pair, it is convenient to remove the inner one first.)

Hence the given expression—

$$= a - b + c + \left\{b + a - c\right\} - \left\{a - b - c\right\}$$
$$= a - b + c + b + a - c - a + b + c = a + b + c.$$

Ex. 3. Simplify the expression $\frac{3x}{4} - \frac{x - 7}{2} + \frac{6x - 9}{8}$.

The line separating the numerator and denominator of a fraction is a species of vinculum, since it serves to show that the *whole* numerator is to be divided by the *whole* denominator. Hence, on breaking up the two latter fractions into fractions having one term only in the numerator, we have—

$$\frac{3x}{4} - \frac{x - 7}{2} + \frac{6x - 9}{8} = \frac{3}{4}x - \frac{1}{2}x + \frac{7}{2} + \frac{6}{8}x - \frac{9}{8} = x + \frac{19}{8}.$$

23. As it is often necessary to inclose quantities within brackets, we shall now show how this is done.

The following rule needs no explanation :—

RULE.—When a number of terms is inclosed within brackets, if the sign placed before the brackets be +, the terms must be written down with their signs of affection *unchanged*; but, if the sign placed before the brackets be −, the sign of affection of every term placed within the brackets must be *changed*.

Thus we may express $a + b - c - d$ in any of the following ways :—

$$a + b - c - d = a + (b - c - d) = (a + b) - (c + d)$$
$$= a - (-b + c + d) = (a + b - d) - c, \text{ \&c.}$$

(When the word *sign* is used in future, the student is to understand *sign of affection*, unless otherwise expressed.)

Ex. V.

Simplify the expressions—

1. $3x - (2x - y) + (5x + 3y - 2z) - (7x - 4y - 3z)$.

2. $(2a^3 + 2a^2b + 2ab^2) - (2a^3 + a^2b + ab^2 - b^3) + (a^3 - a^2b - ab^2 - 2b^3)$.

3. $1 - (1 - 2x) - \left\{3 - (4 - 3x)\right\} + \left\{5 - (4x - \overline{7 - x})\right\}$.

4. $6x^2 - (2 - 3x + x^2) + \left\{-7 + (5x - \overline{8x - 2})\right\} - (3 - \overline{3 - 3x})$.

Group together the terms of the four following expressions, so that—

(i.) The first two and last two are inclosed in brackets.

(ii.) The last three are inclosed in brackets.

(iii.) The first three are so inclosed, and an inner pair of brackets used to inclose the second and third.

5. $a - b + c - d$.

6. $-6a + 7b - 3c + 5d$.

7. $-4x^3 + 12x^2y - 12xy^2 + 4y^3$.

8. $a^3 - b^3 - c^3 + 3abc$.

Add together—

9. $ax^2 + bxy + cy^2$, $-dx^2 - axy + ey^2$, $bx^2 + 3cxy + fy^2$, $-2ax^2 - 2by^2$.

10. $ax - cy - ez$, $-bx + dy + fz$, $cx - ey - gz$, $-dx + fy + hz$.

Subtract—

11. $(2a + b)x - (3b + c)y + (4c + d)z$ from $(3a - d)x + (4b - a)y + (5c - b)z$.

12. $(y - z)a^2 + (z - x)ab + (x - y)b^2$ from $(y - x)a^2 - (y - z)ab - (z - x)b^2$.

Multiplication.

24. Remembering the definition (Art. 14) of a coefficient, it follows that the product of two terms having coefficients is found by multiplying the product of the coefficients by the product of the remaining factors.

Thus, the product of $4a$ and $3b$, or $4a \times 3b = 12ab$.

Again, the terms to be multiplied may contain like letters.

Now, $a^3 = aaa$, $a^5 = aaaaa$, and hence it follows that

$$a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8.$$

And, generally, we have

$$a^m \times a^n = aaa\ldots\text{to } m \text{ factors} \times aaa\ldots\text{to } n \text{ factors},$$
$$= aaa\ldots\text{to } (m + n) \text{ factors} = a^{m+n}.$$

It therefore follows that the product of the *powers* of like letters is found by adding together the *exponents* of the like letters.

Thus, $a^4 \times a^2 = a^6$, and $a^3 \times a = a^4$, &c.

And, further, as regards the sign of the product, we have seen, in Art. 12, that *like signs* give +, and *unlike signs* give −.

There are three things, then, which must be attended to in the multiplication of algebraical terms, viz.:—

1. The *signs*.—Like signs give +, and unlike signs give −.

2. The *coefficients*.—These are to be multiplied like ordinary numbers.

3. The *letters*.—The exponents of like letters are to be added together, and the powers so obtained written side by side with the unlike letters.

Ex. 1.	Ex. 2.	Ex. 3.
$6\,a^2b$	$-\,3\,xyz$	$-\,5\,abd$
$2\,a^3c$	x^2yz^3	$-\,2\,bcd^2$
Ans. $12\,a^5bc$	Ans. $-\,3\,x^3y^2z^4$	Ans. $10\,ab^2cd^3$

25. Whenever the multiplicand or multiplier, or both, contains more than one term, it is evident the product is found by multiplying each term of the multiplicand separately by each term of the multiplier, and adding together the separate products.

Ex. 4.	Ex. 5.
$3\,a^2 - 7\,ab + 5\,b^2$	$x^3 - 3\,x^2y + 3\,xy^2 - y^3$
$6\,ab$	$-\,2\,xy$
Ans. $18\,a^3b - 42\,a^2b^2 + 30\,ab^3$	$-\,2\,x^4y + 6\,x^3y^2 - 6\,x^2y^3 + 2\,xy^4$

Ex. 6.

$$\begin{array}{r}
x^2 + xy + y^2 \\
x^2 - xy + y^2 \\
\hline
x^4 + x^3y + x^2y^2 \\
\quad - x^3y - x^2y^2 - xy^3 \\
\qquad\qquad x^2y^2 + xy^3 + y^4 \\
\hline
\text{Ans. } x^4 \qquad + x^2y^2 \qquad + y^4.
\end{array}$$

Ex. 7.

$$\begin{array}{r}
x^2 + (a + b)\,x + ab \\
x + c \\
\hline
x^3 + (a + b)\,x^2 + abx \\
\qquad cx^2 + (ac + bc)\,x + abc \\
\hline
x^3 + (a + b + c)\,x^2 + (ab + ac + bc)\,x + abc.
\end{array}$$

It will be observed that the terms in the above examples are all arranged according to the power of some letter. Thus, in Ex. 4, they are arranged according to the descending powers of a; and in Exs. 5, 6, 7, they are arranged according to the descending powers of x. It matters not whether they are arranged according to ascending or descending powers, and the result would be the same if they are not so arranged. It is then, however, much easier to collect like terms, as they generally fall under each other. When we come to division we shall find it *necessary* to arrange the terms according to the power of some letter.

Ex. VI.

1. Multiply $3a + 2b$ by $4a - 3b$, and $6x + 7y$ by $3x - 5y$.

2. Multiply $x^2 + 2xy - 2y^2$ by $x - 2y$, and $15x^2 + 17xy - 4y^2$ by $2x + y$.

3. Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$, and $a^2 + b^2$ by $a^2 - b^2$.

4. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$, and the product by $x^4 + y^4$.

5. Multiply $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

6. Find the continued product of $x^4 + y^4$, $x^2 + y^2$, $x + y$, $x - y$.

7. Develop the expressions $(a + b)^3$, and $(x - y)^4$.

8. Multiply $5a^3 + 15a^2 + 45a + 135$ by $a^2 - 2a - 3$.

9. Multiply $1 + 3x - 7x^2$ by $x - 2$, and $a - x$ by $x^2 + ax + a^2$.

10. Multiply $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$ by $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4$.

11. Multiply $a + bx + cx^2 + dx^3$ by $cx + f$.

12. Find the product of $x^2 + px + q$ and $x - a$.

13. Find the continued product of $x + a$, $x + b$, $x + c$.

26. We have explained in the last Article the general method to be pursued in Multiplication. There are, however, many algebraical expressions which may be multiplied by inspection.

I. Expressions of the form $(a + b)^2$.

It is easily found by long multiplication that

$$(a + b)^2 = a^2 + 2ab + b^2,$$

and this, expressed in ordinary language, may be read thus :—

The square of the sum of two quantities is the square of the first, plus twice their product, plus the square of the second.

(This rule is evidently true whether the quantities are *positive* or *negative*.)

Ex. 1. $(3a + 7b)^2 = (3a)^2 + 2(3a)(7b) + (7b)^2 = 9a^2 + 42ab + 49b^2$.

Ex. 2. $(6x - 5)^2 = (6x)^2 + 2(6x)(-5) + (-5)^2 = 36x^2 - 60x + 25$.

Ex. 3. $$\begin{aligned}(a + b + c)^2 &= \{(a + b) + c\}^2 \\ &= (a + b)^2 + 2(a + b)c + c^2 \\ &= (a^2 + 2ab + b^2) + 2(ac + bc) + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

REMARK.—It is a very common mistake with beginners to write down $a^2 + b^2$ as the square of $(a + b)$, thereby leaving out *twice the product of the quantities a and b*. They should impress this formula, viz. :—

$$(a + b)^2 = a^2 + 2ab + b^2$$

thoroughly upon the mind.

II. The form $(a + b)(a - b)$.

It may be easily found by long multiplication that

$$(a + b)(a - b) = a^2 - b^2,$$

which, in ordinary language, may be thus expressed :—

The product of the sum and difference of two quantities is the difference of the squares of the quantities.

This formula may be applied in all cases where the terms of the quantities to be multiplied are of the same magnitude, but when *some* of them differ in sign.

Ex. 1. $(a + b + c + d)(a + b - c - d)$

$= \{(a+b)+(c+d)\}\{(a + b) - (c + d)\}$

$= (a + b)^2 - (c + d)^2 = (a^2 + 2ab + b^2) - (c^2 + 2cd + d^2)$

$= a^2 + b^2 - c^2 - d^2 + 2ab - 2cd.$

Ex. 2. $(a^3 - 3a^2b + 3ab^2 - b^3)(a^3 + 3a^2b + 3ab^2 + b^3)$

$= \{(a^3 + 3ab^2) - (3a^2b + b^3)\}\{(a^3 + 3ab^2) + (3a^2b + b^3)\}$

$= (a^3 + 3ab^2)^2 - (3a^2b + b^3)^2$

$= (a^6 + 6a^4b^2 + 9a^2b^4) - (9a^4b^2 + 6a^2b^4 + b^6)$

$= a^6 - 3a^4b^2 + 3a^2b^4 - b^6.$

The principle to be adopted in all such cases is to find what terms in the given quantities are *exactly alike*, and put them *first* in brackets, when the remaining terms will fall into the proper form. Thus—

$(3a + 7b + 5c - d)(3a - 7b + 5c + d) = \{(3a + 5c) + (7b - d)\}\{(3a + 5c) - (7b - d)\}.$

And $(3a - 7b - 5c + d)(3a + 7b + 5c + d) = \{(3a + d) - (7b + 5c)\}\{(3a + d) + (7b + 5c)\}.$

(III.) The form $(x + a)(x + b)$.

By multiplication we find that—

$$(x + a)(x + b) = x^2 + (a + b)x + ab,$$

which in ordinary language may be thus expressed :—

The product of two binomials containing a common term and an uncommon term, is the square of the common term, plus the sum of the uncommon terms multiplied into the common term, plus the product of the uncommon term.

Thus—

$(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \times 3 = x^2 + 5x + 6.$

$(x + 6)(x - 1) = x^2 + (6 - 1)x + 6(-1) = x^2 + 5x - 6.$

$(x - 5)(x - 7) = x^2 - (5 + 7)x + (-5)(-7) = x^2 - 12x + 35.$

$(x + 3a)(x - 5a) = x^2 + (3a - 5a)x + (3a)(-5a) = x^2 - 2ax - 15a^2.$

And so,

$$(x + a + b)(x + c + d) = (x + \overline{a + b})(x + \overline{c + d}) = x^2 + (a + b + c + d)x + (a + b)(c + d).$$

We may extend this formula to any number of factors.

Thus, by multiplication, we find that—

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc.$

$(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2 + (abc + abd + acd + bcd)x + abcd.$

Law of Formation of the Terms.

1. It will be seen that the coefficient of the first term is in each case *unity*; that of the second term, the sum of the uncommon letters taken *singly*; that of the third term, the sum of the uncommon letters taken *two together*; that of the fourth term, the sum of the uncommon letters taken *three together*, &c.

2. The power of the common letter is in the first term that of the number of the binomials, and it sinks *one* every term.

Ex. $(x + 1)(x + 2)(x + 3) = x^3 + (1 + 2 + 3)x^2 + (1 + 1 \times 2 + 1 \times 3 + 2 \times 3)x + 1 \times 2 \times 3 = x^3 + 6x^2 + 11x + 6.$

(IV.) The form $(a + b + c + d + \&c.)^2$.

By ordinary multiplication or otherwise we have $(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$; and a similar result follows if we take a larger number of terms.

We may hence deduce the following rule:—

The square of the sum of any number of quantities is the sum of the squares of the quantities together with twice the sum of the products formed by multiplying the first quantity

into all that follow separately, then the second into all that follow, &c.

Ex. 1. $(1 + 2x + 3x^2)^2 = 1 + 4x^2 + 9x^4$
$\quad + 4x + 6x^2 + 12x^3$
$= 1 + 4x + 10x^2 + 12x^3 + 9x^4.$

Ex. 2. $(a^3 + 3a^2b + 3ab^2 + b^3)^2$
$= a^6 + 9a^4b^2 + 9a^2b^4$
$\quad + 6a^5b + 6a^4b^2 + 2a^3b^3 + b^6$
$\quad + 18a^3b^3 + 6a^2b^4 + 6ab^5$
$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$

Ex. VII.

1. Find by inspection the squares of $x - y$, $3a - 5b$, $4c^2 + d^2$, $3x^2 - 2y^2$.

2. Find the continued product of $a + b$, $a^2 + b^2$, $a^4 + b^4$, and $a - b$.

3. Multiply $mx + ny$ by $mx - ny$, and $5a^2 - 3b^2$ by $10a^2 + 6b^2$.

4. Find the value of $(a + b + c + d)(a - b + c - d)$, and of $(a + b - c - d)(a + b + c + d)$.

5. Show that $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2) = x^4 - 2x^2y^2 + y^4$.

6. Multiply $x + 5$ separately by $x - 1$, $x + 2$, $x - 3$, $x - 5$.

7. What is the continued product of $x - 2$, $x + 3$, $x - 6$, $x + 5$?

8. If $2s = a + b + c$, find the value of $s(s - a)(s - b)(s - c)$.

9. If $2s = a + b + c + d$, what is the value of—
$(2s - 2a)^2 + (2s - 2b)^2 + (2s - 2c)^2 + (2s - 2d)^2$?

Prove that—

10. $(ax + by)^2 + (cx + dy)^2 + (ay - bx)^2 + (cy - dx)^2 = (a^2 + b^2 + c^2 + d^2)(x^2 + y^2)$.

11. $\{(ac - bd)x + (ad + bc)y\}^2 + \{(ac - bd)y - (ad + bc)x\}^2 = (a^2 + b^2)(c^2 + d^2)(x^2 + y^2)$.

12. $(ax + by + cz)^2 = (a + b + c)(ax^2 + by^2 + cz^2) - ab(x - y)^2 - ac(x - z)^2 - bc(y - z)^2$.

13. $(x + a)(x + b)(x + c) = (x - a)(x - b)(x - c) + 2\{(a + b + c)x^2 + abc\}$.

14. $(m + n + p + q)^2 = (m + n)^2 + (m + p)^2 + (m + q)^2 + (n + p)^2 + (n + q)^2 + (p + q)^2 - 2(m^2 + n^2 + p^2 + q^2)$.

15. $(a^2 + b^2 + c^2)(m^2 + n^2 + p^2 + q^2) = (am + bn + cp)^2 + (an - bm + cq)^2 + (ap - bq - cm)^2 + (aq + bp - cn)^2$.

16. $(a - b)(b - c) + (b - c)(c - a) + (c - a)(a - b) = 3(ab + ac + bc) - (a + b + c)^2$.

17. $(x + y + z + a)^2 - (x - y - z + a)^2 = 4(x + a)(y + z)$.

18. $\{(a - b)^2 + 4ab\}\{(a + b)^2 - 4ab\}\{a^4 - b^4 + 2ab(a^2 - b^2)\} = (a + b)^5(a - b)^3$.

19. $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$.

20. $4(a^2 + b^2)(c^2 + d^2) = (a + b)^2(c + d)^2 + (a - b)^2(c - d)^2 + (a + b)^2(c - d)^2 + (a - b)^2(c + d)^2$.

21. $8(a^4 + b^4) = (a + b)^4 + 6(a^2 - b^2)^2 + (a - b)^4$.

22. $(x - y)^2 + (y - z)^2 + (z - x)^2 + 2(x - y)(y - z) + 2(x - y)(z - x) + 2(y - z)(z - x) = 0$.

23. $(a + b + c)(b + c - a)(a + c - b)(a + b - c) = 4a^2b^2$, when $a^2 + b^2 = c^2$.

24. $2\{(ax^m + by^n)^2 + (ay^n - bx^m)^2\} = (a^2 + b^2)\{(x^m + y^n)^2 + (x^m - y^n)^2\}$.

Division.

27. As in multiplication, we have especially three things to attend to, viz. :—

1. The *signs*.

We have learnt (Art. 13) that like signs give +, and unlike signs give −.

2. The *coefficients*.

Understanding here the *numerical* coefficients, it is plain that they may be divided as ordinary arithmetical quantities.

3. The *letters*.

As the product of the quantities a and b is expressed by ab, it follows that the quotient of ab by either of the factors a or b will give the remaining factor b or a respectively.

Thus, $ab \div a$ or $\frac{ab}{a} = b$, and $ab \div b$ or $\frac{ab}{b} = a$.

And so, $xyz \div xz = y$, and $pqrs \div qs = pr$.

We may then conclude that *when the divisor is contained as a factor in the dividend, the quotient is found by omitting from the dividend those of its factors which constitute the divisor.*

If the divisor be *not* contained as an exact factor in the dividend, we may then express the quotient symbolically.

Thus, $xy \div ab = \frac{xy}{ab}$.

When, however, the dividend and divisor have a common factor, it is plain that we may, as in arithmetic, strike out of the numerator and denominator of the symbolical quotient this common factor.

Thus, $5\ abc \div bd = \frac{5\ abc}{bd} = \frac{5\ ac}{d}$.

And $16\ xyz \div 10\ axz = \frac{16\ xyz}{10\ axz} = \frac{8\ y}{5\ a}$.

28. A power of a quantity is divided by any other power of the same quantity by subtracting the index of the divisor from that of the dividend, the quotient being that power of the quantity whose index is the remainder so obtained.

1. Let the power of the quantity in the dividend be the *higher.*

We have $a^5 = aaaaa$, and $a^3 = aaa$.

$$\therefore a^5 \div a^3 = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Or, generally, m being greater than n, since

$a^m = aaa\ldots$to m factors, and $a^n = aaa\ldots$to n factors,

we have—

$$a^m \div a^n = \frac{aaa\ldots\text{to } m \text{ factors}}{aaa\ldots\text{to } n \text{ factors}} = aaa\ldots(m - n) \text{ factors} = a^{m-n}.$$

2. Let the power of the quantity in the dividend be the *lower.*

Suppose we have to divide a^4 by a^7.

We have $a^4 \div a^7 = \dfrac{aaaa}{aaaaaaa} = \dfrac{1}{aaa} = \dfrac{1}{a^3}$.

We may, however, so express the result that it shall agree exactly with the proposition at the head of this article.

For we may conceive of a^3 as representing the *product* of *unity* and the quantity a^3. We shall therefore be perfectly consistent if we allow a^3 to represent the *quotient* of *unity* by the quantity a^3.

We shall then have $a^{-3} = \dfrac{1}{a^3}$, and hence we get from the above result—

$$a^4 \div a^7 = a^{-3} = a^{4-7}.$$

Or, generally, m being less than n, we have—

$$\frac{a^m}{a^n} = \frac{aaa.....\text{to } m \text{ factors}}{aaa.....\text{to } n \text{ factors}} = \frac{1}{aaa.....(n - m) \text{ factors}}$$

$= \dfrac{1}{a^{n-m}}$; or, using the notation just explained,

$= a^{-(n-m)} = a^{-n+m} = a^{m-n}$.

3. Let the powers of the quantities in the dividend and divisor be *equal*.

It is evident that their quotient is *unity*.

Thus, $\dfrac{a^4}{a^4} = \dfrac{aaaa}{aaaa} = \dfrac{1}{1} = 1$.

And so, $\dfrac{a^m}{a^m} = 1$.

If, however, we *assume* the principle proved in the two cases above to hold here, we have—

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

It follows therefore that $a^0 = 1$.

Cor.—From the above interpretation of negative indices it follows that the same rules for multiplication and division of quantities involving them may be applied as in the case of positive indices.

Thus, $a^5 \times a^{-2} = a^5 \times \dfrac{1}{a^2} = \dfrac{a^5}{a^2} = a^{5-2}$.

And so, $a^4 \div a^{-3} = a^{4-(-3)} = a^7$.

Ex. 1. $\dfrac{20\,x^2yz}{4\,xz} = 5\,xy.$

Ex. 2. $\dfrac{-\,45\,a^3b^2c^4}{15\,ab^2c^2} = -\,3\,a^2c^2.$

Ex. 3. $\dfrac{-\,7\,a^3 + 21\,a^2b - 14\,ab^2}{-\,7\,a} = a^2 - 3\,ab + 2\,b^2.$

Ex. 4. $\dfrac{4\,x^5yz^3 - 6\,x^2y^3z^4 + 42\,xy^4z^4}{3\,x^2y^4z^5}$

$$= \frac{4\,x^3}{3\,y^3z^2} - \frac{2}{yz} + \frac{14}{xz}, \text{ or}$$

$$= \frac{4}{3}x^3y^{-3}z^{-2} - 2\,y^{-1}z^{-1} + 14\,x^{-1}z^{-1}.$$

Ex. 5. Divide $3\,a^2 + 13\,ab - 10\,b^2$ by $a + 5\,b$.

When the divisor, as in this example, contains more than one term, it is generally convenient to follow the method of arithmetical long division. Thus—

$$\begin{array}{l} a + 5\,b)\ 3\,a^2 + 13\,ab - 10\,b^2 (3\,a - 2\,b \\ \quad\ \underline{3\,a^2 + 15\,ab} \\ \qquad\quad -\ 2\,ab - 10\,b^2 \\ \qquad\quad \underline{-\ 2\,ab - 10\,b^2} \end{array}$$

Ex. 6. Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.

$$\begin{array}{l} a^2 + ab + b^2)\ a^4 + a^2b^2 + b^4\ (a^2 - ab + b^2 \\ \qquad\quad \underline{a^4 + a^3b + a^2b^2} \\ \qquad\quad -\ a^3b + b^4 \\ \qquad\quad -\ a^3b - a^2b^2 - ab^3 \\ \qquad\quad\ \ \underline{a^2b^2 + ab^3 + b^4} \\ \qquad\quad\ \ a^2b^2 + ab^3 + b^4 \end{array}$$

It will be seen that in the last two examples care has been taken to keep the terms of the divisor, dividend, and successive remainders arranged according to the ascending or descending powers of some letter. In these cases we have arranged the terms according to the descending powers of a, and, as therefore follows, according to the ascending powers of b. Want of care in this respect will often render the operation of finding the true quotient *tiresome*, if not impossible. The next two examples will illustrate this point. They may be

attempted first by the student, keeping the terms in the order as given.

Ex. 7. Divide $2 - 7x - 15x^2$ by $5x - 1$.

$$
\begin{array}{l}
-1+5x)\,2-7x-15x^2(-2-3x \\
\qquad\quad 2-10x \\
\hline
\qquad\qquad 3x-15x^2 \\
\qquad\qquad 3x-15x^2 \\
\hline
\end{array}
$$

Or thus,

$$
\begin{array}{l}
5x-1)\,-15x^2-7x+2(-3x-2 \\
\qquad -15x^2+3x \\
\hline
\qquad\qquad -10x+2 \\
\qquad\qquad -10x+2 \\
\hline
\end{array}
$$

Ex. 8. Divide $x^3 + y^3$ by $x^{-1} + y^{-1}$.

Here we have the powers of x in the dividend descending, while in the divisor they are ascending. Arranging them in the divisor as in the dividend, the operation is easy. Thus—

$$
\begin{array}{l}
y^{-1}+x^{-1})\,x^3+y^3\ (x^3y-x^2y^2+xy^3 \\
\qquad x^3+x^2y \\
\hline
\qquad\quad -x^2y+y^3 \\
\qquad\quad -x^2y-xy^2 \\
\hline
\qquad\qquad\quad xy^2+y^3 \\
\qquad\qquad\quad xy^2+y^3 \\
\hline
\end{array}
$$

We shall work the next example in two ways to illustrate, firstly, the above point again ; and, secondly, to show how the operation may be sometimes abbreviated by the use of brackets.

Ex. 9. Divide $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

$$
\begin{array}{l}
x+y+z)\,x^3-3xyz+y^3+z^3(x^2-xy-xz+y^2-yz+z^2 \\
\qquad x^3+x^2y+x^2z \\
\hline
\qquad\quad -x^2y-x^2z-3xyz \\
\qquad\quad -x^2y-xy^2-xyz \\
\hline
\qquad\qquad\quad -x^2z+xy^2-2xyz \\
\qquad\qquad\quad -x^2z\qquad\quad -xyz-xz^2 \\
\hline
\qquad\qquad\qquad\quad xy^2-xyz+xz^2+y^3 \\
\qquad\qquad\qquad\quad xy^2\qquad\qquad\quad +y^3+y^2z \\
\hline
\qquad\qquad\qquad\qquad -xyz+xz^2-y^2z \\
\qquad\qquad\qquad\qquad -xyz\qquad\quad -y^2z-yz^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad xz^2+yz^2+z^3 \\
\qquad\qquad\qquad\qquad\qquad\quad xz^2+yz^2+z^3 \\
\hline
\end{array}
$$

Or thus, inclosing the last two terms of the divisor in brackets—

$$
\begin{array}{l}
x + (y+z) \;\big)\, x^3 - 3xyz + y^3 + z^3 \big(x^2 - (y+z)x + (y^2 - yz + z^2) \\
\qquad\qquad x^3 + (y+z)\,x^2 \\
\qquad\qquad \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\quad -(y+z)\,x^2 - 3xyz \\
\qquad\qquad\quad -(y+z)\,x^2 - (y^2 + 2yz + z^2)\,x \\
\qquad\qquad\quad \overline{\qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad (y^2 - yz + z^2)\,x + y^3 + z^3 \\
\qquad\qquad\qquad\qquad \underline{(y^2 - yz + z^2)\,x + y^3 + z^3}
\end{array}
$$

It will be seen in both the above operations that we have brought down the terms of the dividend only when the subtrahends indicated they were required. This often prevents much useless repetition.

Ex. VIII.

Find the quotient of—

1. $28\,a^2b - 7\,ab^2 + 14\,b^3$ by $7\,b$; $3\,x^2y^2 - 12\,xy^3$ by $3\,xy$.

2. $-6\,a^5b + 15\,a^4b^2 - 20\,a^3b^3$ by $-3\,ab$; $4\,x^3y^2 + 6\,x^2y^3 + 4\,xy^4$ by $2\,xy^2$.

3. $ax^{2m} + bx^my^n + cy^{2n}$ by x^{m+n}; $ax^{2m-2n} + bx^{m-n}y^n + cy^{2n}$ by x^{m-n}.

4. $30\,x^2 + 2\,xy - 12\,y^2$ by $5\,x - 3\,y$; and by $6\,x + 4\,y$.

5. $1 + 2\,x + 3\,x^2 + 2\,x^3 + x^4$ by $1 + x + x^2$.

6. $12 - 19\,x - 21\,x^2$ by $7\,x - 3$; and by $3\,x + 4$.

7. $x^4 - 4\,x^2y^2 + 12\,xy^3 - 9\,y^4$ by $x - 3\,y$; and by $x + y$.

8. $x^5 - y^5$ by $x - y$; and $x^5 + y^5$ by $x + y$.

9. $acx^{m+n} + adx^my^m + bcx^ny^n + bdy^{m+n}$ by $cx^n + dy^m$.

10. $a^6 - a^4b^2 - a^2b^4 + b^6$ by $a^3 - a^2b - ab^2 + b^3$.

11. $b^2 + ab + bc + ac$ by $a + b$; and $a^2 + ab + bc + ac$ by $a + c$.

12. $a + (a+b)\,x + (a+b+c)\,x^2 + (a+b+c)\,x^3 + (b+c)\,x^4 + cx^5$ by $1 + x + x^2 + x^3$.

13. $a^5 - pa^4 + qa^3 - qa^2 + pa - 1$ by $a - 1$.

14. $a^4 + b^4 + c^4 + d^4 - 2\,(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - 8\,abcd$ by $a - b + c + d$; and that of this quotient by $a - b - c - d$.

15. Show that the remainder, after the division of $x^4 - px^3 + qx^2 - rx + s$ by $x - a$, is $a^4 - pa^3 + qa^2 - ra + s$.

16. Divide $x^4 - y^4$ by $x^{-2} - y^{-2}$, and $x^2 + x^{-2} + 2$ by $x + x^{-1}$.

17. Show that the quotient of 1 by $1 + x$, is $1 - x + x^2 - x^3 +$ &c. *ad infinitum*.

18. Show that the quotient of 1 by $1 - 3x + 3x^2 - x^3$ is $1 + 3x + 6x^2 + 10x^3 + 15x^4 +$ &c. *ad infinitum*.

19. Divide $(x + y)^4 - 2\,(x + y)^2 z^2 + z^4$ by $x + y - z$.

20. Divide $(a - c)^3 - 3\,(a - c)^2\,(b - d) + 3\,(a - c)\,(b - d)^2 - (b - d)^3$ by $a - b - c + d$.

Factors.

29. The *ordinary* method of finding the quotient of two algebraical quantities having been explained in the last article, we shall now proceed to show how, in certain cases, this method may be avoided, and the quotient written down at sight. It may be remarked at the outset, that the resolution of algebraical expressions into their elementary factors is a subject of very great importance, and one which the student will do well to thoroughly master.

(I.) The form $x^2 + 2\,ax + a^2$.

We have seen (Art. 26) that $x^2 + 2\,ax + a^2 = (x + a)^2$. Hence the *sum of the squares* of two quantities, together with twice the product of the quantities, is equal to the *square of the sum* of the quantities.

And hence, any algebraical expression, which can be thrown into the form $(x^2 + 2\,ax + a^2)$, is of necessity a perfect square.

Thus—

$x^2 + 6x + 9 = x^2 + 2\,(3)\,x + 3^2 = (x + 3)^2$.

$a^2 - 10\,ab + 25\,b^2 = a^2 + 2\,a\,(-5\,b) + (-5\,b)^2 = (a - 5\,b)^2$.

$16\,a^2x^2 - 56\,abxy + 49\,b^2y^2 = (4\,ax)^2 + 2\,(4\,ax)\,(-7\,by) + (-7\,by)^2 = (4\,ax - 7\,by)^2$.

$3\,ab^2x^2 - 6\,abcxy + 3\,ac^2y^2 = 3\,a\,(b^2x^2 - 2\,bcxy + c^2y^2) = 3\,a\{(bx)^2 - 2\,(bx)\,(cy) + (cy)^2\} = 3\,a\,(bx - cy)^2.$

(II.) The form $a^2 - b^2$.

We have seen (Art. 26, II.) that $a^2 - b^2 = (a + b)\,(a - b)$.

Hence the *difference of the squares* of two quantities is equal to the product of the sum and difference of the quantities. Thus—

$a^2x^2 - b^2y^2 = (ax)^2 - (by)^2 = (ax + by)\,(ax - by).$

$x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)\,(x^2 - y^2)$
$= (x^2 + y^2)\,(x + y)\,(x - y).$

$a^2 + b^2 - c^2 - d^2 + 2\,ab - 2\,cd$
$= (a^2 + 2\,ab + b^2) - (c^2 + 2\,cd + d^2)$
$= (a + b)^2 - (c + d)^2$
$= \{(a + b) + (c + d)\}\,\{(a + b) - (c + d)\}$
$= (a + b + c + d)\,(a + b - c - d).$

$x^4 + x^2y^2 + y^4 = (x^4 + 2\,x^2y^2 + y^4) - x^2y^2$
$= (x^2 + y^2)^2 - x^2y^2$
$= \{(x^2 + y^2) + xy\}\,\{(x^2 + y^2) - xy\}$
$= (x^2 + xy + y^2)\,(x^2 - xy + y^2).$

(III.) The form $x^2 + px + q$.

This form evidently includes both the preceding, for the first form—viz., $x^2 + 2\,ax + a^2$ is included, since q may be the square of half p; and the second is included—viz., $x^2 - a^2$, since we may have $p = o$, and q a negative square quantity.

Now, the resolution into elementary rational factors of the quantity $x^2 + px + q$ is not always possible; but, since (Art. 26, III.),

$$x^2 + (a + b)\,x + ab = (x + a)\,(x + b),$$

we have the following rule, when the quantity admits of resolution.

RULE.—If the third term q of the quantity $x^2 + px + q$ can be broken up into two factors, a and b, such that the sum of these will give the coefficient of x, then the elementary factors of $x^2 + px + q$ are $x + a$ and $x + b$.

Thus, $x^2 + 7\,x + 12 = (x + 3)\,(x + 4)$; for the *product* of 3 and 4 is 12, and their *sum* is 7, the coefficient of x.

$x^2 - x - 30 = (x - 6)(x + 5)$; for the *product* of -6 and 5 is -30, and their *sum* is -1.

And so, $x^2 - 18x + 32 = (x - 2)(x - 16)$,

And $a^2 + 3ab - 108b^2 = a^2 + (12b - 9b)a + (12b)(-9b) = (a + 12b)(a - 9b)$.

(IV.) The form $ax^2 + bx + c$.

This is the *general* form of a trinomial. The following remarks, though equally applying to each of the three preceding forms, are especially intended to be *practically* applied to trinomials not included by them.

The above form will include such expressions as the following:—$20x^2 + 11x - 42$, $6x^2 - 37x + 55$.

It is evident that the product of the first terms of the factors will be the first term of the given trinomial, and that the product of the last terms of the factors will be the third term of the given trinomial.

And, further, when the third term is *negative*, the last term of one factor must have the sign $+$, and the last term of the other the sign $-$; but, when the third term is *positive*, the last terms of the factors must have the same sign as the middle term.

Thus, $12x^2 - 31x - 30 = (4x + 3)(3x - 10)$.

Here the factors of $12x^2$ are either $3x$ and $4x$, $6x$ and $2x$, $12x$ and x, and the factors of 30 either 5 and 6, 3 and 10, 2 and 15, 1 and 30; and we must give a $+$ sign to one of each of these latter pairs, and a $-$ sign to the other. It is easily found on trial that, in order to obtain $-31x$ as the middle term, the factors of the trinomial must be $4x + 3$ and $3x - 10$.

So we have $10a^2 - 41ab + 21b^2 = (5a - 3b)(2a - 7b)$, and $acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$.

(V.) The forms $x^n + y^n$ and $x^n - y^n$.

We shall show in the next article that a rational integral algebraical expression, involving x, contains $x - a$ as a factor when it vanishes on substituting a for x.

Hence, $x^n + y^n$ and $x^n - y^n$ must each vanish on putting y for x, if they contain $x - y$ as a factor, n being an integer.

The former becomes $y^n + y^n$ or $2y^n$, and the latter $y^n - y^n$ or 0. We therefore conclude that—

$x^n + y^n$ does not contain $x - y$ as a factor, and that—

$x^n - y^n$ does contain $x - y$ as a factor, whether n be *even* or *odd*.

Again, on the same principle, they must each vanish if they contain $x + y$ as a factor, on putting $-y$ for x.

The former becomes $(-y)^n + y^n$, which vanishes when n is *odd*, and the latter becomes $(-y)^n - y^n$, which vanishes when n is *even*.

Hence we conclude that—

$x^n + y^n$ contains $x + y$ as a factor when n is *odd*, and

$x^n - y^n$ contains $x + y$ as a factor when n is *even*.

Now, the quotient of either of these quantities by $x + y$ or $x - y$ can in any particular case be found by long division.

We thus find that—

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4),$$
$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3),$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

The law of formation of the co-factor in each case is easy to see; and if we may *assume* this apparent law as generally true, we may conclude that, when an algebraical quantity is of the form $x^n + y^n$ or $x^n - y^n$, and it contains $x + y$ or $x - y$ as a factor, the law of formation of the co-factor is as follows:—

Law of Formation of Co-Factor.

1. The terms are homogeneous, and of dimensions one degree lower than the given expression, the power of x in the first term being $n - 1$, and diminishing each successive term by *unity;* and the power of y increasing each successive term by *unity*, and first appearing in the second term.

2. The coefficient of every term is *unity*.

3. The *signs* are alternately + and − , when $x + y$ is the corresponding elementary factor; and are all +, when $x - y$ is the corresponding elementary factor.

Ex. 1. $a^5 + 32 = a^5 + 2^5 = (a + 2)(a^4 + a^3 \cdot 2 + a^2 \cdot 2^2 + a \cdot 2^3 + 2^4) = (a + 2)(a^4 + 2a^3 + 4a^2 + 8a + 16)$.

Ex. 2. $a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 + b^3)(a^3 - b^3) = (a + b)(a^2 - ab + b^2) \cdot (a - b)(a^2 + ab + b^2 = (a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$.

30. *The remainder of the division of a rational integral function of* x *by* x − a *may be found by putting* a *for* x *in the given function.*

DEF.—A **function** of x is an algebraical expression involving x; and a rational integral function of x is an expression of the form $ax^n + bx^{n-1} + \&c. + sx + t$, where all the powers of x are integral and positive.

Let $f(x)$* be a rational integral function of x, and suppose Q to be the quotient, and R the remainder on dividing the function by $x - a$.

Then, evidently—

$$Q(x - a) + R = f(x) \text{ identically.}$$

And this identity must hold for all values of x, and therefore holds when $x = a$.

In this case we have $Q(a - a) + R = f(a)$

$$0 + R = f(a)$$

$$\text{or } R = f(a).$$

Now $f(a)$ is the result of putting a for x in the given function, and is, as we have just shown, the remainder on dividing the given function by $x - a$.

COR. 1. When there is no remainder, we must, of course, have $f(a) = 0$. Hence, a given rational integral function of x vanishes when a is put for x, if it be divisible by $x - a$.

Ex. 1. The remainder, after the division of $2x^3 - 5x^2 + 6x + 7$ by $x - 2$ is 15.

For, putting $x = 2$, we have—

$$2x^3 - 5x^2 + 6x + 7 = 2 \cdot 2^3 - 5 \cdot 2^2 + 6 \cdot 2 + 7 = 15.$$

Ex. 2. The function—

$x^3 - 2x^2 + 5x - 52$ is divisible by $x - 4$.

For, putting $x = 4$, we have—

$$x^3 - 2x^2 + 5x - 52 = 4^3 - 2 \cdot 4^2 + 5 \cdot 4 - 52 = 0.$$

COR. 2. Any rational integral function of x is divisible by $x - 1$, when the sum of the coefficients of the terms is *zero*.

For, putting $x = 1$ in the given function, it is evident that it is reduced to the sum of its coefficients, which sum must be zero if the function be divisible by $x - 1$.

Ex. Each of the following functions is divisible by $x - 1$, viz.:—

$3x^4 + 7x^3 - x^2 + 12x - 21$, $5x^7 - 2x - 3$,

$(a - b)x^2 + (b - c)x + (c - a)$, $(a + b)^2x^2 - 4abx - (a - b)^2$.

* The expression $f(x)$ must not be considered to mean the product of f and x, but as a *symbol* used for convenience.

COR. 3. Any rational integral function of x is divisible by $x + 1$, when the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers.

(The term independent of x is always to be considered as the coefficient of an *even* power).

Let $ax^n + ax^{n-1} + \&c. + rx^2 + sx + t$ be a rational integral function of x.

Put $x = -1$, then we have, if the function be divisible by $x + 1$—

$$a(-1)^n + b(-1)^{n-1} + \&c. + r(-1)^2 + s(-1) + t = 0.$$

Suppose n to be even, then evidently $(-1)^n = (-1)(-1)(-1)\ldots$to an *even* number of factors $= +1$.

And so $(-1)^{n-1} = (-1)(-1)(-1)\ldots$to an *odd* number of factors $= -1$; and so on.

Hence we get $a - b + \&c. + r - s + t = 0$, and this must evidently require the condition that the sum of the positive quantities is equal to the sum of the negative, and, therefore, that the sum of the coefficients of the *even* powers of x is equal to the sum of the coefficients of the *odd* powers. And a similar result will follow if we suppose n to be *odd*.

Ex. Each of the following functions is divisible by $x + 1$, viz.:—

$$x^3 + 5x^2 + 7x + 3,\ 5x^5 - 4x^4 + 8x^2 - 2x - 1,$$

$$a^2x^4 - (a + 1)(a + 2)x^2 + 2x + 3a + 4,\ px^3 + (q + r)x^2 + (q + r)x + p.$$

Ex. IX.

Resolve into elementary factors—

1. $x^2 - 9a^2,\ 16y^4 - 25z^4,\ 24a^2 - 54b^2,\ 8x^3 - 27y^3.$

2. $x^4 - xy^3,\ a^8 - b^8,\ xy^4 + x^4y,\ 2x^3y^2z - 8xy^2z^3.$

3. $a^4 - 4b^4,\ x^4 + x^2y^2 + y^4,\ a^4 - 2a^2b^2 + b^4,\ a^2 + b^2 - c^2 + 2ab.$

4. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd,\ a^2 - b^2 - c^2 + d^2 + 2bc + 2ad,\ a^2 - (b - c)^2.$

5. $(x + 7)^2 - (x + 2)^2,\ (x + 5)^2 - (x + 2)^2,\ (2a + b)^2 - (a - c)^2.$

6. $(x^3 - y^3)^2 + 4(x^4 + x^2y^2 + y^4)x^2y^2,\ (x^2 + y^2)^4 - 5(x^2 + y^2)^2x^2y^2 + 4x^4y^4.$

7. $x^2 - 3x - 70$, $x^2 + 11x + 10$, $a^2 - 15ab + 56b^2$, $x^2 - 4x - 192$.

8. $a^2x^2 + abxy - 42b^2y^2$, $3ax^2 - 24ax - 60a$, $24ac - 5ac^2 + ac^3$.

9. $6x^2 - 11x - 35$, $8x^2 + 6x - 135$, $18x^2 - 21x - 72$, $20x^2 - 11x - 42$.

10. $3x^3y + 10x^2y^2 + 3xy^3$, $20x^3 + 12ax^2 + 25bx^2 + 15abx$, $m^2x^2 + (mq + mp)x + pq$.

Write down the quotient of—

11. $x^4 - 16$ by $x - 2$, $3x^5 + 96$ by $x + 2$, $x^6 - 27$ by $x^2 - 3$.

12. $(a + b)^3 - (c + d)^3$ by $a + b + c + d$, $a^2 - b^2 + c^2 - d^2 + 2ac + 2bd$ by $a + b + c - d$.

Find the remainder after the division of—

13. $x^4 + px^3 + qx^2 + rx + s$ by $x - a$, $x^4 + a^4$ by $x^2 - a^2$.

14. $x^3 - 5x^2 + 7x - 9$ by $x + 3$, $x^4 - 3x + 7$ by $x - 2$.

Show that—

15. $5x^5 - 3x^3 + 7x^2 - 8x - 1$ is divisible by $x - 1$.

16. $2x^4 - 3x^3 + x^2 - 7x - 13$ is divisible by $x + 1$.

CHAPTER III.

INVOLUTION AND EVOLUTION.

Involution.

31. Involution is the operation by which we obtain the *powers* of quantities. This can of course be done by multiplication, but the results obtained by the actual multiplication of simple forms enable us to develop without multiplication more complex forms. As the subject requires the aid of the Binomial Theorem, we shall here show how to develop a few only of the more simple expressions.

32. The power of a single term is obtained by raising the

coefficient of the term to the power in question, and *multiplying* the exponents of the letters of the term by the exponent of the power in question.

Thus, $(a^3)^2 = a^3 \times a^3 = a^{3+3} = a^{3\times 2}$,

$(a^m)^n = a^m \times a^m \times a^m \ldots\ldots$ to n factors $= a^{m+m+m+\ldots \text{ to } n \text{ terms}}$ $= a^{mn}$.

And so, $(4\,a^2b^3c^4)^3 = 4^3\,a^{2\times 3}b^{3\times 3}c^{4\times 3} = 64\,a^6b^9c^{12}$.

33. Development of the *third*, *fourth*, and *fifth* powers of $a + b$.

We know (Art. 26, I.) that $(a + b)^2 = a^2 + 2ab + b^2$.

$\therefore (a + b)^3 = (a^2 + 2ab + b^2)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$;

also, $(a + b)^4 = (a^3 + 3ab^2 + ab^2 + b^3)(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$;

and $(a + b)^5 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4)(a + b)$
$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

The following law of the formation of the terms is evident:—

Law of Formation of Terms.

1. The first term contains a raised to the given power, and the power of a decreases by *unity* in each successive term, while the power of b (which first appears in the second term) increases by unity in each successive term, till it reaches the power of the given quantity.

2. The first coefficient is unity, and the coefficient of any term is found by multiplying the previous coefficient by the exponent of a in the previous term, and dividing the product by the number of terms hitherto developed.

Ex. 1. $(2x + 3y)^3 = (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 = 8x^3 + 36x^2y + 54xy^2 + 27y^3$.

Ex. 2. $(a + b + c)^3 = \overline{(a + b} + c)^3 = (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3 = (a^3 + 3a^2b + 3ab^2 + b^3) + 3(a^2 + 2ab + b^2)c + 3(a + b)c^2 + c^3 = a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3ab^2 + 3b^2c + 3ac^2 + 3bc^2 + 6abc$.

(In the following examples the above law may be *assumed* as *generally* true.)

Ex. X.

1. Find the values of $(a^2b^3)^2$, $(-3ab^3)^3$, $(2a^2bc)^4$, $(-x^3yz^2)^3$.

Expand—

2. $(a + 3b)^4$, $(2a + b)^4$, $(a - b)^5$, $(3a - 4b)^3$.

3. $(2m + 1)^4$, $(5x + 2)^3$, $(3a - 4c)^4$, $(-a - b)^3$.

4. $(x^2 + x + 1)^2$, $(3a - b + 4c - d)^2$, $(a + 2b - c)^2$ $(3a + 3b + 3c)^2$.

5. $(1 + x - x^2)^3$, $(ax + by + cz)^3$, $\{(a+b)x-(c+d)y\}^3$.

6. $(1 + x)^7$, $(1 + x + x^2)^4$, $(a + bx + cx^2)^4$.

7. $(ax - by)^8$, $(3x + y)^3 (3x - y)^3$, $(x^2 + xy + y^2)^3 (x - y)^3$.

8. $(x^4 + x^2y^2 + y^4)^2 (x + y)^2 (x - y)^2$,

$\{(a + b)^2 - 4ab\}^3 \{(a - b)^2 + 4ab\}^3$.

Simplify—

9. $(a + b + c)^3 - 3(a + b + c)^2 c + 3(a + b + c)c^2 - c^3$.

10. $(a - b)^3 + 3(a - b)^2 (b - c) + 3(a - b)(b - c)^2 + (b - c)^3$.

11. $(1 + x + 3x^2 + 3x^3)^3 + (1 - x + 3x^2 - 3x^3)^3$.

12. $\{(x + y)^3 - (x^3 + y^3)\}^3 - 27x^3y^3 (x^3 + y^3)$.

Evolution.

34. Evolution is the operation by which we obtain the *roots* of quantities.

Since the *square* or *second power* of a^3 is $(a^3)^2$ or a^6, we call (Art. 17) a^3 the *square root* or the *second root* of a^6.

And so, since the *cube* or *third power* of a^2 is $(a^2)^3$ or a^6, we call a^2 the *cube root* or the *third root* of a^6.

So, generally, since the nth power of $a^m = (a^m)^n = a^{mn}$, we call a^m the nth root of a^{mn}.

Thus, we have $\sqrt{a^6} = a^3$, $\sqrt[3]{a^6} = a^2$, $\sqrt[n]{a^{mn}} = a^m$.

Hence, in the case of quantities consisting of a single letter with a given exponent, when the given exponent contains as a factor the *number* indicating the root, we must divide the *given exponent* by this number, the quotient being the exponent of the root.

(We shall see farther on that this rule holds when the given exponent is not so divisible, the root in this case being called a surd).

35. Since the product of an *even* number of *negative factors* must give a *positive* result, and the product of an *odd* number of *negative* factors a *negative* result, it follows that—

I. When the root is indicated by an *even* number—

1. The root of a *positive* quantity may be written either with a + or − sign.

Thus, $\sqrt[2]{9\,a^2} = \pm 3\,a$, $\sqrt[4]{16\,b^4} = \pm 2\,b$.

2. The root of a *negative* quantity is impossible.

Thus, $\sqrt[2]{-\,a^2}$, $\sqrt[4]{-\,a^4b^{12}}$, &c., are impossible quantities.

II. When the root is indicated by an *odd* number, the root has always the sign of the given quantity.

Thus, $\sqrt[3]{-\,27\,b^6} = -\,3\,b^2$, $\sqrt[5]{32\,x^{10}y^5} = 2\,x^2y$.

(It may be remarked that the theory of *impossible quantities* forms an important branch of Algebra, which the student cannot yet enter upon. According to that theory, all quantities have as many roots as the number indicating the root.)

Square Root.

36. We shall now develop the method of finding the square root of a given quantity.

Ex. 1. Find the square root of $a^2 + 2\,ab + b^2$.

We know that $a^2 + 2\,ab + b^2 = (a + b)^2$.

Hence, $a + b$ is the square root of $a^2 + 2\,ab + b^2$ or $a^2 + (2\,a + b)\,b$.

Now, it is evident that the first term a of the root is the square root of the first term a^2 of the given quantity; and if this term be subtracted, there remains $2\,ab + b^2$, from which to determine b the second term of the root. Now, b is contained in $2\,ab + b^2$ or $(2\,a + b)\,b$ exactly $(2\,a + b)$ times. Hence it follows that the second term of the root is found by dividing the remainder by twice the first term of the root, and, if we wish to arrange our work in a way similar to long division, it is evident that we first take for our divisor $2\,a + b$, that is, twice the first term of the root added to the second term, which multiplied by b the second term, and subtracted, leaves no remainder.

Thus the whole process may be arranged as follows :—

$$\begin{array}{l|l} & a^2 + 2ab + b^2 (a + b \\ & a^2 \\ \hline 2a + b & \quad 2ab + b^2 \\ & \quad 2ab + b^2 \\ \hline \end{array}$$

Ex. 2. Find the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Now we know (Art. 26) that—

$$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \text{ or } (a + b)^2 + 2(a + b)c + c^2 = (a + b + c)^2.$$

And if we compare the form $(a + b)^2 + 2(a + b)c + c^2$ with the form $a^2 + 2ab + b^2$, it is evident that, having obtained as in the last example the first two terms $a + b$, we shall by continuing the process obtain the third term. Thus—

$$\begin{array}{l|l} & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 (a + b + c \\ & a^2 \\ \hline 2a + b & \quad 2ab + b^2 \\ & \quad 2ab + b^2 \\ \hline 2a + 2b + c & \quad\quad 2ac + 2bc + c^2 \\ & \quad\quad 2ac + 2bc + c^2 \\ \hline \end{array}$$

We may deduce from the above examples the following general rule :—

Rule.

1. Arrange the terms of the given quantity according to the ascending or descending powers of some letter, and take the square root of the first term for the first term of the quotient.

2. Subtract the square of the quotient, and bring down the next two terms of the given quantity.

3. Double the quotient, and place the result as a *trial divisor;* then, dividing the first of the terms brought down by this trial divisor to obtain the second term of the root, add the quotient so obtained to the first term of the root, and also to the trial divisor, to obtain a *complete* divisor.

4. Multiply the complete divisor by the second term of the root, and subtract the product, as in long division, from the terms brought down.

5. If there be any remainder or more terms to bring down, double the whole quotient for a trial divisor, and divide the

remainder by the new trial divisor, to obtain the third term of the root; and so on.

Ex. 3. Find the square root of $1 - 4x + 10x^2 - 12x^3 + 9x^4$. The terms are here arranged according to the ascending powers of x. Then proceeding according to rule, we have—

$$
\begin{array}{l|l}
 & 1 - 4x + 10x^2 - 12x^3 + 9x^4 \,(1 - 2x + 3x^2 \\
 & 1 \\
\hline
2 - 2x & \quad - 4x + 10x^2 \\
 & \quad - 4x + \;\; 4x^2 \\
\hline
2 - 4x + 3x^2 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
 & \qquad\qquad 6x^2 - 12x^3 + 9x^4 \\
\hline
\end{array}
$$

The student will observe that twice the quotient is most easily obtained by bringing down the previous complete divisor with its last term doubled.

Square Root of Numerical Quantities.

37. It is easy to apply the above method to numerical quantities.

Since $1^2 = 1$, $10^2 = 100$, $100^2 = 10{,}000$, $1{,}000^2 = 1{,}000{,}000$, &c., it is evident that the square roots of numbers having less than three figures must contain one figure only;

That those having not less than three and less than five must contain two figures and two only;

Those having not less than five and less than seven must contain three figures and three only; and so on.

Hence it follows that, if a dot be placed over the units' figure, and over every alternate figure to the left, the number of dots will give the number of figures in the square root.

Thus, the square roots of the numbers $\dot{1}4\dot{1}37\dot{6}$ and $\dot{1}5\dot{2}2\dot{7}5\dot{6}$ have three and four figures respectively.

In the number $\dot{1}4\dot{1}37\dot{6}$ we call 14, 13, 76 respectively the first, second, and third periods. So in the number $\dot{1}5\dot{2}2\dot{7}5\dot{6}$, the first, second, third, and fourth periods are respectively 1, 52, 27, 56.

It is evident that the number of periods correspond to the number of figures in the square root, and it will be seen that the figures of each period are used in the operation for the corresponding figure of the root.

Ex. Find the square root of 565504.

Pointing off the number, we find the first period to be 56. Now, the greatest square in 56 is 49, and the square next greater is 64. Hence the number lies between the square of 700 and 800; and, following the algebraical method, 700 will be the first term of the root.

The operation will stand thus—

$$\begin{array}{ll} & \dot{5}\dot{6}\dot{5}\dot{5}0\dot{4}\ (\overset{a}{7}00 + \overset{b}{5}0 + \overset{c}{2} = 752 \\ & 490000 = a^2 \\ \hline 2a + b = 1400 + 50 = 1450 & 75504 \\ & 72500 = 2ab + b^2 \\ \hline 2a + 2b + c = 1400 + 100 + 2 = 1502 & 3004 \\ & 3004 = 2ac + 2bc + c^2 \end{array}$$

Or, omitting the useless ciphers, and bringing down one period of figures at a time, the operation will stand thus—

$$\begin{array}{ll} & \dot{5}6\dot{5}\dot{5}0\dot{4}(752 \\ & 49 \\ \hline 145 & 755 \\ & 725 \\ \hline 1502 & 3004 \\ & 3004 \\ \hline \end{array}$$

Ex. 1. Find the square root of 6091024.

$$\begin{array}{ll} & \dot{6}0\dot{9}1\dot{0}2\dot{4}(2468 \\ & 4 \\ \hline 44 & 209 \\ & 176 \\ \hline 486 & 3310 \\ & 2916 \\ \hline 4928 & 39424 \\ & 39421 \\ \hline \end{array}$$

Ex. 2. Find the square root of 83521.

$$\begin{array}{ll} & \dot{8}\dot{3}5\dot{2}1(289 \\ & 4 \\ \hline 48 & 435 \\ & 384 \\ \hline 569 & 5121 \\ & 5121 \\ \hline \end{array}$$

It will be observed that the second remainder, 51, is greater than the previous complete divisor, 48, and it might be supposed, therefore, that the second figure in the root should be 9 instead of 8.

Now, the square of $(a + 1)$ exceeds the square of a by $2a + 1$.

Thus, $(a + 1)^2 - a^2 = (a^2 + 2a + 1) - a^2 = 2a + 1$.

Hence it follows that, so long as any remainder is less than twice the corresponding number in the root + 1, we may be certain that we have taken the figure of the root sufficiently large.

Thus, since the remainder is less than $28 \times 2 + 1$ or 57, we may be certain that 8 is the correct figure and not 9.

Square Root of a Decimal.

38. It is evident that the *square* of any number containing one, two, three, &c., decimal figures, will contain *two*, *four*, *six*, &c., decimal figures respectively; and, hence, conversely, every decimal considered as a square must contain an even number of decimal figures, and its square root must contain half this even number of figures. It will then be necessary to add a cipher when the given number of decimal figures is odd.

Further, since decimals and integers follow the same system of notation, it is evident that if a dot be placed over the *units'* figure of the given number, the pointing off may be performed with regard to the integral part exactly as in integers, there being no necessity to point off the decimals, only taking care to bring them down in pairs, and putting a decimal point in the quotient when the first pair is brought down.

And again, if an integer be given which is not a perfect square, we may, by affixing to the right of it a decimal point and an even number of ciphers, gradually approximate to the square root as nearly as we please.

Ex. 3. Find the square root of 1·8225.

	1·8225(1·35
	1
23	82
	69
265	1325
	1325

Ex. 4. Find the square root of 247 to four places of decimals.

Instead of adding a decimal point and eight ciphers to the right of the given number, we will proceed in the ordinary way till we arrive at a remainder. Then putting a decimal point in the quotient, we shall add two ciphers to this and each successive remainder.

	247(15·7162
	1
25	147
	125
307	2200
	2149
3141	5100
	3141
31426	195900
	188556
314322	734400
	628644
	105756

39. It will be shown hereafter that *when* n + 1 *figures of a square root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is* 2 n + 1.

Ex. Find the square root of 29 to six places of decimals.

The square root required will evidently contain *seven* figures. We shall therefore find the first *four* figures by the ordinary method, and the other *three* by the above method. Thus—

	29(5·385164
	25
103	400
	309
1068	9100
	8544
10765	55600
	53825
10770 ,	17750 then by division,
	10770
	69800
	64620
	51800
	43080
Ans. 5·385164.	8720

Cube Root.

40. We will next develop the method of finding the cube root of a quantity.

Ex. 1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We know (Art. 33) that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Hence $a + b$ is the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

We see then that, the quantity being arranged according to the powers of a, the first term a of the cube root is the cube root of the first term of the given quantity; and if this term a^3 be subtracted, there remains $3a^2b + 3ab^2 + b^3$.

We see again that if this remainder be divided by $3a^2$, its first term gives b the second term of the root, and, further, if it be divided by b, we get $3a^2 + 3ab + b^2$ as a quotient. If we wish therefore to arrange the whole process in a way

similar to ordinary division, it is evident that we must write as a divisor $3a^2 + 3ab + b^2$, in order that after multiplication by the quotient figure b we may obtain a quantity which when subtracted shall leave no remainder. The operation will then stand thus—

$$\begin{array}{r|l} & a^3 + 3a^2b + 3ab^2 + b^3(a + b \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & \quad 3a^2b + 3ab^2 + b^3 \\ & \quad 3a^2b + 3ab^2 + b^3 \\ \hline \end{array}$$

We call $3a^2$ the *trial divisor*, because by means of it we search for the second term of the cube root. Having obtained this second term, we then form *the complete divisor* $3a^2 + 3ab + b^2$.

Ex. 2. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$\begin{array}{r|l} & 8x^3 - 36x^2y + 54xy^2 - 27y^3(2x - 3y \\ & 8x^3 \\ \hline 12x^2 - 18xy + 9y^2 & \quad - 36x^2y + 54xy^2 - 27y^3 \\ & \quad - 36x^2y + 54xy^2 - 27y^3 \\ \hline \end{array}$$

Explanation.—We find the cube root of $8x^3$ to be $2x$. This is then the first term of the quotient, and corresponds to a in the previous example. We now require $3a^2$ for a trial divisor. This, of course, $= 3(2x)^2 = 12x^2$. Subtracting the first term of the given quantity and dividing the first term of the remainder by this trial divisor, we obtain $-3y$ for the quotient. This forms the second term of the root, and corresponds to b in the last example. We now easily obtain $3ab$ and b^2.

Thus, $3ab = 3(2x)(-3y) = -18xy$, and $b^2 = (-3y)^2 = 9y^2$.

Hence, the complete trial divisor, corresponding to $3a^2 + 3ab + b^2$ in the last example, $= 12x^2 - 18xy + 9y^2$.

Multiplying now by $-3y$ the quotient, we obtain $-36x^2y + 54xy^2 - 27y^3$, which subtracted leaves no remainder.

Hence, $2x - 3y$ is the cube root.

Ex. 3. Find the cube root of—

$$x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6.$$

$$\begin{array}{l|l}
x^6 + 6x^5y - 40x^3y^3 + 96xy^5 - 64y^6 & x^2 + 2xy - 4y^2 \\
\underline{x^6} & \\
3x^4 + 6x^3y + 4x^2y^2 & 6x^5y - 40x^3y^3 \\
& \underline{6x^5y + 12x^4y^2 + 8x^3y^3} \\
& -12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6 \\
& \underline{-12x^4y^2 - 48x^3y^3 + 96xy^5 - 64y^6}
\end{array}$$

$$\begin{array}{rl}
3(x^2+2xy)^2 = & 3x^4 + 12x^5y + 12x^2y^2 \\
3(x^2+2xy)(-4y^2) = & -12x^2y^2 - 24xy^3 \\
(-4y^2)^2 = & 16y^4 \\
\hline
3x^4 + 12x^3y & -24xy^3 + 16y^4
\end{array}$$

It will be evident that the first two terms are obtained exactly as in Exs. 1 and 2. To obtain the next term we treat the terms already found as corresponding to a in Ex. 1. Then we obtain the trial divisor $3(x^2 + 2xy)^2$ corresponding to $3a^2$, and afterwards having divided the first term of the remainder by the first term of the trial divisor in order to get the term corresponding to b, we obtain in order $3(x^2 + 2xy)(-4y^2)$ corresponding to $3ab$, then $(-4y^2)^2$ corresponding to b^2, and lastly, by addition, $3x^4 + 12x^3y - 24xy^3 + 16y^4$, being the complete divisor, corresponding to $3a^2 + 3ab + b^2$ of Ex. 1. We then conclude the operation as before.

Cube Root of Numerical Quantities.

41. We shall now apply the above method to numerical quantities. It may be shown, as in Art. 34, that, if we place a dot over the *units' figure*, and over every *third* figure to the left, the number of periods so formed will be the number of figures in the root.

Ex. 1. Find the cube root of 262144.

$$\begin{array}{rcl} & & 26\dot{2}144\dot{(}\overset{a}{6}0 + \overset{b}{4} = 64 \\ & & 216000 \\ & & \hline 46144 \\ 3\,a^2 & = & 10800 \\ 3\,ab & = & 720 \\ b^2 & = & 16 \\ \hline 3\,a^2 + 3\,ab + b^2 & = & 11536 \qquad 46144 \end{array}$$

Explanation.—Pointing off the given number, we find the first period to be 262, and that the cube root consists of two figures. Now, the greatest perfect cube in 262 is 216, which is the cube of 6. Hence, the given number lies between the cubes of 60 and 70 ; and following the algebraical method, 60 will be the first term of the cube. This, we see, corresponds to a in the algebraical method.

We first then subtract the cube of a—viz., 216000, which leaves as a remainder 46144.

We now write down $3\,a^2$ or $3\,(60)^2 = 10800$, which is the trial divisor for determining b. Dividing then by this value of $3\,a^2$, we find $b = 4$, which is the second term of the cube root. We next obtain $3\,ab = 3\,(60)\,(4) = 720$, and $b^2 = 4^2 = 16$, and so by addition we get $3\,a^2 + 3\,ab + b^2 = 11536$, which is the complete divisor. Multiplying this then by the quotient figure, we subtract the product, and, there being no remainder, we find the cube root to be 64.

We may omit the useless ciphers in the above operation, if, remembering the local value of figures when numbers are expressed in ordinary notation, we take care to place the right-hand figure of the value of $3\,ab$ one place to the right of the corresponding figure of the value of $3\,a^2$; and also to place the right hand figure of b^2 one figure further still to the right.

The operation will then stand thus—

$$\begin{array}{rcl} & & 26\dot{2}144\dot{(}64 \\ & & 216 \\ & & \hline 46144 \\ 3 \times 6^2 & = & 108 \\ 3 \times 6 \times 4 & = & 72 \\ 4^2 & = & 16 \\ \hline & & 11536 \qquad 46144 \end{array}$$

Ex. 2. Find the cube root of 102503232.

$$\begin{array}{lrl}
 & & 10\dot{2}50\dot{3}23\dot{2}(468 \\
 & & \underline{64} \\
3 \times 4^2 & = 48 & 38503 \\
3 \times 4 \times 6 & = 72 & \\
6^2 & = 36 & \\
 & \underline{5556} & 33336 \\
 & 36 & \underline{5167232} \\
 & \overline{6348} & \\
3 \times 46^2 & = & \\
3 \times 46 \times 8 & = 1104 & \\
8^2 & = 64 & \\
 & \overline{645904} & 5167232
\end{array}$$

Explanation.—The first two figures of the root are obtained as in Ex. 1. We then treat the number they form, viz., 46, as corresponding to a in the algebraical model, omitting useless ciphers. Obtaining then $3a^2$ or $3 \times 46^2 = 6348$, we find 8 to be the next figure of the root. Then writing under this, $3ab$ or $3 \times 46 \times 8 = 1104$, and afterwards b^2 or $8^2 = 64$, taking care as to the positions of the right-hand figures, and adding, we get 645904 as the complete divisor. Then as before.

Remark.—It is unnecessary to be at the trouble to find the value of 3×46^2 by ordinary multiplication. For referring to the algebraical model, and writing here the successive terms of the complete divisor, and adding, we have—

$$\begin{array}{rl}
 & 3a^2 \\
 & 3ab \\
 & b^2 \\
\text{Sum} = & \overline{3a^2 + 3ab + b^2} \\
 & b^2 \\
 & \overline{3a^2 + 6ab + 3b^2}, \text{ or } 3(a^2 + 2ab + b^2) = 3(a + b)^2.
\end{array}$$

If we now again write down b^2 under this sum, and then add up *the last four lines*, we get—

This is three times the square of the first two terms of the root.

It therefore follows that, if, as in the above example, after completing the operation for finding the first two figures of the cube root, we write under the complete divisor just obtained the value of the square of the second figure, and then add together the last four lines thus obtained, we get three

times the square of the quotient for a partial divisor by which to determine the next figure of the root.

The four lines to be added are in the above example bracketed. This method will be found to materially shorten the work, for it may be similarly applied to find the trial divisor when the cube root consists of any number of figures

Cube Root of a Decimal.

42. We know that the *cube* of any number containing one, two, three, &c., decimal figures will contain three, six, nine, &c., decimal figures respectively, and hence, conversely, every decimal considered as a cube must contain a number of decimal figures which is a multiple of *three*, and the number of decimal figures in the *cube root* must be *one-third* of the number contained in the given *cube*. It will then be necessary to add ciphers when the given number of decimal figures is not a multiple of 3.

And by continuing the reasoning of Art. 37, if a dot be placed over the units' figure and over every third figure to the left, it will be sufficient to bring down the decimal figures *three* at a time, putting a decimal point in the quotient when the first three are brought down.

And further, if an integer be given which is not a perfect cube, we may proceed in the ordinary way till we arrive at a remainder, and then, putting a decimal point in the quotient, by affixing *three* ciphers to this and each successive remainder, approximate to the cube root as nearly as we please.

Ex. 3. Find the cube root of 395·446904.

$$
\begin{array}{rrrl}
 & & & \dot{3}\dot{9}\dot{5}\cdot 446904(7\cdot 34 \\
 & & & 343 \\
 & & & \overline{52446} \\
 & 3 \times 7^2 & = & 147 \\
3 \times 7 \times & 3 & = & 63 \\
 & 3^2 & = & 9 \\
 & & & \overline{15339} \qquad 46017 \\
 & & & 9 \qquad\quad \overline{6429904} \\
 & & & \overline{15987} \\
 & 3 \times 73^2 & = & \\
3 \times 73 \times & 4 & = & 876 \\
 & 4^2 & = & 16 \\
 & & & \overline{1607476} \qquad 6429904
\end{array}
$$

43. We shall show farther on that *when* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *figures more may be obtained by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is* 2 n + 2.

We may apply this principle with advantage when we require the cube root of number to a given number of decimals.

Ex. Find the cube root of 87 to five places of decimals.

The required cube root will evidently contain 6 figures, and since 6 here corresponds to $2n + 2$ above, it is evident that $n = 2$. Hence, we shall find 4 (that is, $n + 2$) figures by the ordinary method, and then 2 more by division.

The operation will stand thus—

		87(4·43104	
		64	
$3 \times 4^2 =$	48	23000	
$3 \times 4 \times 4 =$	48		
$4^2 =$	16		
	5296	21184	
	16	1816000	
$3 \times 44^2 =$	5808		
$3 \times 44 \times 3 =$	396		
$3^2 =$	9		
	584769	1754307	
	9	61693000	
$3 \times 443^2 =$	588747		
$3 \times 443 \times 1 =$	1329		
$1^2 =$	1		
	58887991	58887991	
	1	280500900	
	58901283	235605132	
		44895768	

Ans. 4·43104.

Ex. XI.

Find the square roots of—

1. $4x^2y^4z^6$, $16a^4y^4$, $x^4 + 2a^2x^2 + a^4$.
2. $4x^6 - 12x^5y + 29x^4y^2 - 30x^3y^3 + 25x^2y^4$.

3. $25\,a^4 - 30\,a^3b + 19\,a^2b^2 - 6\,ab^3 + b^4$.

4. $1 - 4\,x + 10\,x^2 - 20\,x^3 + 25\,x^4 - 24\,x^5 + 16\,x^6$.

5. $a^2 + 2\,abx + (2\,ac + b^2)\,x^2 + 2\,(ad + bc)\,x^3 + (2\,bd + c^2)\,x^4 + 2\,cdx^5 + d^2x^6$.

6. $a^4x^{2n} - 6\,a^3x^{2n-1} + 17\,a^2x^{2n-2} - 24\,ax^{2n-3} + 16\,x^{2n-4}$.

7. $x^2 + 2 + x^{-2}$, $a^2x^{-2} - 2 + a^{-2}x^2$.

8. $9\,x^{2m} - 3\,a^2x^m + 25\,a^2 - 30\,ax^m + \frac{a^4}{4} + 5\,a^3$.

Find the square roots of—

9. 1296, 6241, 42849, 83521.

10. 10650·24, ·000576, $\dot{1}$, $\frac{48}{243}$.

11. $\sqrt{17}$, $\sqrt{1{\cdot}5}$, $\frac{2}{\sqrt{5}+1}$, $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

Give the values correct to four places of decimals of—

12. $\frac{\frac{5}{8}\text{ of }{\cdot}31416}{\sqrt{93}}$, $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}} + \frac{\sqrt{10}-2}{\sqrt{10}+2}$, 3·1416 of $\sqrt{\frac{40}{32{\cdot}16}}$.

Find the cube roots of—

13. $8\,a^3b^6y^{12}$, $125\,x^{12}y^3$, $a^3 + 6\,a^2b + 12\,ab^2 + 8\,b^3$.

14. $x^{12} + 9\,x^{10} + 6\,x^8 - 99\,x^6 - 42\,x^4 + 441\,x^2 - 343$.

15. $x^3 + 3\,x^2y + 3\,xy^2 + y^3 - 6\,cxy - 3\,cx^2 - 3\,cy^2 + 3\,c^2x + 3\,c^2y - c^3$.

16. $x^3 + x^{-3} + 3\,(x + x^{-1})$, $x^3y^{-3} + 3\,x^2y^{-2} + 3\,xy^{-1} + 1$.

Find the cube roots of—

17. 5849513501832, 1371·330631.

18. 20·346417; ·037, $\frac{108}{1372}$.

Give the value of the following correct to four places of decimals:—

19. $\frac{\sqrt[3]{5{\cdot}12} + \sqrt[3]{{\cdot}03375}}{\sqrt[3]{80} - \sqrt[3]{{\cdot}01}}$, $\frac{1}{\sqrt[3]{4} + \sqrt[3]{2} + 1.}$

20. $\frac{\sqrt[3]{{\cdot}625} + \sqrt[3]{{\cdot}04}}{\sqrt{{\cdot}05} + \sqrt{{\cdot}04}}$ of $\frac{\sqrt{5}+2}{7}$.

21. $(\sqrt{7} + 2)(\sqrt{7} - 1)$, $(5 + \sqrt{3})(4 + \sqrt{12})$.

22. $\sqrt{11 + 6\sqrt{2}}$, $\sqrt[3]{6 + 15\sqrt{3}}$.

23. $\sqrt{\frac{3}{4} - \frac{6 + 2\sqrt{5}}{16}} \cdot \frac{\sqrt{5} + 1}{4}$.

24. $a^3(b - c) - b^3(a - c) + c^3(a - b)$, where $a = \sqrt{1 \cdot 2}$, $b = -\sqrt{\cdot 3}$, and $c = -\sqrt[6]{\cdot 027}$.

CHAPTER IV.

GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE.

Greatest Common Measure.

44. In Arithmetic (page 24) we defined the G.C.M. of two or more numbers as their highest common factor. In Algebra the same definition will suffice, provided we understand by the term *highest common factor, the factor of highest dimensions* (Art. 18). This, it need hardly be remarked, does not necessarily correspond to the factor of highest numerical value.

45. To find the G.C.M. of two quantities.

RULE.—Let A and B be the quantities, of which A is not of lower dimensions than B. Divide A by B, until a remainder is obtained of lower dimensions than B. Take this remainder as a new divisor, and the preceding divisor A as a new dividend, and divide till a remainder is again obtained of lower dimensions than the divisor; and so on. The last divisor is the G.C.M.

Before giving the general theory of the G.C.M. we shall work out a few examples.

Ex. 1. Find the G.C.M. of $x^2 - 6x - 27$ and $2x^2 - 11x - 63$.

According to the above rule, the operation is as follows:—

$$
\begin{array}{l}
x^2 - 6x - 27)\,2x^2 - 11x - 63\,(2 \\
\qquad\quad 2x^2 - 12x - 54 \\
\qquad\quad \hline \\
x - 9)\,x^2 - 6x - 27\,(x + 3 \\
\qquad\quad x^2 - 9x \\
\qquad\quad \hline \\
\qquad\quad 3x - 27 \\
\qquad\quad 3x - 27 \\
\qquad\quad \hline
\end{array}
$$

$\therefore$ The G.C.M. is $x - 9$.

Ex. 2. Find the G.C.M. of $10x^3 + 31x^2 - 63x$ and $14x^3 + 51x^2 - 54x$.

We may tell by inspection that x is a common factor, which we therefore strike out of both, only *taking care to reserve* it. The quantities then become—

$$10x^2 + 31x - 63, \text{ and } 14x^2 + 51x - 54.$$

We may now proceed according to rule, taking the former as divisor. We see, however, that the coefficient of the first term of the dividend is not exactly divisible by the coefficient of the first term of the divisor. Multiply therefore (to avoid fractions) the dividend by such a number as will make it so divisible, viz., by 5. This will not affect the G.C.M., as 5 is not *a factor* of the first expression, viz., $10x^2 + 31x - 63$.

It may as well be here mentioned that the G.C.M. of two quantities cannot be affected by the multiplication or division of *one* of the quantities by any quantity which is not a measure of the other. We shall, for a similar reason, reject certain factors or introduce them into any of the remainders or dividends during the operation. (See Art. 47).

$$\begin{array}{r} 14x^2 + 51x - 54 \\ 5 \\ \hline 10x^2 + 31x - 63)70x^2 + 255x - 270(7 \\ 70x^2 + 217x - 441 \\ \hline 38x + 171 \end{array}$$

Rejecting the factor 19 of this remainder, we have—

$$\begin{array}{r} 2x + 9)10x^2 + 31x - 63(5x - 7 \\ 10x^2 + 45x \\ \hline - 14x - 63 \\ - 14x - 63 \\ \hline \end{array}$$

Hence, $2x + 9$ is the last divisor, and multiplying this by x, the common measure struck out at the commencement, we find the G.C.M to be $x(2x + 9)$ or $2x^2 + 9x$.

Ex. 3. Find the G.C.M. of $x^6-7x^5-3x^4-5x^3+42x^2-34x-21$, and $x^5-11x^4+25x^3+19x^2-49x-21$.

$$x^5-11x^4+25x^3+19x^2-49x-21)\ x^6-7x^5-3x^4-5x^3+42x^2-34x-21\ (x+4$$
$$x^6-11x^5+25x^4+19x^3-49x^2-21x$$
$$4x^5-28x^4-24x^3+91x^2-13x-21$$
$$4x^5-44x^4+100x^3+76x^2-196x-84$$
$$16x^4-124x^3+15x^2+183x+63$$

Multiplying the preceding divisor by 16, and taking the result for a dividend, we have—

$$16x^4-124x^3+15x^2+183x+63)\ 16x^5-176x^4+400x^3+304x^2-784x-336(x$$
$$16x^5-124x^4+15x^3+183x^2+63x$$
$$-52x^4+385x^3+121x^2-847x-336$$

(Multiplying this remainder by 4)

$$4$$
$$-208x^4+1540x^3+484x^2-3388x-1344(-13$$
$$-208x^4+1612x^3-195x^2-2379x-819$$
$$-72x^3+679x^2-1009x-525$$

Multiplying the preceding divisor by 9, and taking the result for a dividend, we have—

$$-72x^3+679x^2-1009x-525)\ 144x^4-1116x^3+135x^2+1647x+567(-2x$$
$$144x^4-1358x^3+2018x^2+1050x$$
$$242x^3-1883x^2+597x+567$$

(Multiplying this remainder by 36)

$$36$$
$$8712x^3-67788x^2+21492x+20412(121$$
$$8712x^3-82159x^2+122089x+63525$$
$$14371x^2-100597x-43113$$

Dividing this remainder by 14371, and taking the quotient for a new divisor, we have—

$$\begin{array}{r} x^2 - 7x - 3) - 72x^3 + 679x^2 - 1009x - 525 (- 72x + 175 \\ - 72x^3 + 504x^2 + 216x \\ \hline 175x^2 - 1225x - 525 \\ 175x^2 - 1225x - 525 \\ \hline \end{array}$$

$\therefore\ x^2 - 7x - 3$ is the G.C.M.

It will be seen that we have introduced and rejected factors during the operation in order to avoid fractional coefficients. This, as will be seen from the general theory, will not affect the result, provided that no factor thus introduced or rejected is a measure of the corresponding divisor or dividend, as the case may be.

Theory of the Greatest Common Measure.

46. Let A and B be the two algebraical quantities, and the operation as indicated by the rule (Art. 45) be performed. Thus, let A be divided by B, with quotient p and remainder C. Then let B be divided by C, with quotient q, and remainder D. Lastly, let C be divided by D, with quotient r, and remainder zero.

$$\begin{array}{r} B)A(p \\ pB \\ \hline C)B(q \\ qC \\ \hline D)C(r \\ rD \\ \hline 0 \end{array}$$

Then we are required to show that D is the G.C.M. of A and B.

(1.) D is a common measure of A and B.

Now, we have $C = rD$, $B = qC + D$, $A = pB + C$. Hence, D is a measure of C, and therefore of qC. It is therefore a measure of $qC + D$ or B. Hence, also, D is a measure of pB, and since it is also a measure of C, it must be a measure of $pB + C$ or A. But we have shown it to be a measure of B. Hence, D is a common measure of A and B.

(2.) D is the G.C.M. of A and B.

For every measure of A and B will divide $A - pB$ or C; and hence every measure of A and B will divide $B - qC$ or D. Now, D cannot be divided by any quantity higher than D, and, therefore, there cannot exist a measure of A and B higher than D. Hence, D is the G.C.M. of A and B.

47. *A factor which does not contain any factor common to both A and B may be rejected at any stage of the process.*

Let the operation stand thus:—

$B = mB'$ suppose,

$B')A(p$ $\qquad$ $C')B'(q$

$\underline{pB'}$ $\qquad$ $\underline{qC'}$

$C = nC'$ suppose, $\qquad$ $D)C'(r$

$\underline{rD}$

0

where neither m nor n contains any factor common to A and B.

It will be an exercise for the student to show that D is the G.C.M. of A and B.

48. *A factor, which has no factor that the divisor has, may be introduced into the dividend at any stage of the process.*

The operation may stand thus—

$B)mA(p$, where m has no factor that B has;

$\underline{pB}$

$C)nB(q$, where n has no factor that C has;

$\underline{qC}$

$D)C(r$

$\underline{rD}$

0

As in Arts. 46, 47, it may be easily shown that D is the G.C.M.

Both the above principles are made use of in working out Ex. 3 Art. 45.

49. When a common factor can be found by inspection, it is advisable to strike it out of the given expressions. Then, having found by the ordinary process the G.C.M. of the resulting quantities, we must multiply the G.C.M. so found by the rejected factor.

Thus, $4x$ is common to the quantities $4x^3 - 20x^2 + 24x$, and $4x^3 + 16x^2 - 84x$.

Rejecting it, we get $x^2 - 5x + 6$, and $x^2 + 4x - 21$, whose G.C.M. is easily found to be $x - 2$.

Multiplying by $4x$, we find the required G.C.M. to be $4x^2 - 8x$.

50. By a little ingenuity on the part of the student in breaking up the given expressions into factors, the ordinary and often tedious process of finding the G.C.M. may be avoided. The limits of our space will allow us only one example.

Ex. Find the G.C.M. of $3x^3 + 4x^2 - 10x + 3$, and $15x^3 + 47x^2 + 13x - 12$.

The first expression contains $x - 1$ as a factor (Art. 30), for the sum of its coefficients is *zero*. The other factor may be obtained thus—

$$\begin{aligned} 3x^3 + 4x^2 - 10x + 3 &= 3x^3 - 3x^2 + 7x^2 - 7x - 3x + 3 \\ &= 3x^2(x - 1) + 7x(x - 1) - 3(x - 1) \\ &= (3x^2 + 7x - 3)(x - 1). \end{aligned}$$

Now, $3x^2 + 7x - 3$ is not further resolvable, and $x - 1$ is evidently (Art. 30) not a factor of $15x^3 + 34x^2 + 13x - 12$. It is, therefore, very probable that $3x^2 + 7x - 3$ is the G.C.M. required.

We may test it thus—

$$\begin{aligned} 15x^3 + 47x^2 + 13x - 12 &= 15x^3 + 35x^2 - 15x + 12x^2 + 28x - 12 \\ &= 5x(3x^2 + 7x - 3) + 4(3x^2 + 7x - 3) \\ &= (5x + 4)(3x^2 + 7x - 3). \end{aligned}$$

Hence, $3x^2 + 7x - 3$ is the G.C.M. required.

G.C.M. of Three or More Quantities.

51. The G.C.M. of three or more quantities may be found thus—

RULE.—Find the G.C.M. of any two of the quantities, then the G.C.M. of the G.C.M. so found and a third quantity, and so on. The last found G.C.M. will be the G.C.M. required.

Ex. XII.

Find the G.C.M. of the following—

1. $x^2 - 5x + 6$ and $x^2 + 3x - 18$.
2. $x^3 + 6x^2 + 11x + 6$ and $x^3 + 5x^2 + 7x + 3$.
3. $2x^3 + 10x^2 - 18x - 90$ and $3x^3 + 16x^2 - 26x - 141$.
4. $x^2 + (a + b\ x + ab$ and $x^2 + (a + c)x + ac$.
5. $a^3 - b^3$ and $a^3 + a^2b + ab^2$.

6. $x^3 - 4x + 3$ and $x^3 + 4x^2 - 5$.

7. $4x^3 - 32x^2 + 85x - 75$ and $3x^3 - 15x^2 + 15x + 9$.

8. $9x^2 - 3xy + 2y - 4$ and $6x^4 - 4x^3 - 9xy^2 + 6y^2$.

9. $48x^4 + 8x^3 + 31x^2 + 15x$ and $24x^4 + 22x^3 + 17x^2 + 5x$.

10. $15a^3 + a^2b - 3ab^2 + 2b^3$ and $54a^2b^2 - 24b^4$.

11. $3x^3 - (3c + d + 1)x^2 - (2a + b - 3c - d + 2)x + 2a + b$ and $2x^2 - (a + b + 2)x + a + b$.

12. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.

13. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ and $9ac + 2a^2 - 5ab + 4c^2 + 8bc - 12b^2$.

14. $e^x x^2 + e^x + x^2 + 1$ and $e^{2x}x^4 - e^{2x} + x^4 - 1$.

15. $ax^4 + (b + c)x^2 - ax - b - c$ and $ex^3 - (f - g)x^2 + (f - e)x - g$.

16. $4x^4 + 2x^3 + 4x^2 + 39x - 9$, $8x^4 + 20x^2 + 51x + 9$, and $2x^4 + x^3 + 3x^2 + 18x$.

17. $ax^3 - (c + 1)x^2 + (c + 1)x - a$, $bx^4 - (b + d)x^3 + (c + d)x^2 - (c + e)x + e$, and $(c + 1)x^5 + (d + 2)x^4 - (d + 1)x^3 - (c + 2)x^2$.

18. $a^3 - b^3 + c^3 + 3abc$ and $a^2 - b^2 + c^2 + 2ac$.

Least Common Multiple.

52. When two or more algebraical expressions are arranged according to the powers of some letter, the expression of lowest dimensions which is divisible by each of the given expressions is called the L.C.M.

53. The L.C.M. of monomials and of expressions whose factors are apparent may be found by inspection.

Ex. 1. Find the L.C.M. of ab, ac, ad, bc, bd, cd.

If we form an expression, whose elementary factors contain each of the elementary factors of the given quantities, we shall evidently have a *common multiple*; and if no elementary factor of this expression is of a higher power than the highest power of the same factor in the given quantities, we shall get the L.C.M.

Hence, the required L.C.M. $= abcd$.

Ex. 2. Find the L.C.M. of—

$(a - b)(b - c), (a - b)(c - a), (b - c)(c - a)$.

Ans. $(a - b)(b - c)(c - a)$.

Ex. 3. Find the L.C.M. of $a(x + 1)$, $b(x^2 - 1)$, $c(x^2 + 2x - 3)$, $d(x^2 + 4x - 3)$. We may write the given expressions thus—

$a(x + 1)$, $b(x + 1)(x - 1)$,

$c(x - 1)(x + 3)$, $d(x + 1)(x + 3)$.

Hence, the required L.C.M. $= abcd(x - 1)(x + 1)(x + 3)$.

Ex. 4. Find the L.C.M. of $a^2 - ax + x^2$, $a^2 + ax + x^2$, $a^3 + x^3$, $a^3 - x^3$.

Now (Art. 29) $a^3 + x^3 = (a + x)(a^2 - ax + x^2)$,

and $a^3 - x^3 = (a - x)(a^2 + ax + x^2)$.

Hence the required L.C.M.—

$= (a + x)(a - x)(a^2 + ax + x^2)(a^2 - ax + x^2) = a^6 - x^6$.

54. *The L.C.M. of two quantities is found by dividing their product by their G.C.M.*

Let a and b be the two quantities, and d the G.C.M. ;

And suppose $a = pd$ and $b = qd$.

It is evident that p and q contain no common factor. Hence pq is the L.C.M. of p and q ; and, therefore, no expression of lower dimensions than pqd can possibly be divisible by pd and qd.

Hence pqd is the L.C.M. of pd and qd, or of a and b.

Now $pqd = pd \times qd \div d = a \times b \div d$, and hence the rule.

55. To find the L.C.M. of *three* or more quantities.

RULE.—Find the L.C.M. of two of the quantities, then the L.C.M. of the expression thus obtained and a third quantity, and so on. The last expression so found is the L.C.M. required.

We shall prove this rule in the case of *three* quantities.

Let a, b, c be the quantities, and m be the L.C.M. of a and b.

Then the L.C.M. of m and c is the L.C.M. required.

For every common multiple of m and c is a common multiple of a, b, c. And every common multiple of a and b

must contain the m, their least common multiple. Hence, every common multiple of a, b, c must be a common multiple of m and c, and the converse is also true. Hence, the L.C.M. of m and c is the L.C.M. of a, b, c.

Ex. XIII.

Find the L.C.M. of—

1. axy^2, $3\,a^2x^3y$, $4\,a^3y^3$, $6\,x^2y^2$.
2. $5\,a^2b^2$, $6\,a^2c^2$, $4\,b^2c^2$.
3. $(a - b)\,(b - c)$, $(b - a)\,(a - c)$, $(c - a)\,(c - b)$.
4. $ax\,(x + a)$, $a^2(x - a)$, $x^2 - a^2$.
5. $x^2 + 3x + 2$, $x^2 + 4x + 3$, $x^2 + 5x + 6$.
6. $x^2 - x - 30$, $x^2 - 11x + 30$, $x^2 - 25$.
7. $6x^2 + 37x + 56$, $8x^2 + 38x + 35$, $12x^2 + 47x + 40$.
8. $5\,(x^2 - x + 1)$, $6\,(x^2 + 1)$, $7\,(x^3 + 1)$.
9. $x^4 + a^2x^2 + a^4$, $a^2x^2 + a^3x + a^4$, $ax^2 - a^2x + a^3$.
10. $x^2 + (a + b)\,x + ab$, $x^2 + (a + c)\,x + ac$, $x^2 + (b + c)\,x + bc$.
11. $1 - x$, $1 + x$, $1 + x^2$, $1 + x^4$, $1 + x^8$.
12. $x^3 + 6x^2 + 11x + 6$, $x^3 - 6x^2 - 25x + 150$.
13. $a^3 - 3ab\,(a - b) - b^3$, $a^3 - b^3$, $a^3 + a^2b + ab^2$.
14. $x^4 - 1$, $6x^5 + 5x^4 + 8x^3 + 4x^2 + 2x - 1$.
15. $a^4 - 2a^2b^2 + b^4$, $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$.
16. $3x^3 - 4x + 1$, $2x^3 - 7x + 5$, $4x^4 + 6x^2 + 10x$.
17. $3x^2 + 6x - 24$, $x^3 - 12x + 16$, $5x^4 - 22x - 36$.
18. $a^3 - ab^2$, $b^3 - a^2b$, $ab^2 - b^3$, $a^2b - a^3$.
19. $3x^4 - 48$, $5x^2 - 20$, $3x^2 - 16x + 20$.
20. $x^8 - y^8$, $x^4 - 2x^2y^2 + y^4$, $x^3 + x^2y + xy^2 + y^3$.
21. $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^4x + a^5$ and $x^5 - ax^4 + a^2x^3 - a^3x^2 + a^4x - a^5$.
22. $a^2 + b^2 - c^2 - d^2 + 2ab - 2cd$ and $a^2 - b^2 - c^2 + d^2 + 2ad - 2bc$.

23. $a^3 + b^3 + c^3 - 3abc$ and $(a + b)^2 + 2(a + b)c + c^2$.

24. $(a + b)^2 - (c + d)^2$, $(a + c)^2 - (b - d)^2$, $(a + d)^2 - (b - c)^2$, $(c + d)^2 - (a - b)^2$, $(b + d)^2 - (a - c)^2$, and $(b + c)^2 - (a - d)^2$.

CHAPTER V.

Fractions.

56. It is unnecessary to repeat here the propositions relating to fractions which were proved in Arithmetic, Chap. II. of this work. The student will see that, by substituting general symbols for the particular figures there used, the reasoning will equally hold. We shall work out a few examples to show the method of dealing with them in algebra.

Ex. 1. Simplify the fraction $\dfrac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15}$.

By inspection (Art. 30) we see that $x - 3$ is a factor of numerator and denominator. We have then—

$$\frac{x^3 - 2x^2 + x - 12}{x^2 + 2x - 15} = \frac{x^2(x - 3) + x(x - 3) + 4(x - 3)}{(x - 3)(x + 5)}$$

$$= \frac{(x^2 + x + 4)(x - 3)}{(x - 3)(x + 5)} = \frac{x^2 + x + 4}{x + 5}. \quad \text{Ans.}$$

Ex. 2. Find the value of $\dfrac{1}{a + b} + \dfrac{1}{a - b} - \dfrac{2a}{a^2 + b^2}$.

$$\frac{1}{a + b} + \frac{1}{a - b} - \frac{2a}{a^2 + b^2} = \frac{(a - b) + (a + b)}{(a + b)(a - b)} - \frac{2a}{a^2 + b^2}$$

$$= \frac{2a}{a^2 - b^2} - \frac{2a}{a^2 + b^2} = 2a\left(\frac{1}{a^2 - b^2} - \frac{1}{a^2 + b^2}\right)$$

$$= 2a \cdot \frac{(a^2 + b^2) - (a^2 - b^2)}{(a^2 - b^2)(a^2 + b^2)} = 2a \cdot \frac{2b^2}{a^4 - b^4} = \frac{4ab^2}{a^4 - b^4}.$$

Ex. 3. Find the value of $\frac{bc}{(a-b)(a-c)} + \frac{ac}{(b-a)(b-c)} + \frac{ab}{(c-a)(c-b)}$.

The second denominator has a factor, $(b - a)$, which differs from a factor, $(a - b)$, of the first denominator in sign only. We shall therefore change the sign of the second fraction, and also of its first factor. This will not alter its value.

And, similarly, we find that by changing the signs of each of the factors of the third denominator we shall have them in a form corresponding to factors of the first and second denominators. The sign of the third fraction will not be changed, as the sign of the denominator will, on the whole, be unchanged.

The given expression then will stand thus—

$$= \frac{bc}{(a-b)(a-c)} - \frac{ac}{(a-b)(b-c)} + \frac{ab}{(a-c)(b-c)}$$

$$= \frac{bc(b-c) - ac(a-c) + ab(a-b)}{(a-b)(a-c)(b-c)}$$

$$= \frac{bc(b-c) - a^2c + ac^2 + a^2b - ab^2}{(a-b)(a-c)(b-c)}, \text{ or, re-arranging,}$$

$$= \frac{a^2(b-c) - a(b^2-c^2) + bc(b-c)}{(a-b)(a-c)(b-c)}, \text{ then, dividing numerator and denominator by } b - c,$$

$$= \frac{a^2 - a(b+c) + bc}{(a-b)(a-c)} = \frac{(a-b)(a-c)}{(a-b)(a-c)} = 1.$$

Ex. 4. Simplify—

$$\left\{a^2 - \frac{4a^2x - 3ax^2 + x^3}{a+x}\right\} \times \left\{a^2 + \frac{4a^2x + 3ax^2 + x^3}{a-x}\right\}.$$

The given expression—

$$= \frac{a^2(a+x) - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^2(a-x) + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 + a^2x - 4a^2x + 3ax^2 - x^3}{a+x} \times \frac{a^3 - a^2x + 4a^2x + 3ax^2 + x^3}{a-x}$$

$$= \frac{a^3 - 3a^2x + 3ax^2 - x^3}{a + x} \times \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a - x}$$

$$= \frac{(a - x)^3}{a + x} \times \frac{(a + x)^3}{a - x} = \frac{(a - x)^2}{1} \times \frac{(a + x)^2}{1} = (a^2 - x^2)^2.$$

Ex. 5. Divide—

$$(a+b)\left\{\frac{1}{x - a} + \frac{1}{x + b}\right\} \text{ by } (a - b)\left\{\frac{a}{(x-a)^2} + \frac{b}{(x + b)^2}\right\}.$$

Now—

$$\frac{(a + b)\left\{\frac{1}{x - a} - \frac{1}{x + b}\right\}}{(a - b)\left\{\frac{a}{(x - a)^2} + \frac{b}{(x+b)^2}\right\}} = \frac{(a+b) \cdot \frac{(x + b) - (x - a)}{(x - a)(x + b)}}{(a - b) \cdot \frac{a(x + b)^2 + b(x - a)^2}{(x-a)^2 (x+b)^2}}.$$

or, reducing—

$$= \frac{(a + b) \cdot \frac{a + b}{(x - a)(x + b)}}{(a - b) \cdot \frac{(a + b)x^2 + ab(a + b)}{(x - a)^2 (x + b)^2}} = \frac{(a + b) \cdot \frac{1}{1}}{(a - b) \cdot \frac{x^2 + ab}{(x - a)(x + b)}}$$

$$= \frac{(a + b)(x - a)(x + b)}{(a - b)(x^2 + ab)}.$$

Ex. XIV.

Simplify the following expressions:—

1. $\frac{x^2 - 5x + 4}{x^2 + 2x - 24}, \frac{x^3 - 3x + 2}{x^3 + 4x^2 - 5}.$

2. $\frac{6x^2 + 29x + 35}{14x^2 + 39x + 10}, \frac{2x^3 + 7x - 9}{5x^3 - 3x^2 - 4x + 2}.$

3. $\frac{a^4 - 2a^2b^2 + b^4}{a^3 - 4a^2b + 4ab^2 - b^3}, \frac{24a^3 - 28a^2b + 6ab^2 - 7b^3}{6a^2 + 11ab - 21b^2}.$

4. $\frac{x^2(y^2 - z^2) - xy(2y^2 + yz - z^2) + y^3(y + z)}{x^2(y + z)^2 - xy(2y^2 + 3yz + z^2) + y^3(y + z)}, \frac{x^4 + x^2y^2 + y^4}{x^6 - y^6}.$

5. $\frac{1}{a + b} + \frac{1}{a - b}, \frac{1}{a - b} - \frac{1}{a + b}.$

6. $\dfrac{a}{a+b} + \dfrac{b}{a-b} - \dfrac{ab}{ab-b^2} + \dfrac{ab}{a^2+ab}$.

7. $\dfrac{1}{4(x-1)} - \dfrac{x-1}{4(x^2+1)} - \dfrac{x-1}{2(x^2+1)^2}$.

8. $\dfrac{5}{x+1} - \dfrac{2}{(x+2)^2} - \dfrac{18}{5(x+2)} - \dfrac{7x+1}{5(x^2+1)}$.

9. $\dfrac{11}{x+1} - \dfrac{11}{x+3} - \dfrac{14}{(x+1)^2}$.

10. $\dfrac{8}{(x-1)^4} - \dfrac{4}{(x-1)^3} - \dfrac{2}{(x-1)^2} + \dfrac{16x+14}{3(x^2-1)} - \dfrac{16x-8}{3(x^2-x+1)}$.

11. $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-a)(b-c)} + \dfrac{1}{(c-a)(c-b)}$.

12. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-a)(b-c)} + \dfrac{c}{(c-a)(c-b)}$.

13. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-a)(b-c)} + \dfrac{c^2}{(c-a)(c-b)}$.

14. $\dfrac{a^3}{(a-b)(a-c)} + \dfrac{b^3}{(b-a)(b-c)} + \dfrac{c^3}{(c-a)(c-b)}$.

15. $\dfrac{1}{\left(\frac{a}{b}-1\right)\left(\frac{a}{c}-1\right)} + \dfrac{1}{\left(\frac{b}{a}-1\right)\left(\frac{b}{c}-1\right)} + \dfrac{1}{\left(\frac{c}{a}-1\right)\left(\frac{c}{b}-1\right)}$.

16. $\dfrac{(a+b)^2+(b-c)^2+(a+c)^2}{(a+b)(b-c)(a+c)} - \dfrac{2}{a+c} - \dfrac{2}{b-c} + \dfrac{2}{a+b}$.

17. $\left\{\dfrac{1}{a+x} + \dfrac{1}{a-x} + \dfrac{2a}{a^2+x^2}\right\} \cdot \left\{\dfrac{1}{a+x} - \dfrac{1}{a-x} - \dfrac{2x}{a^2+x^2}\right\}$.

18. $\left\{\dfrac{a^2+b^2}{a^2-b^2} + \dfrac{a-b}{a+b}\right\} \div \left\{2 + \dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a-b}{a+b}\right\}$.

19. $\left\{1 - \dfrac{2x^2}{a^2} + \dfrac{2x^4}{a^2(a^2+x^2)}\right\} \times \left\{\dfrac{2a^4}{x^2(a^2-x^2)} - \dfrac{2a^2}{x^2} - 1\right\}$.

20. $\dfrac{a}{c} + \dfrac{b}{a} + \dfrac{c}{b} - \dfrac{a^2\left(\frac{b}{c}+1\right) + b^2\left(\frac{c}{a}+1\right) + c^2\left(\frac{a}{b}+1\right)}{ab+bc+ca}$.

21. $\left\{\left(\frac{x+y}{x-y}\right)^2+1\right\}\cdot\left\{\left(\frac{x+z}{x-z}\right)^2+1\right\}\cdot\left\{\left(\frac{y+z}{y-z}\right)^2+1\right\}\times$

$$\frac{x^2(y-z)+y^2(z-x)+z^2(x-y)}{x^4y^2+x^2y^4+x^4z^2+x^2z^4+y^4z^2+y^2z^4+2x^2y^2z^2}.$$

22. $\left\{\frac{1-x^2}{1-x^3}+\frac{1-x}{1-x+x^2}\right\}\div\left\{\frac{1+x}{1+x+x^2}-\frac{1-x^2}{1-x^3}\right\}.$

23. $\frac{a^2}{(a+b)(a+c)(x-a)}+\frac{b^2}{(a+b)(b-c)(x+b)}-$

$$\frac{c^2}{(a+c)(b-c)(x+c)}.$$

24. $\left\{\frac{9x^2}{a^2}-\frac{24x}{a}+16-\frac{25a^2}{x^2}\right\}\div\left\{\frac{3x}{a^2}-\frac{4}{a}-\frac{5}{x}\right\}.$

25. $\left\{x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)\right\}\times\left\{\frac{1}{x}+\frac{1}{x^3}+\frac{1}{x^3(x^2-1)}\right\}^3.$

26. $$\frac{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2-2}{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2+2}.$$

27. $\frac{b^2}{(a+b)^2}+\frac{a^2}{(a-b)^2}-\frac{2b}{a+b}-\frac{2a}{a-b}+2.$

28. $\frac{x^2}{(x-y)(a+x)}+\frac{y^2-2xy}{(x-y)(a+y)}+\frac{y^2}{(a+y)^2}.$

29. $$\frac{\frac{x}{y}+\frac{y}{x}-1}{\frac{x^3}{y^3}-1}\times\frac{\frac{x}{y}+\frac{y}{x}+1}{\frac{x^3}{y^3}+1}\times\left(\frac{x^4}{y^4}-\frac{x^2}{y^2}\right).$$

30. $$\left\{\frac{1}{x}-\frac{1}{x+\frac{1}{y+\frac{1}{z}}}\right\}\left(y+\frac{1}{x}+\frac{1}{z}\right).$$

CHAPTER VI.

Simple Equations.

57. When two algebraical expressions are connected by the sign (=), they are said to form an equation.

When the equality is such that it is true for *all* values of the letters in the given expressions, it is called *an identity.*

Thus, $(x + a)(x + b) = x^2 + (a + b)x + ab$
and $(a + b)^2 - (a - b)^2 = 4ab$ } are identities.

58. When the condition of equality is such that some one or other of the letters must have particular values or a limited number of values, the statement of equality is termed an *equation of condition*, or, more briefly, *an equation.*

Thus, it may be found on trial that the equality

$$4x + 2 = 3x + 5$$

is true only when $x = 3$. Such an expression is therefore an equation.

59. The letters of an equation to which particular or a limited number of values must be given are termed *unknown* quantities.

Equations may contain *one*, *two*, *three*, or more unknown quantities.

The determination of the particular value or values of the unknown quantities is called the *solution* of the equation, and each of the values which *satisfies* the equation is said to be a *root* of the equation.

60. The expressions on the left and right sides of the sign (=) are termed the first and second sides respectively. It follows, therefore, that—

1. *If both sides of an equation be multiplied by the same quantity, the equation still subsists.*

2. *If both sides be divided by the same quantity, the equation still holds.*

3. *Any term may be transposed from one side to the other if the sign of the term be changed.*

Thus, if $3x + a = b$, we must have also

$$3x = b - a,$$

for this results from subtracting a from each side of the equation.

4. *The equation holds if every term on both sides has its sign changed.*

Thus, if $ax + b = cx - d$, we may reason as follows:—

The quantity $(ax + b)$ looked upon as a whole is given equal to the quantity $(cx - d)$ looked upon as a whole. If we change the *qualities* of these quantities, they will evidently be still equal.

$$\text{Hence, } -(ax + b) = -(cx - d)$$
$$\text{or, } -ax - b = -cx + d.$$

Now, this is the result of changing the sign of every term on both sides of the given equation.

5. *The sides of an equation may be reversed without destroying the equality.*

Thus, if $mx + n = px + q$, it must also follow that

$$px + q = mx + n.$$

6. *The sides of an equation may be raised to the* SAME POWER, *or we may extract the same root of both sides, and the equation still subsists.*

61. Simple equations are those in which the unknown quantities are not higher than the first degree, when the equations are reduced to a rational integral form.

The following is the general method adopted in solving a simple equation involving only one unknown quantity—

1. *Clear of fractions if necessary.*

2. *Transpose all the terms involving the unknown quantity to the first side of the equation, and all the remaining terms to the second side.*

3. *Simplify both sides if necessary, and divide both sides by the coefficient of the unknown quantity.*

Ex. 1. Solve the equation $5x + 6 = 3x + 12$.

Transposing the terms, we have—

$$5x - 3x = 12 - 6.$$

Now, simplifying, we get—

$$2x = 6;$$

and dividing each side by the coefficient of the unknown quantity, viz., by 2, we have—

$$x = 6 \div 2 = 3.$$

VERIFICATION.— Putting the value 3 for x in each side of the given equation, the first side becomes $5 \times 3 + 6$ or 21 ; and the second side becomes $3 \times 3 + 12$ or 21. The value of x found therefore *satisfies* the given equation.

Ex. 2. Given $\frac{x-2}{2} + \frac{x}{3} = 20 - \frac{x-6}{2}$, find x.

Clearing of fractions, by multiplying *every term* on each side by the L.C.M. of the denominators, viz., by 6, we get—

$$3(x-2) + 2x = 20 \times 6 - 3(x-6).$$

(Beginners often neglect to multiply integral terms such as 20 by the L.C.M.)

or $3x - 6 + 2x = 120 - 3x + 18$, or, transposing,

$$3x + 2x + 3x = 120 + 18 + 6, \text{ or, simplifying,}$$

$$8x = 144,$$

or dividing each side by 8, the coefficient of x, we have—

$$x = 18, \text{ the value required.}$$

(It will be good practice for the student to verify this result as in the last example).

Ex. 3. $\frac{4x-21}{7} + 7\frac{5}{6} + \frac{7x-28}{3} = x + 3\frac{3}{4} - \frac{9-7x}{8} + 1\frac{1}{2}$.

It is sometimes convenient to first *partially* clear off fractions. Thus, multiplying each side by 72, we have—

$$\frac{72(4x-21)}{7} + 47 \times 12 + 24(7x - 28)$$

$$= 72x + 15 \times 18 - 9(9 - 7x) + 6;$$

$$\text{or } \frac{288x}{7} - 72 \times 3 + 564 + 168x - 672,$$

$$= 72x + 270 - 81 + 63x + 6;$$

or, transposing,

$$41\tfrac{1}{7}x + 168x - 72x - 63x$$

$$= 270 - 81 + 6 + 216 - 564 + 672;$$

or, simplifying,

$74\frac{1}{7}x = 519$; or, multiplying each side by 7,

$519x = 519 \times 7$;

$\therefore x = 7.$

Ex. XV.

1. $5x + 2 = 2x + 11.$

2. $\frac{x}{4} + \frac{x}{6} + \frac{x}{8} = 2\frac{1}{6}.$

3. $2x + a = 3x - b.$

4. $3(x-7) + 4x = 2(2x - 4) + 2.$

5. $\frac{x-1}{2} + \frac{x+3}{4} = \frac{2x-7}{6} + \frac{8x-1}{12}.$

6. $\frac{7x-8}{13} + \frac{3x-2}{7} = \frac{10x+3}{11} - \frac{7x-3}{18}.$

7. $\frac{4x-15}{9} - \frac{2x+3}{6} = \frac{5x-1}{12} - 3\frac{1}{1}\frac{1}{8}.$

8. $\frac{8(5x+2)}{3} = \frac{2x-1}{8} = \frac{17x-2}{4} + \frac{5\frac{3}{8} + 80x}{7}.$

9. $ax + bc = bx + ac.$

10. $\frac{x}{a} + \frac{a}{b} = \frac{x}{b} + \frac{b}{a}.$

11. $\frac{x-a}{b} + \frac{x-b}{a} = 2 + \frac{cx - c^2}{ab}.$

12. $abx + b^3 = b^2x + a^3.$

13. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = ab + ac + bc.$

14. $\frac{x+b}{a} = \frac{a+x}{b}.$

15. $\frac{ax + bx + cx}{abc} = a + b + c.$

16. $x - \frac{a^2 - 3bx}{a^2} - b^2 = \frac{bx}{a} + \frac{6bx - 5a^2}{2a^2} - \frac{bx + 4a}{4a}.$

17. $\cdot 15x + \cdot 025 = \cdot 075x + \cdot 175.$

18. $\frac{3 - \cdot 125x}{8} + \cdot 1\dot{6} = \frac{2 + \cdot 1875x}{7} - \cdot 08\dot{3}.$

19. $\dfrac{1}{2x} + \dfrac{1}{12x} = \dfrac{1}{6x} + 1\frac{2}{3}$.

20. $(x + a)(x + b) = (x + c)(x + d)$.

21. $(x - a)(x - b) = (x - \overline{a + b})^2$.

22. $\dfrac{1}{ax} + \dfrac{1}{bx} + \dfrac{1}{cx} = \frac{1}{2}(a + b + c)^2 - \frac{1}{2}\left(\dfrac{a}{bcx} + \dfrac{b}{acx} + \dfrac{c}{abx}\right)$.

23. $\dfrac{x - a}{bc} + \dfrac{x - b}{ac} + \dfrac{x - c}{ab} = 2\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right)$.

24. $\dfrac{1 - ax}{bc} + \dfrac{1 - bx}{ac} + \dfrac{1 - cx}{ab} = \left(\dfrac{2}{a} + \dfrac{2}{b} + \dfrac{2}{c}\right)x$.

Abbreviated Methods for Particular Cases.

62. When the unknown quantity is involved in both numerator and denominator, it is often convenient to reduce such fractions to *mixed numbers*.

Ex. Solve the equation $\dfrac{6x - 7}{x + 2} - \dfrac{12x + 18}{3x - 5} = 2$.

By division we get $\dfrac{6x - 7}{x + 2} = 6 - \dfrac{19}{x + 2}$;

and $\dfrac{12x + 18}{3x - 5} = 4 + \dfrac{38}{3x - 5}$.

Hence the given equation becomes—

$$\left(6 - \frac{19}{x + 2}\right) - \left(4 + \frac{38}{3x - 5}\right) = 2;$$

$$\text{or } 6 - \frac{19}{x + 2} - 4 - \frac{38}{3x - 5} = 2;$$

$$\text{or, transposing, } -\frac{19}{x + 2} = \frac{38}{3x - 5} + 2 - 6 + 4;$$

$$\text{or } -\frac{19}{x + 2} = \frac{38}{3x - 5};$$

or, dividing each side by -19, we have—

$$\frac{1}{x + 2} = -\frac{2}{3x - 5}.$$

Hence, multiplying each side by the L.C.M. of the denominators—

$$3x - 5 = -2(x + 2) = -2x - 4,$$
$$\text{or } 3x + 2x = -4 + 5,$$
$$\text{or } 5x = 1,$$
$$\therefore x = \tfrac{1}{5}.$$

63. *When each side of an equation consists solely of a single fraction, the numerator of either fraction may change places with the denominator of the other.*

Let $\frac{a}{b} = \frac{p}{q}$ be the equation.

Multiply each side by b, then, by Art. 60 (1.)—

$$\frac{a}{b} \times b = \frac{p}{q} \times b, \text{ or } a = \frac{pb}{q}.$$

Divide each side by p, then, by Art. 60 (2.)—

$$\frac{a}{p} = \frac{pb}{q} \div p, \text{ or } \frac{a}{p} = \frac{b}{q}.$$

Here the denominator b of the first side of the given equation has changed places with the numerator p of the second side.

And similarly we may show that $\frac{q}{b} = \frac{p}{a}$, where the other numerator and denominator have changed places.

COR. *The two sides of an equation of the form* $\frac{a}{b} = \frac{p}{q}$ *may be inverted.*

For interchanging p and b in the last result, viz., $\frac{q}{b} = \frac{p}{a}$, we get $\frac{q}{p} = \frac{b}{a}$, and therefore also, by Art. 60 (5.), we have $\frac{b}{a} = \frac{q}{p}$.

(The student is *cautioned* against inverting the *separate terms* of the two sides of an equation when there are more than one term on each side.)

64. When each side of an equation consists solely of a single fraction, we may perform the following operations :—

1. *We may add or subtract the numerator and denominator of* EACH *fraction for a new numerator or denominator, and retain either the original numerator or denominator for the other term of the fraction, both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{p}{q}$, we have—

$$\text{(i.)}\ \frac{a+b}{b} = \frac{p+q}{q}, \quad \text{(ii.)}\ \frac{a-b}{b} = \frac{p-q}{q},$$

$$\text{(iii.)}\ \frac{a+b}{a} = \frac{p+q}{p}, \quad \text{(iv.)}\ \frac{a-b}{a} = \frac{p-q}{p},$$

or, (v.) we may have equations formed by inverting each of these.

These results are easily obtained—

For, since $\frac{a}{b} = \frac{p}{q}$, we have, adding unity to each side—

$$\frac{a}{b} + 1 = \frac{p}{q} + 1 \text{ or } \frac{a+b}{b} = \frac{p+q}{q}.$$

And so, by subtracting unity from each side, we get—

$$\frac{a-b}{b} = \frac{p-q}{q}, \text{ and so on.}$$

2. *We may take the* SUMS *of the numerator and denominator of each for new numerators or denominators, and the* DIFFERENCES *for the other terms of the fraction; and* VICE VERSA, *both sides being always similarly treated.*

Thus, if $\frac{a}{b} = \frac{c}{d}$, we have also $\frac{a+b}{a-b} = \frac{c+d}{c-d}$,

$$\text{and}\quad \frac{a-b}{a+b} = \frac{c-d}{c+d},$$

for we have just shown that $\frac{a+b}{b} = \frac{p+q}{q}$,

$$\text{and}\quad \frac{a-b}{b} = \frac{p-q}{q}.$$

Hence, dividing equals by equals, we get—

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{p+q}{q} \div \frac{p-q}{q},$$

or $\dfrac{a+b}{a-b} = \dfrac{p+q}{p-q}$, which is the first result.

And inverting each side, we have, by Art. 63 (Cor.)—

$$\frac{a-b}{a+b} = \frac{p-q}{p+q},$$

Ex. 1. Solve the equation $\dfrac{mx+a+b}{nx-c-d} = \dfrac{mx+a+c}{nx-b-d}$.

By Art. 63, we have $\dfrac{mx+a+b}{mx+a+c} = \dfrac{nx-c-d}{nx-b-d}$.

Then, by Art. 64 (1.), retaining the numerators and taking the *differences* for new denominators, we have—

$$\frac{mx+a+b}{b-c} = \frac{nx-c-d}{b-c};$$

or, multiplying each side by $(b-c)$—

$mx+a+b = nx-c-d$; or, transposing—

$$(m-n)\,x = -(a+b+c+d);\ \therefore\ x = -\frac{a+b+c+d}{m-n}.$$

Ex. 2. Solve $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = a.$

We may consider the quantity a as a fraction whose denominator is *unity*, or as $\frac{a}{1}$.

Then, Art. 64 (2.), taking the *sum* and *difference*, we have—

$$\frac{2\sqrt{a+x}}{2\sqrt{a-x}} = \frac{a+1}{a-1};\ \text{or—}$$

$$\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{a+1}{a-1};\ \text{or, squaring—}$$

$$\frac{a+x}{a-x} = \frac{a^2+2a+1}{a^2-2a+1}.$$

Again, taking the *difference* and *sum*, we have—

$$\frac{2x}{2a} = \frac{4a}{2a^2 + 2}; \text{ or—}$$

$$\frac{x}{a} = \frac{2a}{a^2 + 1}.$$

$$\therefore x = \frac{2a^2}{a^2 + 1}.$$

65. We now give an example to show that sometimes the easy solution depends on an advantageous arrangement of the terms on the two sides of the equation.

Ex. Solve $\sqrt{x + 4} + \sqrt{x - 3} = 7$.

Transposing, we have—

$\sqrt{x + 4} = 7 - \sqrt{x - 3}$; squaring, then—

$$x + 4 = 49 - 14\sqrt{x - 3} + (x - 3);$$

subtracting x from each side and transposing, then—

$$14\sqrt{x - 3} = 49 - 3 - 4 = 42.$$

$\therefore \sqrt{x - 3} = 3$; or squaring,

$$x - 3 = 9; \quad \therefore x = 3 + 9 = 12.$$

Should the student commence by squaring at once, he will render the equation more complicated.

Ex. XVI.

1. $\dfrac{3x + 7}{x + 4} = \dfrac{3x - 13}{x - 4}$.

2. $(x - a)(x - b) = (x - c)(x - d)$.

3. $\dfrac{3x + 13}{15} - \dfrac{3x + 10}{5x - 50} = \dfrac{x}{5}$.

4. $\dfrac{1 - 25x}{15} - \dfrac{3 - 2\frac{1}{2}x}{14(x-1)} = \dfrac{28 - 5x}{3} - \dfrac{10x - 11}{30} + \dfrac{x}{3}$.

5. $\dfrac{x - 4}{6x + 5} + \dfrac{3x - 13}{18x - 6} = \dfrac{1}{3}$.

6. $\dfrac{3 - 2x}{1 - 2x} - \dfrac{5 - 2x}{7 - 2x} = 1 - \dfrac{4x^2 - 2}{7 - 16x + 4x^2}$.

7. $\dfrac{18x - 22}{13 - 2x} + 6x + \dfrac{1 + 16x}{8} = 13\frac{1}{4} - \dfrac{101 - 64x}{8}$.

8. $\dfrac{6x + 5}{3x + 1} + \dfrac{58\frac{1}{2} + 14x}{9 + 2x} = 9$.

9. $\dfrac{3}{4x - 9} + \dfrac{1\frac{1}{2}}{6x - 21} = \dfrac{1}{x - 2\frac{1}{2}}$.

10. $\dfrac{4x - 7}{2x - 9} + \dfrac{2 - 14x}{7} + \dfrac{3\frac{1}{3} + x}{14} = \dfrac{10 - 3\frac{6}{7}x}{2} - \dfrac{19}{21}$.

11. $\dfrac{6x^2 - 7x - 6\frac{2}{3}}{2x - 3} - \dfrac{9x^2 - 12x - 19}{3x - 5} = \dfrac{17}{6x - 7}$.

12. $\dfrac{ax + m + 1}{ax + m - 1} + \dfrac{ax + n}{ax + n - 2} = \dfrac{ax + m}{ax + m - 2} + \dfrac{ax + n + 1}{ax + n - 1}$.

13. $\dfrac{\sqrt{x - a + b} - \sqrt{x + a - b}}{\sqrt{x - a + b} + \sqrt{x + a - b}} = \dfrac{a - b}{a + b}$.

14. $\dfrac{1 - \sqrt{1 - \sqrt{1 - x}}}{1 + \sqrt{1 - \sqrt{1 - x}}} = b$.

15. $\sqrt{2x + 10} + \sqrt{2x - 2} = 6$.

16. $\sqrt{8 - x} - \dfrac{3}{\sqrt{1 - x}} = \sqrt{1 - x}$.

17. $\sqrt[3]{1 + \sqrt{x}} + \sqrt[3]{1 - \sqrt{x}} = 2$.

18. $\dfrac{ax + 1 + \sqrt{a^2x^2 - 1}}{ax + 1 - \sqrt{a^2x^2 - 1}} = b$.

19. $\dfrac{a}{\sqrt{x} - \sqrt{b}} - \dfrac{b}{\sqrt{x} - \sqrt{a}} = \dfrac{a - b}{\sqrt{x}}$.

20. $\dfrac{a}{\sqrt{x} + \sqrt{a}} = \dfrac{a}{\sqrt{x} - \sqrt{a}} + \sqrt{a}$.

21. $\sqrt{\dfrac{\sqrt{x + a}}{\sqrt{x} - c}} - \sqrt{\dfrac{\sqrt{x} - a}{\sqrt{x} + a}} = \sqrt{x - a^2}$.

22. $\dfrac{1}{x-2} - \dfrac{2 + \frac{5}{2}x^2 - \frac{1}{2}x^3}{6 - 5x + x^2} - \frac{1}{2}x = \dfrac{5}{x-3}$.

23. $\dfrac{4x+5}{x+1} + \dfrac{x+5}{x+4} = \dfrac{2x+5}{x+2} - \dfrac{x^2-10}{x+3} + x$.

24. $\dfrac{ax+b}{cx+d} + \dfrac{cx+e}{ax+f} = \dfrac{a^2+c^2}{ac}$.

Problems producing Simple Equations involving One Unknown Quantity.

66. To solve an algebraical problem we represent the required or unknown quantity by a letter, as x, and then express the given conditions in algebraical language. Thus we form an equation, the solution of which gives the required value of the unknown quantity.

Ex. 1. My purse and money are together worth 24 shillings, and the money is worth seven times the purse. Find the value of each.

Let x = the value in shillings of the purse,
Then $7x$ = ,, ,, money.

Now, by problem the value of both together is 24 shillings.
Hence we have—

$$x + 7x = 24$$
$$\text{or } 8x = 24$$
$$\therefore x = 3, \text{ the value in shillings of the purse,}$$
$$\text{and } \therefore \text{ also } 7x = 7 \times 3 = 21, \quad ,, \quad \text{money.}$$

Ex. 2. What number is that to which, if 36 be added, the sum shall be equal to 3 times the number?

Let x = the number;
$\therefore x + 36$ = the sum when 36 is added,
and $3x$ = 3 times the number.

Hence, by problem—

$$x + 36 = 3x,$$
$$\text{or } x - 3x = -36,$$
$$\text{or } -2x = -36;$$
$$\therefore x = \frac{-36}{-2} = 18, \text{ the number required.}$$

Ex. 3. The distance between two towns is such that a train, whose speed is 30 miles an hour, takes 1 hour more in going 10 miles over 5 times the distance than a train whose speed is 20 miles an hour takes in going within 4 miles of 3 times the distance. Find the distance between the towns.

Let x = the distance required in miles.

Then $5x + 10 = 5$ times the distance together with 10 miles,

and $\frac{5x + 10}{30}$ = time in hours to travel this distance at 30 miles an hour.

And so, $\frac{3x - 4}{20}$ = time in hours to travel 4 miles less than 3 times the required distance, at 20 miles an hour.

But by the problem the former of these times exceed the latter by 1 hour.

Hence $\frac{5x + 10}{30} - \frac{3x - 4}{20} = 1.$

From this we easily find $x = 28$.

Hence 28 miles is the distance required.

Ex. 4. Find the price of an article, when as many can be bought for 1s. 4d. as can be bought for 2s. after the price has been raised 1d.

Let x = the price required in *pence;*

Then $\frac{16}{x}$ = number of articles bought for 1s. 4d.

And $x + 1$ = the raised price in *pence.*

$\therefore \frac{24}{x + 1}$ = number of articles bought for 2s. at the raised price.

But, by the problem, the number of articles in each case is the same.

Hence $\frac{16}{x} = \frac{24}{x + 1}$, from which $x = 2$.

Hence 2d. is the price required.

Ex. 5. A man sells geese at as many shillings each as the number he has, and having returned 5s., finds that if he had

had 2 more to sell on the same condition, and had returned 3s., he would have had 38s. more. How many had he?

Let x = the number required;

Then $x^2 - 5$ = number of shillings received.

Also, on the second supposition,

$(x + 2)^2 - 3$ = number of shillings he would have received.

Now, by the problem, this latter number is 38 more than the former.

Hence $(x + 2)^2 - 3 = x^2 - 5 + 38$, from which we find $x = 8$.

Ex. 6. A waterman finds that he can row 5 miles in $\frac{3}{4}$ hour with the tide, and that it takes him $1\frac{1}{2}$ hours to row the same distance against the tide when it is but half as strong. What is the velocity of the tide?

Let x = the velocity of the tide in miles per hour.

Now the velocity of the boat when going *with* the tide $= 5 \div \frac{3}{4} = \frac{20}{3}$.

$\therefore \frac{20}{3} - x$ = velocity of the boat when there is no tide.

Again, velocity of boat *against* the tide when it is half as strong $= 5 \div 1\frac{1}{2} = \frac{10}{3}$.

$\therefore \frac{10}{3} + \frac{x}{2}$ = velocity of the boat, when there is no tide.

Hence we have—

$$\frac{10}{3} + \frac{x}{2} = \frac{20}{3} - x\text{; from which}$$

$$x = 2\tfrac{2}{9}.$$

$\therefore$ The velocity required is $2\frac{2}{9}$ miles per hour.

Ex. XVII.

1. If I add 25 to 3 times a certain number, I obtain the same result as if I subtract 25 from 8 times the number. Find the number.

2. Divide 70 into 2 such parts that the one shall be as much above half the number as the other is above 15.

3. Divide £720 among A, B, and C, so that B may have twice as much as C, and A as much as B and C together.

4. There are two trains, one of which goes 5 miles an hour faster than the other, and the former performs a journey of 100 miles, while the latter goes 75 miles. Find their respective rates.

5. A horse when let out for hire brings in *a clear gain* of 10s. per day, but costs 1s. 6d. daily for food. At the end of 30 days his master had gained £11. 11s. Required the number of days for which he was hired.

6. A and B have 4 guineas between them, and play at hazard. B loses $\frac{1}{6}$ of his money, and afterwards gains $\frac{1}{10}$ of what he then had. It is then seen that B has as much money as A had at the end of the first game. How much had each at first?

7. A and B have respectively an equal number of florins and crowns. B pays a debt of 4s. to A, and then A's money is just half B's. Find what each had.

8. A workman, instead of adopting the 9 hours' system, worked 10 hours daily, and had a corresponding rise of wages. By this means his wages were increased 4s. weekly. Find his original wages.

9. A person who has regular wages of 26s. weekly, thinking to better himself, takes a job at higher wages. He is, however, put on half-time during 20 weeks of the year, and finds himself at the end of the year £4. 12s. worse off. Required his increased wages.

10. A company of men, arranged in a hollow square 4 deep, numbered 144. What was the number in a side of the square?

11. In an examination paper there were two series of questions, and the questions of the second series carried each 3 marks more than those of the first series. A candidate who attempted 3 of the first series, obtaining half marks for them, and 5 of the second series, obtaining for these full marks, got altogether 80 marks. Find the number of marks attached to each question of the first series.

12. A grocer has tea at 3s. 4d. and at 4s. He sells altogether 64 pounds, thereby realizing £12. How much did he sell of each?

13. If 11 be subtracted from 5 times a certain number, and the remainder divided by 6, the quotient will exceed by 2 the quotient obtained by subtracting 3 from 4 times the number and dividing the remainder by 7. Find the number.

14. A garrison of 1,250 men were provisioned for 64 days; but after 22 days a certain number were called away, and it was found that the remaining provisions lasted the number left for 70 days. Find the number told off.

15. At a railway station £15 was taken for single fares, and £33. 15s. for returns. The number of return tickets exceeded the single tickets by 10, and the price of a return ticket was half as much again as a single ticket. Find the fare for a single journey.

16. In a tour lately made round the world, the distance travelled by water was 20,000 miles, and by land 8,000 miles; and the whole time taken was 220 days. Supposing the rate by water to be two-thirds of that by land, find the number of days travelled by land.

17. The distance between A and B is 32 miles. A person starting from A, at the rate of 4 miles an hour, meets another who started from B half an hour later, at a rate of $3\frac{1}{2}$ miles an hour. At what point will they meet?

18. There are two clocks, one of which gains twice as much per day as the second loses, and they are set right at noon on Monday. When it is noon on Thursday by the first clock, it is 11·50 A.M. by the second. What is the gain per day of the first clock?

19. A draper raises his goods a certain rate per cent., and afterwards reduces them to the original price by lowering them $13\frac{1}{2}\frac{1}{3}$ per cent. Find the original rise per cent.

20. Required the distance between two towns such that a person can perform the journey one hour sooner when he walks 4 miles an hour than when he walks $3\frac{1}{2}$ miles an hour?

21. The sum of £12. 15s. is paid away with an equal number of sovereigns, crowns, and sixpences. Required the number of each.

22. A walks along an inclined plane at a certain rate, and B walks along the base of the plane at a rate of one-third of

a mile per hour less than A. The inclination of the plane is such that A is always vertically over B, and that at the end of half an hour they are exactly five-sixths of a mile apart. Find the respective rates of A and B.

23. There is a direct road over a hill between two stations at the foot of each side. The distance on the one side from the foot to the top is 5 miles, and the road down the other side forms a right angle with the road up. It is also known to be 1 mile less down the hill than the direct distance by tunnel between the two stations. Find the distance down the hill.

24. Two trains, whose respective lengths are 122 and 98 yards, and the former of which is going at the rate of 35 miles an hour, pass each other in 30 seconds. Find the rate and relative direction of the second train.

25. A man bought a number of sheep for £132, and having lost 10, and sold 20 that were diseased at 6s. per head below cost price, disposed of the remainder for £116, thereby realizing his outlay. How many did he buy?

26. A boy spends 10s. in oranges and apples. The oranges were bought at 5 for 6d., and the apples at 3 for 2d.; and their number together amounted to 132. What did he spend on each?

27. If B is allowed 2 hours more time than A takes to do a piece of work, he will do 4 times as much, and if C is allowed 1 hour more than A, he also can do 4 times as much. Moreover, D requires 4 hours more than A to do the piece of work. Also, the work done by A and B together is the same as that done by C and D together in the same time. Required the respective times for A, B, C, D to do a piece of work.

28. A person sells out £1,200 Three and a Half per Cent. stock, and invests the money in Two and a Half per Cents., whose price is 14 lower than the first-named stock. The loss in annual income is £7. Find the price of the first-named stock.

29. The banker's discount on a certain sum of money at 5 per cent. per annum is equal to the true discount on a sum £50 larger. Find the sum.

30. An express train, which ought to perform its journey in $2\frac{1}{2}$ hours, after having gone uniformly 80 miles, finds itself 6 minutes behind. However, by increasing the speed to as many miles per hour as there were miles in half the journey, it just arrived at its destination in time. Find the original speed of the train, and the length of the journey.

31. A vessel contains a quantity of spirit (sp. gr. ·9) and water, and a cylinder of wood (sp. gr. ·92), whose length is 10 inches, floats upright, so as to be just covered by the spirit. Find how much of the cylinder floats in the water.

32. A mixture of hydrogen and oxygen is found to condense when fired, to 16 vols. of steam. Now, every 3 vols. of such a mixture is known to condense to 2 vols., when the original gases are in the proportion of 2 : 1. Find the quantity of each.

33. A mixture of 100 grams of sodic and potassic sulphates yielded a gram of baric sulphate. Now, each gram of sodic sulphate yields b grams of baric sulphate, and each gram of potassic sulphate yields c grams of baric sulphate. Find the amount of sodic and potassic sulphates in the mixture.

34. If a oxen consume b acres of grass in c weeks, and a' oxen consume b' acres of grass in c' weeks, the grass growing uniformly, find the week's growth of an acre.

35. The freezing and boiling points of a common thermometer are marked 32° and 212° respectively; those on the centigrade thermometer are marked 0° and 100°. At what temperature do the graduations agree?

36. A person going at the rate of a miles per hour finds himself b hours too late when he has c miles farther to go. How much must he increase his speed to reach home in time?

Simultaneous Equations of the First Degree with two Unknown Quantities.

67. Suppose we have given the equation $3x - 4y = 5$, then by ascribing to y a series of values we get a corresponding series of values for x.

Thus we may have $\left.\begin{matrix} x = 3 \\ y = 1 \end{matrix}\right\}$, $\left.\begin{matrix} x = 7 \\ y = 4 \end{matrix}\right\}$, $\left.\begin{matrix} x = 11 \\ y = 7 \end{matrix}\right\}$, &c.

Again, if another equation, as $4x + y = 32$, be given, we may in the same way obtain a series of pairs of values which satisfy it. And, further, if the two equations are *distinct* and *compatible*, there is always *a pair of values common to the two equations.* This pair of values then satisfies both equations, and the equations are called **simultaneous** equations.

The methods of solving simultaneous equations will now be explained.

First Method.—*Equalize the coefficients of one of the unknown quantities in both equations, and add or subtract the equations so obtained, so as to obtain an equation with one unknown quantity.*

Ex. $\left.\begin{array}{rrrrl} 4x & + & 3y & = & 17 \ldots\ldots (1) \\ 9x & - & 5y & = & 3 \ldots\ldots (2) \end{array}\right\}$.

From (1), multiplying each side by 9, we get—

$$36x + 27y = 153 \ldots\ldots (3).$$

And from (2), multiplying each side by 4, we get—

$$36x - 20y = 12 \ldots\ldots (4).$$

(3) − (4), then $47y = 141$

$$\therefore y = 3.$$

Hence, substituting in (1), we have—

$4x + 3 \times 3 = 17$, from which we get—

$$x = 2.$$

Hence, the solution required is $x = 2$, $y = 3$.

Second Method.—*Express one of the unknown quantities in terms of the other by means of either equation, and substitute its value in the other.*

Taking the same example, we have—

$$\left.\begin{array}{rrrrl} 4x & + & 3y & = & 17 \ldots\ldots (1) \\ 9x & - & 5y & = & 3 \ldots\ldots (2) \end{array}\right\}.$$

From (1) we have $4x = 17 - 3y$, or $x = \dfrac{17 - 3y}{4} \ldots (3)$.

Substituting this value of x in (2) we have—

$$9\left(\frac{17 - 3y}{4}\right) - 5y = 3,$$

or, $153 - 27y - 20y = 12$, from which—

$$y = 3.$$

Then from (3) we get $x = \dfrac{17 - 3 \times 3}{4} = 2.$

THIRD METHOD.—*Express one of the unknown quantities in terms of the other by means of each equation, and equate the expression.*

We will this time express in each case y in terms of x.

We have $\left.\begin{aligned} 4x + 3y &= 17 \ldots\ldots (1) \\ 9x - 5y &= 3 \ldots\ldots (2) \end{aligned}\right\}$.

From (1) $3y = 17 - 4x$, or $y = \dfrac{17 - 4x}{3} \ldots (3)$.

And from (2) $5y = 9x - 3$, or $y = \dfrac{9x - 3}{5} \ldots\ldots (4)$.

Equating (3) and (4), then $\dfrac{17 - 4x}{3} = \dfrac{9x - 3}{5}$, from which

$$x = 2.$$

Then from (3) by substitution, $y = \dfrac{17 - 4 \times 2}{3} = 3.$

Simultaneous Equations of the First Degree of Three or more Unknown Quantities.

68. In the case of *three* unknown quantities we may obtain, from the three given equations, two equations with two unknown quantities, and then, by a similar method, from the two obtain an equation with one unknown quantity; and a like method may be pursued for more than three unknown quantities.

Ex. $\left.\begin{aligned} 3x + y + 4z &= 25 \ldots\ldots (1) \\ 4x + 3y - 5z &= -3 \ldots\ldots (2) \\ 6x + 7y - 8z &= 1 \ldots\ldots (3) \end{aligned}\right\}$

From (1) $12x + 4y + 16z = 100 \ldots\ldots (4)$

And from (2) $12x + 9y - 15z = -9 \ldots\ldots (5)$

(5) − (4), then $5y - 31z = -109 \ldots\ldots (6)$

Again, from (1) $6x + 2y + 8z = 50 \ldots\ldots (7)$

(3) − (7), then $5y - 16z = -49 \ldots\ldots (8)$

(8) − (6), then $15z = 60$

$\therefore z = 4$

Hence from (8), by substitution—

$$5y - 16 \times 4 = -49,$$
$$\text{from which } y = 3.$$

And hence from (1), by substitution of the known values of y and z—

$$3x + 3 + 4 \times 4 = 25,$$
$$\text{from which } x = 2.$$

Hence the respective values of x, y, z, are 2, 3, 4.

Ex. XVIII.

1. $6x + y = 22, 5x + 3y = 27.$
2. $4x - 3y = 14, 6x + 5y = 40.$
3. $3x + 5y = 44, y - x = 4.$
4. $\frac{x}{4} + \frac{y}{3} = 2\frac{2}{3}, \frac{3x}{5} + \frac{y}{10} = 2\frac{9}{10}.$
5. $\frac{x}{6} + \frac{y}{4} = 4, \frac{x}{4} + \frac{y}{6} = 4\frac{1}{3}.$
6. $\frac{x}{8} + \frac{y}{6} = 1\frac{1}{4}, x + \frac{y}{3} = 4.$
7. $3x + 4y + z = 14, 2x + y + 5z = 19, 5x + 2y + 3z = 18.$
8. $7x + 2y + 3z = 20, 3x - 4y + 2z = 1, 5y - 2x + 7z = 29.$
9. $2x - 5y = 3, 3y - 2z = -1, 4x + 2z = 20.$
10. $ax + by = c, a_1x + b_1y = c_1.$
11. $x + y = a, y + z = b, x + z = c.$
12. $ax + by = d, by + cz = e, ax + cz = f.$
13. $\frac{3x-2y}{4} + \frac{5x+2y}{6} = 5\frac{1}{2}, \frac{x+2y}{7} - \frac{3x-2y}{2} = 1.$
14. $\frac{x+y}{10} + \frac{x-y}{15} = \frac{23}{30}, \frac{x+y}{15} + \frac{x-y}{10} = \frac{7}{30}.$
15. $\frac{3}{x} + \frac{9}{y} = 1\frac{7}{8}, \frac{15}{x} - \frac{11}{y} = 2\frac{3}{8}.$

16. $\dfrac{4}{x-2} + \dfrac{6}{y} = 5\frac{1}{2}, \dfrac{4}{y} - \dfrac{1}{x-2} = 0.$

17. $4\,(x + 3\,y) - 3\,(x + y) = \frac{1}{3}\,(x + y)\,(x + 3\,y),$

$$\frac{20}{x + y} + \frac{9}{x + 3\,y} = 3.$$

18. $73\,y - 5\,x = (x - 5\,y)\,(x + 3\,y),$

$$\frac{2}{x - 5\,y} - \frac{5}{x + 3\,y} = \frac{7}{33}.$$

19. $\dfrac{1}{x} + \dfrac{1}{y} = a, \dfrac{1}{y} + \dfrac{1}{z} = b, \dfrac{1}{x} + \dfrac{1}{z} = c.$

20. $3\,x - 2\,y = 0, 4\,y - 3\,z = 0, 5\,z - 6\,u = 14, 7\,u - 2\,x = 3.$

21. $3\,x - 2\,y = 5\,z - 6\,y = 7\,x - 4\,z = 1.$

22. $x + y + z = a, y + z + u = b, z + u + x = c, u + x + y = d.$

23. $\dfrac{x}{m} + \dfrac{y}{n} = a, \dfrac{y}{n} + \dfrac{z}{p} = b, \dfrac{x}{m} + \dfrac{z}{p} = c.$

24. $x + ay + bz = \alpha, y + az + bx = \beta, z + ax + by = \gamma.$

25. $x + ay + a^2z + a^3 = 0.$
$x + by + b^2z + b^3 = 0.$
$x + cy + c^2z + c^3 = 0.$

26. $x + ay + a^2z + a^3u + a^4 = 0.$
$x + by + b^2z + b^3u + b^4 = 0.$
$x + cy + c^2z + c^3u + c^4 = 0.$
$x + dy + d^2z + d^3u + d^4 = 0.$

27. $x + ay = \alpha.$
$y + bz = \beta.$
$z + ct = \gamma.$
$t + dx = \delta.$
$u + ex = \varepsilon.$

28. $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = d$
$\dfrac{a'}{x} + \dfrac{b'}{y} + \dfrac{c'}{z} = d'$
$\dfrac{a''}{x} + \dfrac{b''}{y} + \dfrac{c''}{z} = d''$

29. $x_1 + 2\,x_2 + 3\,x_3 + \&c. + nx_n = a_1.$
$x_2 + 2\,x_3 + 3\,x_4 + \&c. + nx_1 = a_2.$
$\&c. = \&c.$
$x_n + 2\,x_1 + 3\,x_2 + \&c. + nx_{n-1} = a_n.$

30. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{x}{b} + \frac{y}{c} + \frac{z}{a} = \frac{x}{c} + \frac{y}{a} + \frac{z}{b} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Problems producing Simultaneous Equations of the First Degree.

69. There are certain problems which may be solved with much greater facility by the introduction of more than one unknown quantity.

Ex. 1. Nine men and seven women receive together £3. 2s. 8d., and three men receive 10d. more than four women. Find the receipts of each.

Let x and y respectively represent in pence the receipts of a man and woman.

Then, since £3. 2s. 8d. = 752d., expressing the conditions of the problem in algebraical language—

$$\left.\begin{aligned} 9x + 7y &= 752 \\ 3x - 4y &= 10 \end{aligned}\right\}$$ Solving these equations, we find—

$$x = 54,\ y = 38.$$

Hence, the receipts of a man and woman are respectively 54d. and 38d., or 4s. 6d. and 3s. 2d.

Ex. 2. Find a fraction such that, if we diminish its numerator by 1, it becomes equal to $\frac{1}{7}$; and if we increase its denominator by 1, it becomes equal to $\frac{1}{6}$.

Let $\frac{x}{y}$ be the required fraction.

Then, by problem, $\left.\begin{aligned} \frac{x-1}{y} &= \frac{1}{7} \\ \text{and } \frac{x}{y+1} &= \frac{1}{6} \end{aligned}\right\}$, which equations, when solved, give—

$$x = 6,\ y = 35.$$

$\therefore \frac{6}{35}$ is the fraction required.

Ex. 3. A's money, together with twice B's and thrice C's, amounts to £38; B's money, together with twice C's and thrice A's, to £35; and C's money, together with twice A's and thrice B's, also to £35. Find the money of each.

Let x, y, z be respectively the number of pounds each has. Then we have—

$$\left.\begin{aligned} x + 2y + 3z &= 38 \\ y + 2z + 3x &= 35 \\ z + 2x + 3y &= 35 \end{aligned}\right\}; \text{ from which we get—}$$

$$x = 5,\ y = 6,\ z = 7.$$

Hence, £5, £6, £7 are the respective moneys of A, B, and C.

Ex. XIX.

1. A and B engage in play. A puts down half-a-crown to B's florin. They play twenty games, and then it is found that A has won 4s. How many games did each win?

2. There is a number, the sum of whose two digits is 10, and, if 36 be added to the number, the digits change places. Find the number.

3. A grocer has tea at 3s. 4d. a lb., and sugar at 4s. a stone. He sells £2 worth of the two. If he had raised the tea 20 per cent., and lowered the sugar $12\frac{1}{2}$ per cent., and sold the same quantities of each, his profits would have been 1s. 6d. more. Find the quantity sold of each.

4. Eight times the numerator of a certain fraction exceeds three times the denominator by 3, and five times the numerator added to twice the denominator make 29. Find the fraction.

5. If the number of cows in a field were doubled, there would be 65 cows and horses together; but, if the number of horses be doubled, and that of the cows halved, there would be 46. How many are there of each?

6. Thirty shillings are spent in brandy, and 42s. in gin; 19 bottles being purchased in all. Had the 42s. been spent in brandy, and the 30s. in gin, 17 bottles only would have been bought. Find the cost per bottle of each.

7. A fishmonger receives 240 mackerel. He sells a certain number at 4 for a shilling, but the rest being seized as bad fish, and he being fined 10s., finds himself a loser by 9s. Had he sold them at 3 for a shilling, he would have been a gainer by 5s., if 13 more fish had been seized. How many did he sell, and what did he pay for the lot?

8. Three persons invest their money at 3, 4, 5 per cent: interest respectively. The total amount of interest is £38, and the interest of the first and third together is $2\frac{1}{6}$ that of the second; while the total interest would have been £34 had the rates been 5, 4, 3 per cent. respectively. Find the capital of each.

9. A toll-gate keeper receives 8s. 8d. for the toll of a number of horses, oxen, and sheep, the tolls for each being respectively $1\frac{1}{2}$d. 1d., $\frac{1}{2}$d. Had there been twice as many sheep and the number of horses diminished accordingly, he would have received 7s. 2d. Had the oxen passed through free, and the tolls for a horse and sheep respectively been 2d. and $\frac{3}{4}$d., he would have received 9s. $1\frac{1}{2}$d. Find the number of each.

10. A, B, C start from the same place. B after a quarter of an hour doubles his rate, while C, who falls, after ten minutes diminishes his rate $\frac{1}{6}$th. At the end of half an hour A is $\frac{1}{4}$ mile before B, and $\frac{1}{2}$ mile before C, and it is observed that the total distance which would have been walked by the three, had they each continued to walk uniformly from the first, is $6\frac{1}{4}$ miles. Find the original rate of each.

11. A, B, C, working 3, 4, 5 hours respectively, can do $2\frac{1}{2\frac{1}{4}}$ pieces of work; if they each work an hour more, they can finish an extra $\frac{1}{2\frac{3}{4}}$ of a piece; and, if C does not work, the other two, working for 1 and 6 hours respectively, can together finish 1 piece. Find the time required for A, B, C to finish separately a piece of work.

12. There are three numbers such that, if the first be increased by 6, and the second diminished by 5, the product of the results is the product of the first two numbers; if the second be increased by 2 and the third diminished by 3, the product of the results is the product of the second and third; and, if the first be increased by 3 and the third diminished by 6, the product is that of the first and third. Find the numbers.

13. A person performed a journey of $22\frac{1}{2}$ miles, partly by carriage at 10 miles an hour, partly by train at 36 miles an hour, and the remainder by walking at 4 miles an hour. He did the whole in 1 hour 50 minutes. Had he walked the

first portion, and performed the last by carriage, it would have taken him 2 hours $30\frac{1}{2}$ minutes. Find the respective distances by carriage, train, and walking.

14. A and B start from two places C and D, distant 28 miles, and it is found that A reaches D 3 hours after they meet. Had the distance between C and D been 35 miles, A would have reached a point 28 miles from C 2 hours after he met B. Find the respective rates of A and B.

15. Three trains—a luggage, ordinary, and express—move along three parallel pairs of rails, the distance between the stations being 120 miles. The first two start from the same station, and the express from the opposite. The luggage train, starting 2 hours first, is overtaken by the ordinary in 2 hours; and the express train, starting 1 hour after the ordinary, meets the luggage in 1 hour $7\frac{1}{2}$ minutes. Had all three started from the same station, the ordinary would have been overtaken in 2 hours. Find the respective rates of the trains.

16. If (a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) be the respective compositions by weight of three mixtures of three substances, and d_1, d_2, d_3 be the respective prices of the mixtures, find the price per unit of weight of each substance.

17. By alloying two ingots of gold in two given proportions, we form two new ingots of which the fineness of each is known. What is the fineness of each of the given ingots?

18. A group of n persons play as follows:—The first plays with the second and loses $\frac{1}{a}$ of what he has, the second then plays with the third and loses $\frac{1}{a}$ of what he has, the third then with the fourth, losing $\frac{1}{a}$ of what he has, &c., the nth with the first losing $\frac{1}{a}$ of what he has. At the end they each have b. What had each at first?

MATHEMATICS.

SECOND STAGE.

SECTION I.

GEOMETRY.

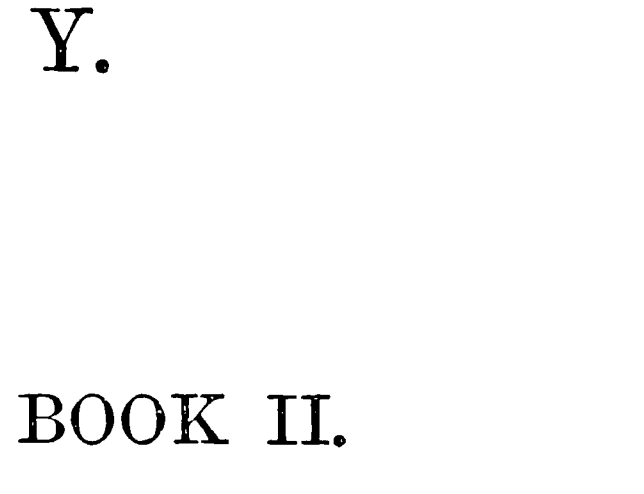
EUCLID'S ELEMENTS, BOOK II.

Definitions.

1. A **rectangle,** or **right-angled parallelogram,** is said to be contained by any two of the straight lines which contain one of the right angles.

2. **In any parallelogram,** the **figure** which is composed of either of the parallelograms about a diameter, together with the two complements, is called a *gnomon.*

Thus the parallelogram HG, together with the complements AF, FC, is a gnomon, which is briefly expressed by the letters AGK, or EHC, which are at the opposite angles of the parallelograms which make the gnomon.



The rectangle under, or contained by two lines, as AB and BC, is concisely expressed thus:—AB, BC.

Proposition 1.—Theorem.

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line, and the several parts of the divided line.

Let A and BC be two straight lines; and let BC be divided into any parts in the points D, E;

The rectangle contained by the straight lines A and BC shall be equal to the rectangle contained by A and BD, together with that contained by A and DE, and that contained by A and EC.

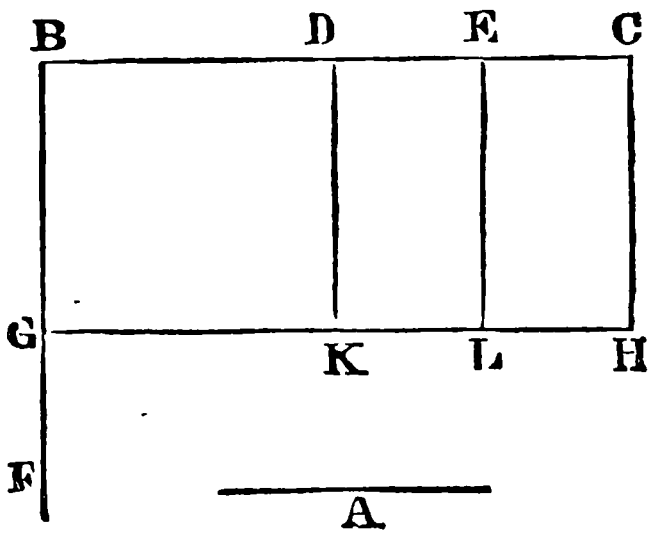
$A \cdot BC = A \cdot BD + A \cdot DE + A \cdot EC.$

Construction.—From the point B draw BF at right angles to BC (I. 11),

And make BG equal to A (I. 3).

Through G draw GH parallel to BC (I. 31),

And through the points D, E, C, draw DK, EL, CH parallel to BG (I. 31).



Proof.—Then the rectangle BH is equal to the rectangles BK, DL, EH.

Since BH = BK + DL + EH.

But BH is contained by A and BC, for it is contained by GB and BC, and GB is equal to A (Const.);

And BK is contained by A and BD, for it is contained by GB and BD, and GB is equal to A;

And DL is contained by A and DE, because DK is equal to BG, which is equal to A (I. 34);

And in like manner EH is contained by A and EC;

Therefore the rectangle contained by A and BC is equal to the several rectangles contained by A and BD, by A and DE, and by A and EC.

Therefore, if there be two straight lines, &c. *Q. E. D.*

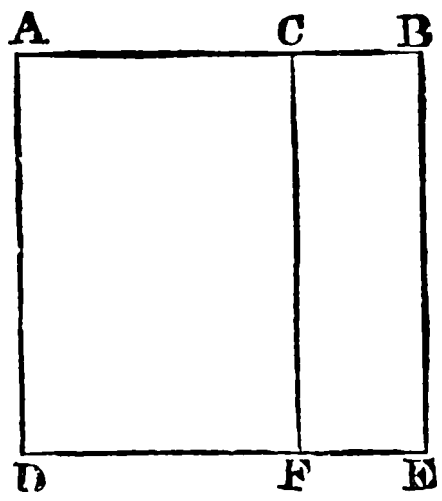
Proposition 2.—Theorem.

If a straight line be divided into any two parts, the rectangles contained by the whole line and each of its parts are together equal to the square on the whole line.

Let the straight line AB be divided into any two parts in the point C;

AB·BC +AB·AC = AB².

The rectangle contained by AB and BC, together with the rectangle contained by AB and AC, shall be equal to the square on AB.



CONSTRUCTION.—Upon AB describe the square ADEB (I. 46).

Through C draw CF parallel to AD or BE (I. 31).

PROOF.—Then AE is equal to the rectangles AF and CE.

But AE is the square on AB;

For AB² is the sum of its parts AF+ CE.

Therefore the square on AB is equal to the rectangles AF and CE.

And AF is the rectangle contained by BA and AC, for it is contained by DA and AC, of which DA is equal to BA;

And CE is contained by AB and BC, for BE is equal to AB.

And ∴ = AB·AC+ AB·BC.

Therefore the rectangle AB, AC, together with the rectangle AB, BC, is equal to the square on AB.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 3.—Theorem.

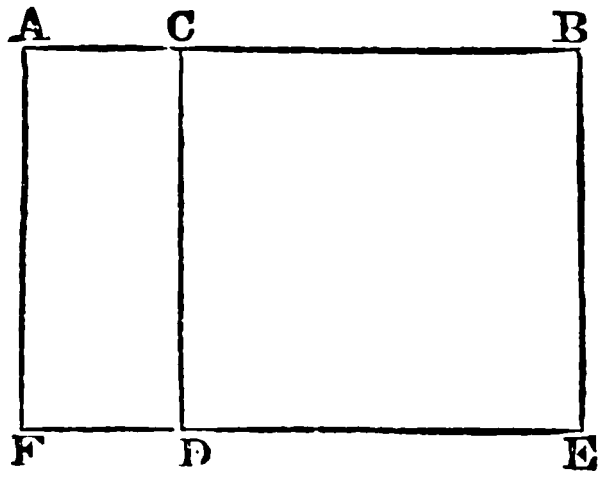
If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the square on that part, together with the rectangle contained by the two parts.

Let the straight line AB be divided into any two parts in the point C;

AB·BC= BC²+ AC·BC.

The rectangle AB·BC shall be equal to the square on BC, together with the rectangle AC·CB.

CONSTRUCTION.—Upon BC describe the square CDEB (I. 46).



Produce ED to F; and through A draw AF parallel to CD or BE (I. 31).

For AE = AD +CE.

PROOF.—Then the rectangle AE is equal to the rectangles AD and CE.

But AE is the rectangle contained by AB and BC, for it is contained by AB and BE, of which BE is equal to BC;

And AD is contained by AC and CB, for CD is equal to CB;

And CE is the square on BC.

Therefore the rectangle AB, BC is equal to the square on BC, together with the rectangle AC, CB.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 4.—Theorem.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

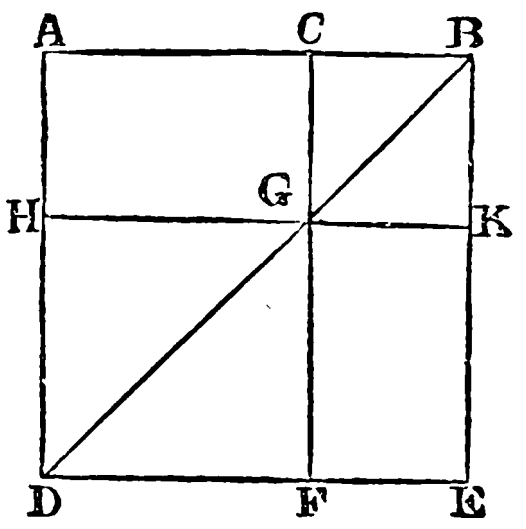
Let the straight line AB be divided into any two parts in C;

The square on AB shall be equal to the squares on AC and CB, together with twice the rectangle contained by AC and CB. $AB^2 = AC^2 + CB^2 + 2AC \cdot CB$.

Construction.—Upon AB describe the square ADEB (I. 46), and join BD.

Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).



Proof.—Because CF is parallel to AD, and BD falls upon them,

Therefore the exterior angle BGC is equal to the interior and opposite angle ADB (I. 29).

Because AB is equal to AD, being sides of a square, the angle ADB is equal to the angle ABD (I. 5);

Therefore the angle CGB is equal to the angle CBG (Ax. 1); Show first that CK is a square $= CB^2$.

Therefore the side BC is equal to the side CG (I. 6).

But CB is also equal to GK, and CG to BK (I. 34);

Therefore the figure CGKB is equilateral.

It is likewise rectangular.

For since CG is parallel to BK, and CB meets them, the angles KBC and GCB are together equal to two right angles (I. 29).

But KBC is a right angle (Const.), therefore GCB is a right angle (Ax. 3).

Therefore also the angles CGK, GKB, opposite to these, are right angles (I. 34).

Therefore CGKB is rectangular; and it has been proved equilateral; therefore it is a square; and it is upon the side CB.

So also $HF=AC^2$

For the same reason HF is also a square, and it is on the side HG, which is equal to AC (I. 34).

Therefore HF and CK are the squares on AC and CB.

And because the complement AG is equal to the complement GE (I. 43),

And $AG+GE=2\,AC\cdot CB$.

And that AG is the rectangle contained by AC and CG, that is, by AC and CB,

Therefore GE is also equal to the rectangle AC, CB;

Therefore AG, GE are together equal to twice the rectangle AC, CB;

And HF, CK are the squares on AC and CB.

Therefore the four figures HF, CK, AG, GE are equal to the squares on AC and CB, together with twice the rectangle AC, CB.

But HF, CK, AG, GE, make up the whole figure ADEB, which is the square on AB;

$\therefore$ whole figure or $AB^2 = AC^2+BC^2, +2\,AC\cdot CB$.

Therefore the square on AB is equal to the squares on AC and CB and twice the rectangle $AC\cdot CB$.

Therefore, if a straight line, &c. *Q. E. D.*

COROLLARY.—From this demonstration it follows that the parallelograms about the diameter of a square are likewise squares.

Proposition 5.—Theorem.

If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Let the straight line AB be bisected in C, and divided unequally in D;

$AD\cdot DB + CD^2 = CB^2$.

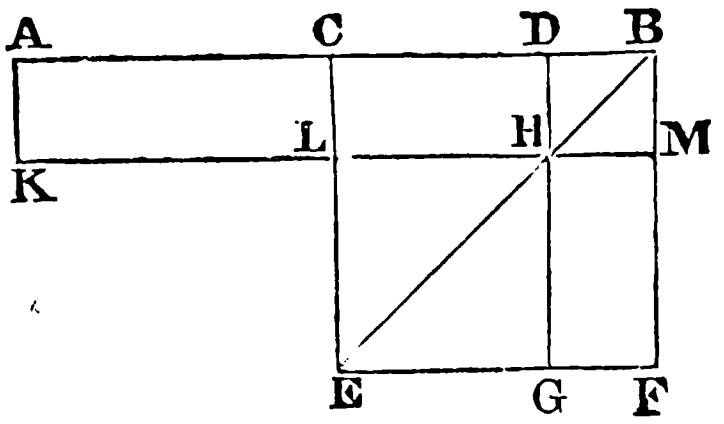
The rectangle AD, BD, together with the square on CD, shall be equal to the square on CB.

CONSTRUCTION.—Upon CB describe the square CEFB (I. 46), and join BE.

Through D draw DHG parallel to CE or BF (I. 31).

Through H draw KLM parallel to CB or EF.

And through A draw AK parallel to CL or BM.



PROOF.—Then the complement CH is equal to the complement HF (I. 43).

To each of these add DM; therefore the whole CM is equal to the whole DF (Ax. 2).

But CM is equal to AL (I. 36), because AC is equal to CB (Hyp.); For AL=CM =DF.

Therefore also AL is equal to DF (Ax. 1).

To each of these add CH; therefore the whole AH is equal to DF and CH (Ax. 2). ∴ AH= DF+CH.

But AH is contained by AD and BD, since DH is equal to DB (II. 4, cor.),

And DF, together with CH, is the gnomon CMG;

Therefore the gnomon CMG is equal to the rectangle AD, DB. ∴ CMG. =AD · DB.

To each of these equals add LG, which is equal to the square on CD (II. 4, cor., and I · 34); Add to each LG or CD^2.

Therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the square on CD.

But the gnomon CMG and LG make up the whole figure CEFB, which is the square on CB;

Therefore the rectangle AD, DB, together with the square on CD, is equal to the square on CB. ∴ CB^2 =AD · DB +CD^2.

Therefore, if a straight line, &c. *Q. E. D.*

COROLLARY.—From this proposition it is manifest that the difference of the squares on two unequal lines AC, CD is equal to the rectangle contained by their sum and difference.

Proposition 6.—Theorem.

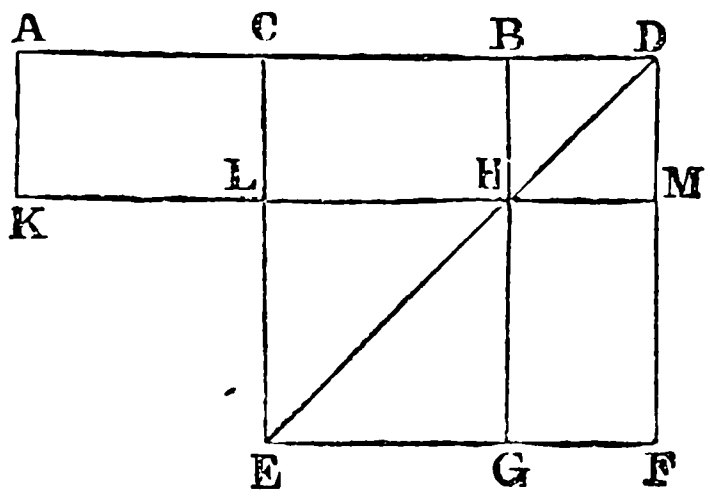
If a straight line be bisected, and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D;

AD · DB + CB² = CD².

The rectangle AD, DB, together with the square on CB, shall be equal to the square on CD.

CONSTRUCTION.—Upon CD describe the square CEFD (I. 46), and join DE.



Through B draw BHG parallel to CE or DF (I. 31).

Through H draw KLM parallel to AD or EF.

And through A draw AK parallel to CL or DM.

For AL = CH = HF.

PROOF.—Because AC is equal to CB (Hyp.), the rectangle AL is equal to CH (I. 36).

But CH is equal to HF (I. 43), therefore AL is equal to HF (Ax. 14).

∴ AM or AD · DB = CMG.

To each of these add CM; therefore the whole AM is equal to the gnomon CMG (Ax. 2).

But AM is the rectangle contained by AD and DB, since DM is equal to DB (II. 4, cor.);

Therefore the gnomon CMG is equal to the rectangle AD, DB (Ax. 1).

Add to each LG or CB².

Add to each of these LG, which is equal to the square on CB (II. 4, cor., and I. 34);

Therefore the rectangle AD, DB, together with the square on CB, is equal to the gnomon CMG and the figure LG.

But the gnomon CMG and LG make up the whole figure CEFD, which is the square on CD;

∴ AD · DB + CB² = CD².

Therefore the rectangle AD, DB, together with the square on CB, is equal to the square on CD.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 7.—Theorem.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part, together with the square on the other part.

Let the straight line AB be divided into any two parts in the point C;

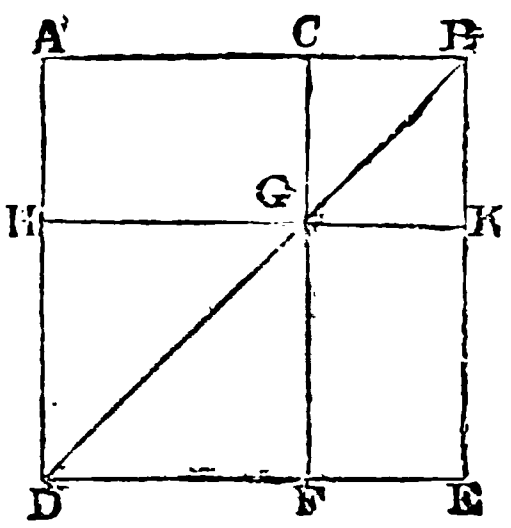
AB² + BC² = 2 AB · BC + AC².

The squares on AB and BC shall be equal to twice the rectangle AB, BC, together with the square on AC.

CONSTRUCTION.—Upon AB describe the square ADEB (I. 46), and join BD.

Through C draw CGF parallel to AD or BE (I. 31).

Through G draw HGK parallel to AB or DE (I. 31).



PROOF.—Then AG is equal to GE (I. 43).

To each of these add CK; therefore the whole AK is equal to the whole CE; For AK=CE.

Therefore AK and CE are double of AK.

But AK and CE are the gnomon AKF, together with the square CK;

Therefore the gnomon AKF, together with the square CK, is double of AK. ∴ AKF + CK = 2AB·BC.

But twice the rectangle AB, BC is also double of AK, for BK is equal to BC (II. 4, cor.);

Therefore the gnomon AKF, together with the square CK, is equal to twice the rectangle AB, BC.

To each of these equals add HF, which is equal to the square on AC (II. 4, cor., and I. 34); Add HF or AC^2 to each equal.

Therefore the gnomon AKF, together with the squares CK and HF, is equal to twice the rectangle AB, BC, together with the square on AC.

But the gnomon AKF, together with the squares CK and HF, make up the whole figure ADEB and CK, which are the squares on AB and BC; ∴ $AB^2 + BC^2 = 2\,AB \cdot BC + AC^2$.

Therefore the squares on AB and BC are equal to twice the rectangle AB, BC, together with the square on AC.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 8.—Theorem.

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole line and the first mentioned part.

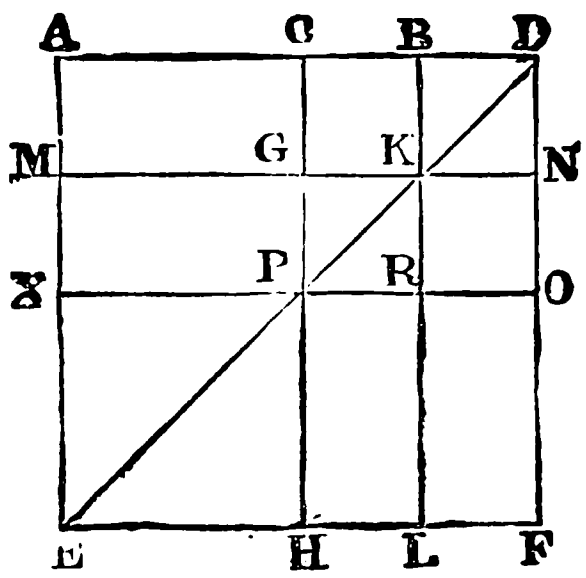
Let the straight line AB be divided into any two parts in the point C;

4 AB·BC +AC² = (AB+BC)².

Four times the rectangle AB, BC, together with the square on AC, shall be equal to the square on the straight line made up of AB and BC together.

CONSTRUCTION.—Produce AB to D, so that BD may be equal to CB (Post. 2, and I. 3).

Upon AD describe the square AEFD (I. 46),



And construct two figures such as in the preceding propositions.

PROOF.—Because CB is equal to BD (Const.), CB to GK, and BD to KN (Ax. 1),

For the same reason PR is equal to RO.

For BN=CK =GR=RN

And because CB is equal to BD, and GK to KN,

Therefore the rectangle CK is equal to BN, and GR to RN (I. 36).

But CK is equal to RN, because they are the complements of the parallelogram CO (I. 43);

Therefore also BN is equal to GR (Ax. 1).

∴ the four together = 4 CK,

Therefore the four rectangles BN, CK, GR, RN are equal to one another, and so the four are quadruple of one of them, CK.

Again, because CB is equal to BD (Const.);

And that BD is equal to BK, that is CG (II. 4, Cor., and I. 34);

And that CB is equal to GK, that is GP (I. 34, and II. 4, cor.);

Therefore CG is equal to GP (Ax. 1).

And because CG is equal to GP, and PR to RO,

And the rectangles AG, MP, PL, RF, are equal to each other, and are together = 4 AG.

The rectangle AG is equal to MP, and PL to RF (I. 36).

But MP is equal to PL, because they are complements of the parallelogram ML (I. 43), and AG is equal to RF (Ax. 1);

Therefore the four rectangles AG, MP, PL, RF are equal to one another, and so the four are quadruple of one of them, AG.

And it was demonstrated that the four CK, BN, GR, and RN are quadruple of CK;

Therefore the eight rectangles which make up the gnomon AOH are quadruple of AK.

And because AK is the rectangle contained by AB and BC, for BK is equal to BC;

∴ Gnomon AOH = 4(CK+AG) = 4 AK = 4 AB·BC

Therefore four times the rectangle AB, BC is quadruple of AK.

But the gnomon AOH was demonstrated to be quadruple of AK;

Therefore four times the rectangle AB, BC is equal to the gnomon AOH (Ax. 1).

To each of these add XH, which is equal to the square on AC (II. 4, cor., and I. 34);

Hence adding XH or AC^2,

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD;

But the gnomon AOH and the square XH make up the figure AEFD, which is the square on AD;

$4\,AB\cdot BC + AC^2 = AF = (AB+BC)^2$.

Therefore four times the rectangle AB, BC, together with the square on AC, is equal to the square on AD, that is, on the line made up of AB and BC together.

Therefore, if a straight line, &c. *Q. E. D.*

Proposition 9.—Theorem.

If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line, and of the square on the line between the points of section.

Let the straight line AB be divided into two equal parts in the point C, and into two unequal parts in the point D;

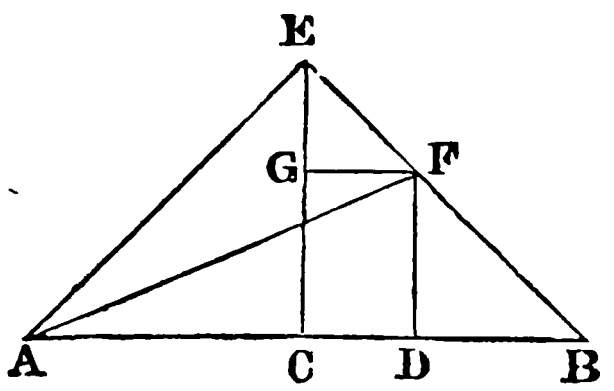
The squares on AD and DB shall be together double of the squares on AC and CD.

$AD^2+DB^2 = 2(AC^2+CD^2)$.

CONSTRUCTION.—From the point C draw CE at right angles to AB (I. 11), and make it equal to AC or CB (I. 3), and join EA, EB.

Through D draw DF parallel to CE (I. 31).

Through F draw FG parallel to BA (I. 31), and join AF.



PROOF.—Because AC is equal to CE (Const.), the angle EAC is equal to the angle AEC (I. 5).

And because the angle ACE is a right angle (Const.), the angles AEC and EAC together make one right angle (I. 32), and they are equal to one another;

For AEC=½ a right ∠ = CEB.

Therefore each of the angles AEC and EAC is half a right angle.

For the same reason, each of the angles CEB and EBC is half a right angle;

∴ AEB is a right ∠.

Therefore the whole angle AEB is a right angle.

And because the angle GEF is half a right angle, and the angle EGF a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle EFG is half a right angle;

EG=GF.

Therefore the angle GEF is equal to the angle EFG, and the side EG is equal to the side GF (I. 6).

Again, because the angle at B is half a right angle, and the angle FDB a right angle, for it is equal to the interior and opposite angle ECB (I. 29),

Therefore the remaining angle BFD is half a right angle;

And DF=DB.

Therefore the angle at B is equal to the angle BFD, and the side DF is equal to the side DB (I. 6).

And because AC is equal to CE (Const.), the square on AC is equal to the square on CE;

Therefore the squares on AC and CE are double of the square on AC.

But $AE^2=2AC^2$.

But the square on AE is equal to the squares on AC and CE, because the angle ACE is a right angle (I. 47);

Therefore the square on AE is double of the square on AC.

Again, because EG is equal to GF (Const.), the square on EG is equal to the square on GF;

Therefore the squares on EG and GF are double of the square on GF.

So also $EF^2=2GF=2\ CD^2$.

But the square on EF is equal to the squares on EG and GF, because the angle EGF is a right angle (I. 47);

Therefore the square on EF is double of the square on GF.

And GF is equal to CD (I. 34);

Therefore the square on EF is double of the square on CD.

But it has been demonstrated that the square on AE is also double of the square on AC;

∴ $AE^2+EF^2=2(AC^2+CD^2)$.

Therefore the squares on AE and EF are double of the squares on AC and CD.

But the square on AF is equal to the squares on AE and EF, because the angle AEF is a right angle (I. 47);

Therefore the square on AF is double of the squares on AC and CD.

But $AE^2+EF^2=AF^2=AD^2+DF^2=AD^2+DB^2$.

But the squares on AD and DF are equal to the square on AF, because the angle ADF is a right angle (I. 47);

Therefore the squares on AD and DF are double of the squares on AC and CD.

And DF is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

$\therefore AD^2+DB=2(AC^2+CD^2)$

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 10.—Theorem.

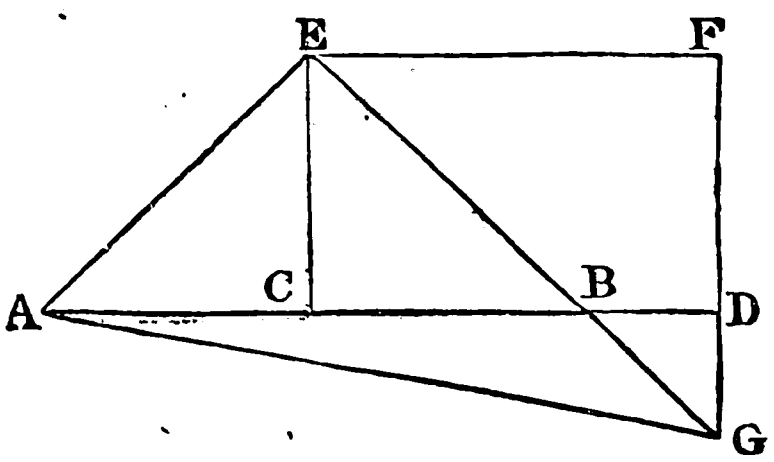
If a straight line be bisected and produced to any point, the square on the whole line thus produced, and the square on the part of it produced, are together double of the square on half the line bisected, and of the square on the line made up of the half and the part produced.

Let the straight line AB be bisected in C, and produced to D; $AD^2+DB^2=2(AC^2+CD^2)$.

The squares on AD and DB shall be together double of the squares on AC and CD.

CONSTRUCTION.—From the point C draw CE at right angles to AB, and make it equal to AC or CB (I. 11, I. 3), and join AE, EB.

Through E draw EF parallel to AB, and through D draw DF parallel to CE (I. 31). Then because the straight line EF meets the parallels EC, FD, the angles CEF, EFD are equal to two right angles (I. 29); therefore the angles BEF, EFD are less than two right angles; therefore EB, FD will meet, if produced towards B and D (Ax. 12).



Let them meet in G, and join AG.

PROOF.—Because AC is equal to CE (Const.), the angle CEA is equal to the angle EAC (I. 5).

And the angle ACE is a right angle; therefore each of the angles CEA and EAC is half a right angle (I. 32).

As in Prop. 9. For the same reason each of the angles CEB and EBC is half a right angle;

AEB is a right ∠. Therefore the whole angle AEB is a right angle.

And because the angle EBC is half a right angle, the angle DBG, which is vertically opposite, is also half a right angle (I. 15);

But the angle BDG is a right angle, because it is equal to the alternate angle DCE (I. 29);

Therefore the remaining angle DGB is half a right angle, and is therefore equal to the angle DBG;

And BD=DG. Therefore also the side BD is equal to the side DG (I. 6).

Again, because the angle EGF is half a right angle, and the angle at F a right angle, for it is equal to the opposite angle ECD (I. 34);

Therefore the remaining angle FEG is half a right angle (I. 32), and therefore equal to the angle EGF;

Also GF=FE. Therefore also the side GF is equal to the side FE (I. 6).

And because EC is equal to CA, the square on EC is equal to the square on CA;

Therefore the squares on EC and CA are double of the square on CA.

Again as in Prop. 9. But the square on AE is equal to the squares on EC and CA (I. 47);

$AE^2 = 2\,AC^2$. Therefore the square on AE is double of the square on AC.

Again, because GF is equal to FE, the square on GF is equal to the square on FE;

Therefore the squares on GF and FE are double of the square on FE.

But the square on EG is equal to the squares on GF and FE (I. 47);

Therefore the square on EG is double of the square on FE.

And FE is equal to CD (I. 34),

Therefore the square on EG is double of the square on CD.

And $EG^2 = 2\,CD^2$. But it has been demonstrated that the square on AE is double of the square on AC;

$\therefore AE^2 + E^2 = 2(AC^2 + CD^2)$. Therefore the squares on AE and EG are double of the squares on AC and CD.

But $AE^2 + EG^2 = AG^2$. But the square on AG is equal to the squares on AE and EG (I. 47);

Therefore the square on AG is double of the squares on AC and CD. $\therefore AG^2 = AD^2 + DB^2$.

But the squares on AD and DG are equal to the square on AG (I. 47); $\therefore AD^2 + DB^2 = 2(AC^2 + CD^2)$.

Therefore the squares on AD and DG are double of the squares on AC and CD.

And DG is equal to DB; therefore the squares on AD and DB are double of the squares on AC and CD.

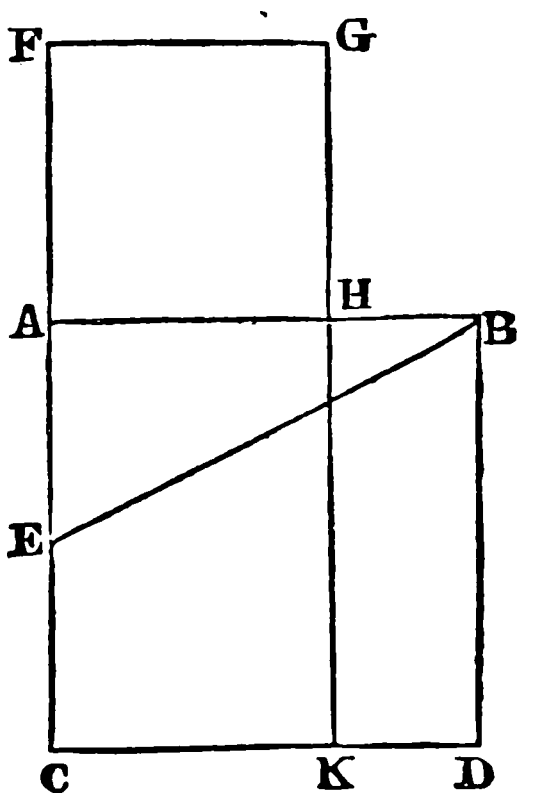
Therefore, if a straight line, &c. *Q.E.D.*

Proposition 11.—Problem.

To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.

Let AB be the given straight line.

It is required to divide AB into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.



Construction.—Upon AB describe the square ABDC (I. 46).

Bisect AC in E (I. 10), and join BE.

Produce CA to F, and make EF equal to EB (I. 3).

Upon AF describe the square AFGH (I. 46).

Produce GH to K.

Then AB shall be divided in H, so that the rectangle AB, BH is equal to the square on AH.

Proof.—Because the straight line AC is bisected in E, and produced to F,

The rectangle CF, FA, together with the square on AE, is equal to the square on EF (II. 6). $CF \cdot FA + AE^2 = EF^2$

But EF is equal to EB (Const.); $= EB^2$

Therefore the rectangle CF, FA, together with the square on AE, is equal to the square on EB.

But the square on EB is equal to the squares on AE and AB, because the angle EAB is a right angle (I. 47); $= AB^2 + AE^2$.

Therefore the rectangle CF, FA, together with the square on AE, is equal to the squares on AE and AB.

Take away the square on AE, which is common to both;

CF·FA =AB². Therefore the remaining rectangle CF, FA is equal to the square on AB (Ax. 3).

But the figure FK is the rectangle contained by CF and FA, for FA is equal to FG;

And AD is the square on AB;

∴FK=AD Therefore the figure FK is equal to AD.

Take away AK, then FH=HD Take away the common part AK, and the remainder FH is equal to the remainder HD (Ax. 3).

or AB·BH =AH². But HD is the rectangle contained by AB and BH, for AB is equal to BD;

And FH is the square on AH;

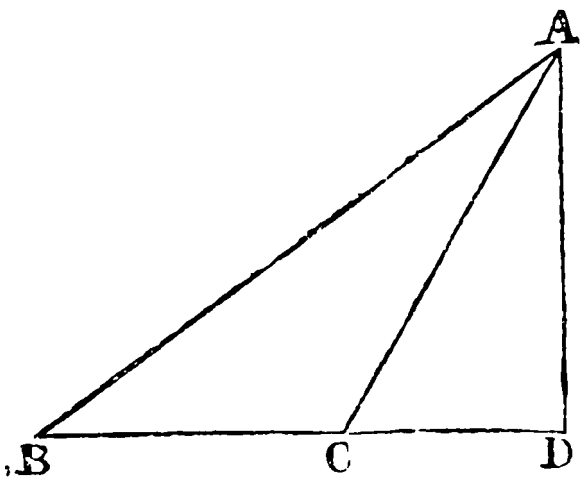
Therefore the rectangle AB,BH is equal to the square on AH.

Therefore the straight line AB is divided in H, so that the rectangle AB, BH is equal to the square on AH. *Q.E.F.*

Proposition 12.—Theorem.

In obtuse-angled triangles if a perpendicular be drawn from either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the straight line intercepted without the triangle, between the perpendicular and the obtuse angle.

Let ABC be an obtuse-angled triangle, having the obtuse angle ACB; and from the point A let AD be drawn perpendicular to BC produced.



For AB²=AC² +CB²+ 2 BC·CD. The square on AB shall be greater than the squares on AC and CB by twice the rectangle BC, CD.

Proof.—Because the straight line BD is divided into two parts in the point C,

BD²=BC² +CD²+ 2BC·CD. The square on BD is equal to the squares on BC and CD, and twice the rectangle BC, CD (II. 4).

To each of these equals add the square on DA;

Therefore the squares on BD and DA are equal to the squares on BC, CD, DA, and twice the rectangle BC, CD.

∴ BD^2+DA^2 or AB^2 $=BC^2+(CD^2+DA^2)+2BC\cdot CD$

But the square on BA is equal to the squares on BD and DA, because the angle at D is a right angle (I. 47);

And the square on CA is equal to the squares on CD and DA (I. 47);

$=BC^2+AC^2+2BC\cdot CD.$

Therefore the square on BA is equal to the squares on BC and CA, and twice the rectangle BC, CD; that is, the square on BA is greater than the squares on BC and CA by twice the rectangle BC, CD.

Therefore, in obtuse-angled triangles, &c. *Q.E.D.*

Proposition 13.—Theorem.

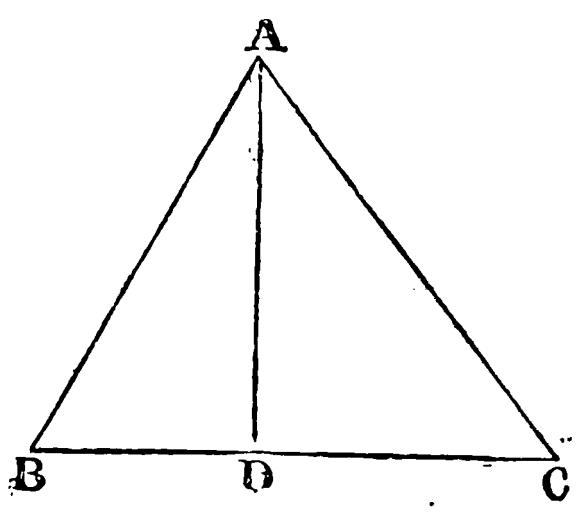
In every triangle, the square on the side subtending an acute angle is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides, and the straight line intercepted between the perpendicular let fall on it from the opposite angle and the acute angle.

Let ABC be any triangle, and the angle at B an acute angle; and on BC, one of the sides containing it, let fall the perpendicular AD from the opposite angle (I. 12).

The square on AC, opposite to the angle B, shall be less than the squares on CB and BA, by twice the rectangle CB, BD.

$AC^2=CB^2+AB^2-2CB\cdot BD.$

Case I.—First, let AD fall within the triangle ABC.



Proof.—Because the straight line CB is divided into two parts in the point D,

The squares on CB and BD are equal to twice the rectangle contained by CB, BD, and the square on DC (II. 7).

For $CB^2+BD^2=2CB\cdot BD+DC^2.$

To each of these equals add the square on DA.

Therefore the squares on CB, BD, DA are equal to twice the rectangle CB, BD, and the squares on AD and DC.

∴ $CB^2+(BD^2+DA^2)=2CB\cdot BD+(AD^2+DC^2)$, or $CB^2+BA^2=2CB\cdot DB+AC^2.$

But the square on AB is equal to the squares on BD and DA, because the angle BDA is a right angle (I. 47);

And the square on AC is equal to the squares on AD and DC (I. 47);

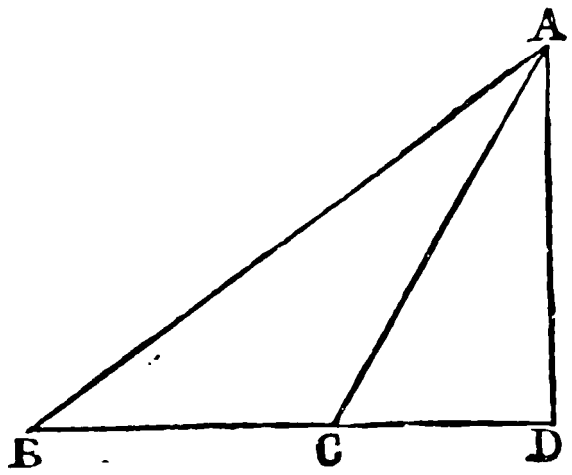
Therefore the squares on CB and BA are equal to the

$\therefore AC^2 = CB^2 + BA^2 - 2CB \cdot BD$.

square on AC, and twice the rectangle CB, BD; that is, the square on AC alone is less than the squares on CB and BA by twice the rectangle CB, BD.

CASE II.—Secondly, let AD fall without the triangle ABC.

PROOF.—Because the angle at D is a right angle (Const.),



$AB^2 = AC^2 + CB^2 + BC \cdot CD$.

the angle ACB is greater than a right angle (I. 16);

Therefore the square on AB is equal to the squares on AC and CB, and twice the rectangle BC, CD (II. 12).

To each of these equals add the square on BC.

$\therefore AB^2 + BC^2 = AC^2 + 2(BC^2 + BC \cdot CD)$.

Therefore the squares on AB and BC are equal to the square on AC, and twice the square on BC, and twice the rectangle BC, CD (Ax. 2).

But because BD is divided into two parts at C,

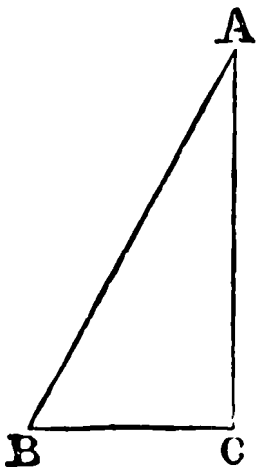
The rectangle DB, BC is equal to the rectangle BC, CD and the square on BC (II. 3);

Now $DB \cdot BC = BC \cdot CD + BC^2$.

And the doubles of these are equal, that is, twice the rectangle DB, BC is equal to twice the rectangle BC, CD and twice the square on BC;

$\therefore AC^2 = AB^2 + BC^2 - 2BC \cdot DB$.

Therefore the squares on AB and BC are equal to the square on AC, and twice the rectangle DB, BC; that is, the square on AC alone is less than the squares on AB and BC by twice the rectangle DB, BC.



CASE III.—Lastly, let the side AC be perpendicular to BC.

PROOF.—Then BC is the straight line between the perpendicular and the acute angle at B; and it is manifest that the squares on AB and BC are equal to the square on AC, and twice the square on BC (I. 47, and Ax. 2).

Therefore, in every triangle, &c. *Q.E.D.*

Proposition 14.—Problem.

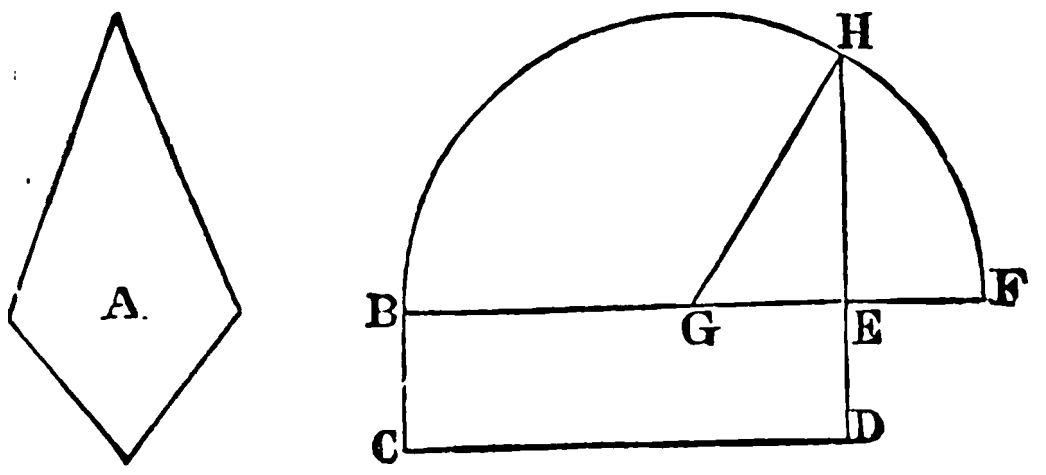
To describe a square that shall be equal to a given rectilineal figure.

Let A be the given rectilineal figure.

It is required to describe a square that shall be equal to A.

CONSTRUCTION.—Describe the rectangular parallelogram BCDE equal to the rectilineal figure A (I. 45).

If then the sides of it, BE, ED, are equal to one another it is a square, and what was required is now done.



But if they are not equal, produce one of them, BE, to F, and make EF equal to ED (I. 3).

Bisect BF in G (I. 10), and from the centre G, at the distance GB, or GF, describe the semicircle BHF;

Produce DE to H, and join GH;

Then the square described upon EH shall be equal to the rectilineal figure A.

PROOF.—Because the straight line BF is divided into two equal parts in the point G, and into two unequal parts in the point E,

The rectangle BE,EF, together with the square on GE, is equal to the square on GF (II. 5). $BE \cdot EF + GE^2 = GF^2$

But GF is equal to GH; $= GH^2$

Therefore the rectangle BE,EF, together with the square on GE, is equal to the square on GH. $= GE^2 + EH^2$.

But the square on GH is equal to the squares on GE and EH (I. 47);

Therefore the rectangle BE,EF, together with the square on GE, is equal to the squares on GE and EH. $\therefore BE \cdot EF$ or $BD = EH^2$.

Take away the square on GE, which is common to both;

Therefore the rectangle BE,EF is equal to the square on EH (Ax. 3).

But the rectangle contained by BE and EF is the parallelogram BD, because EF is equal to ED (Const.);

Therefore BD is equal to the square on EH.

But BD is equal to the rectilineal figure A (Const.);

Therefore the square on EH is equal to the rectilineal figure A. Hence $EH^2 = A$.

Therefore, a square has been made equal to the given rectilineal figure A, viz., the square described on EH. *Q.E.F.*

EXERCISES ON BOOK II.

PROP. 1—11.

1. Divide a given straight line into two such parts that the rectangle contained by them may be the greatest possible.

2. The sum of the squares of two straight lines is never less than twice the rectangle contained by the straight lines.

3. Divide a given straight line into two parts such that the squares of the whole line and one of the parts shall be equal to twice the square of the other part.

4. Given the sum of two straight lines and the difference of their squares, to find the lines.

5. In any triangle the difference of the squares of the sides is equal to the rectangle contained by the sum and difference of the parts into which the base is divided by a perpendicular from the vertical angle.

6. Divide a given straight line into such parts that the sum of their squares may be equal to a given square.

7. If ABCD be any rectangle, A and C being opposite angles, and O any point either within or without the rectangle—$OA^2 + OC^2 = OB^2 + OD^2$.

8. Let the straight line AB be divided into any two parts in the point C. Bisect CB in D, and take a point E in AC such that EC = CD. Then shall $AD^2 = AE^2 + AC \cdot CB$.

9. If a point C be taken in AB, and AB be produced to D so that BD and AC are equal, show that the squares described upon AD and AC together exceed the square upon AB by twice the rectangle contained by AE and AC.

10. From the hypothenuse of a right-angled triangle portions are cut off equal to the adjacent sides. Show that the square on the middle segment is equal to twice the rectangle under the extreme segments.

11. If a straight line be divided into any number of parts, the square of the whole line is equal to the sum of the squares of the parts, together with twice the rectangles of the parts taken two and two together.

12. If ABC be an isosceles triangle, and DE be drawn parallel to the base BC, cutting in D and E either the side or sides produced, and EB be joined; prove that $BE^2 = BC \cdot DE + CE^2$.

PROP. 12—14.

13. In any triangle show that the sum of the squares on the sides is equal to twice the square on half the base, and twice the square on the line drawn from the vertex to the middle of the base.

14. If squares are described on the sides of any triangle, find the difference between the sum of two of the squares and the third square, and show from your result what this becomes when the angle opposite the third square is a right angle.

15. Show also what the difference becomes when the vertex of the triangle is depressed until it coincide with the base.

16. The square on any straight line drawn from the vertex of an isosceles triangle, together with the rectangle contained by the segments of the base, is equal to the square upon a side of the triangle.

17. If a side of a triangle be bisected, and a perpendicular drawn from the middle point of the base to meet the side, then the square of the altitude of the triangle exceeds the square upon half the base by twice the rectangle contained by the side and the straight line between the points of section of the side.

18. In any triangle ABC, if perpendiculars be drawn from each of the angles upon the opposite sides, or opposite sides produced, meeting them respectively in D, E, F, show that—

$$BA^2 + AC^2 + CB^2 = 2\,AE \cdot AC + 2\,CD \cdot CB + 2\,BF \cdot BA;$$

all lines being measured in the *same direction* round the triangle.

19. Construct a square equal to the sum of the areas of two given rectilinear figures.

20. The base of a triangle is 63 ft., and the sides 25 ft. and 52 ft. respectively. Show that the segments of the base, made by a perpendicular from the vertex, are 15 ft. and 48 ft. respectively, and that the area of the triangle is 630 sq. ft.

21. In the same triangle, show that the length of the line joining the vertex with the middle of the base is 22·9 ft.

22. A ladder, 45 ft. long, reaches to a certain height against a wall, but, when turned over without moving the foot, must be shortened 6 ft. in order to reach the same height on the opposite side. Supposing the width of the street to be 42 ft., show that the height to which the ladder reaches is 36 ft.

23. The base and altitude of a triangle are 8 in. and 9 in. respectively; show that its area is equal to a square whose side is 6 in. Prove your result by construction.

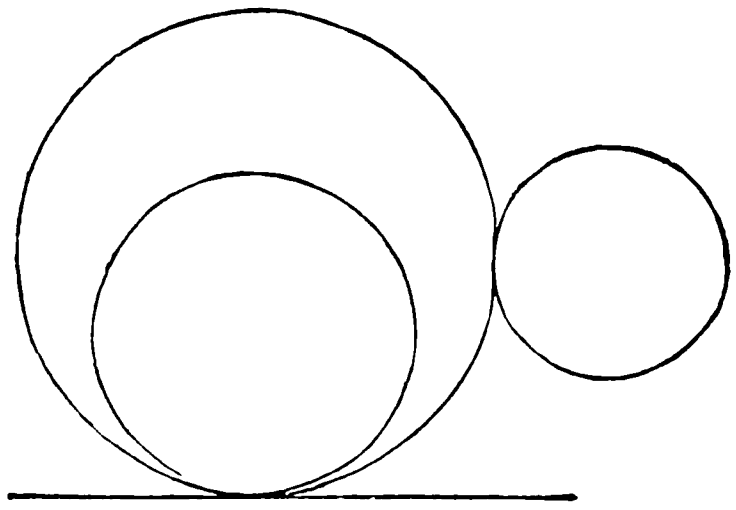
24. On the supposition that lines can be always expressed *exactly* in terms of some unit of length, what geometrical propositions may be deduced from the following algebraical identities?—

(1.) $(a + b)^2 = a^2 + 2\,ab + b^2$
(2.) $(a + b)\,(a - b) + b^2 = a^2$
(3.) $(2\,a + b)\,b + a^2 = (a + b)^2$
(4.) $(a + b)^2 + b^2 = 2\,(a + b)\,b + a^2$
(5.) $4\,(a + b)\,b + b^2 = (a + 2\,b)^2$
(6.) $(a + b)^2 + (a - b)^2 = 2\,a^2 + 2\,b^2$
(7.) $(2\,a + b)^2 + b^2 = 2\,a^2 + 2\,(a + b)^2$

EUCLID'S ELEMENTS, BOOK III.

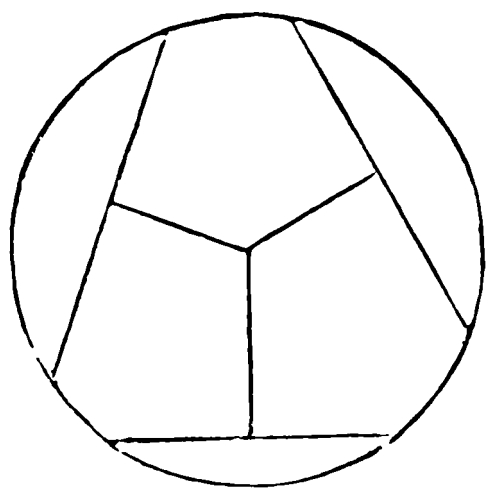
Definitions.

1. **Equal circles** are those of which the diameters are equal, or from the centres of which the straight lines to the circumferences are equal.



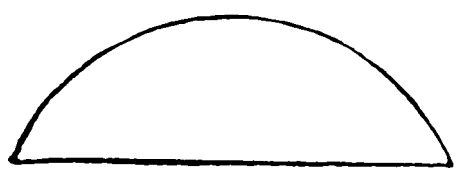
2. **A straight line is said to touch a circle,** or to be a tangent to it, when it meets the circle, and being produced does not cut it.

3. **Circles** are said to **touch one another,** which meet, but do not cut one another.



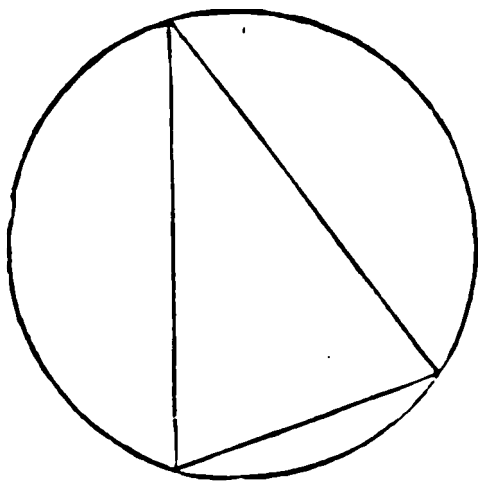
4. Straight lines are said to be **equally distant** from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

5. And the straight line on which the greater perpendicular falls, is said to be **farther** from the centre.



6. **A segment of a circle** is the figure contained by a straight line and the circumference which it cuts off.

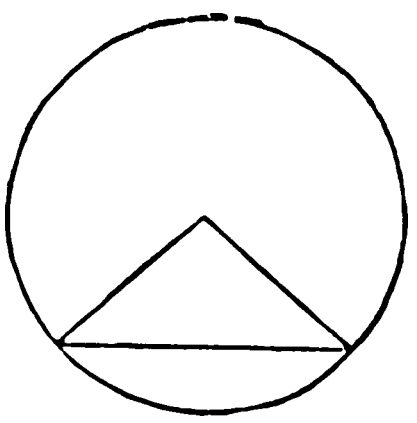
7. The **angle of a segment** is that which is contained by the straight line and the circumference.



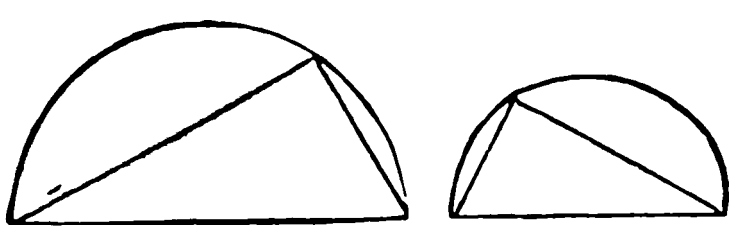
8. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line, which is the base of the segment.

9. An angle is said to **insist** or **stand upon** the circumference intercepted between the straight lines that contain the angle.

10. A **sector** of a circle is the figure contained by two straight lines drawn from the centre and the circumference between them.



11. **Similar segments** of circles are those which contain equal angles.



[Any portion of the circumference is called an *arc*, and the *chord* of an arc is the straight line joining its extremities.]

Proposition 1.—Problem.

To find the centre of a given circle.

Let ABC be the given circle.

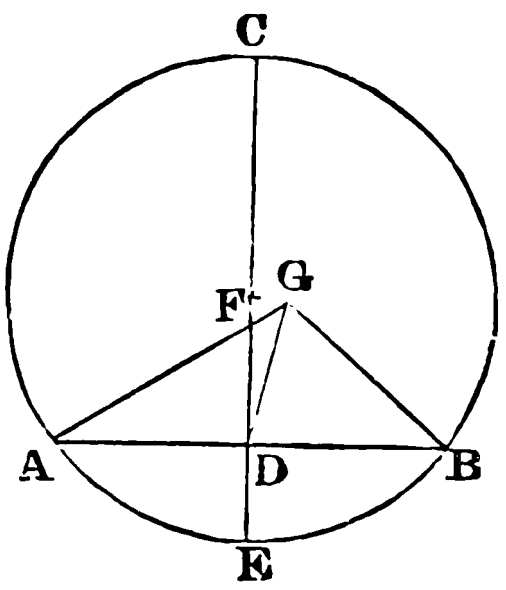
It is required to find its centre.

CONSTRUCTION.—Draw within the circle any chord AB, and bisect it in D (I. 10).

From the point D draw DC at right angles to AB (I. 11).

Produce CD to meet the circumference in E, and bisect CE in F (I. 10).

Then the point F shall be the centre of the circle ABC.



Suppose G the centre.

PROOF.—For if F be not the centre, if possible let G be the centre; and join GA, GD, GB.

Then, because DA is equal to DB (Const.), and DG common to the two triangles ADG, BDG;

The two sides AD, DG are equal to the two sides BD, DG, each to each;

And the base GA is equal to the base GB, being radii of the same circle;

Therefore the angle ADG is equal to the angle BDG (I. 8). ∴ ∠ ADG = ∠ BDG.

But when a straight line, standing on another straight line, makes the adjacent angles equal to one another, each of the angles is called a right angle (I. Def. 10).

Therefore the angle GDB is a right angle.

But the angle FDB is also a right angle (Const.);

∴ ∠GDB = ∠FDB. Therefore the angle GDB is equal to the angle FDB (Ax. 11), the less to the greater; which is impossible.

Therefore G is not the centre of the circle ABC.

In the same manner it may be shown that no point which is not in CE can be the centre.

And since the centre is in CE, it must be in F, its point of bisection.

Therefore F is the centre of the circle ABC: which was to be found.

COROLLARY.—From this it is manifest that, if in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

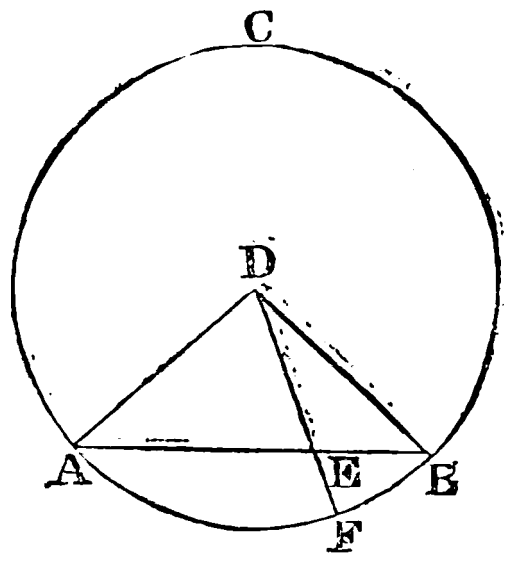
Proposition 2.—Theorem.

If any two points be taken in the circumference of a circle, the straight line which joins them shall fall within the circle.

Let ABC be a circle, and A and B any two points in the circumference.

The straight line drawn from A to B shall fall within the circle.

CONSTRUCTION.—Find D the centre of the circle ABC (III. 1), and join DA, DB.



In AB take any point E; join DE, and produce it to the circumference in F.

PROOF.—Because DA is equal to DB, the angle DAB is equal to the angle DBA (I. 5).

∠DBE = ∠DAB, and ∴ ∠DEB > DBE. And because AE, a side of the triangle DAE, is produced to B, the exterior angle DEB is greater than the interior and opposite angle DAE (I. 16).

But the angle DAE was proved to be equal to the angle DBE;

Therefore the angle DEB is also greater than DBE.

But the greater angle is subtended by the greater side (I. 19);

∴ DB > DE. Therefore DB is greater than DE.

But DB is equal to DF; therefore DF is greater than DE, and the point E is therefore within the circle.

In the same manner it may be proved that every point in AB lies within the circle.

Therefore the straight line AB lies within the circle.

Therefore, if any two points, &c. *Q.E.D.*

Proposition 3.—Theorem.

If a straight line drawn through the centre of a circle bisect a straight line in it which does not pass through the centre, it shall cut it at right angles; and conversely, if it cut it at right angles, it shall bisect it.

Let ABC be a circle, and let CD, a straight line drawn through the centre, bisect any straight line AB, which does not pass through the centre.

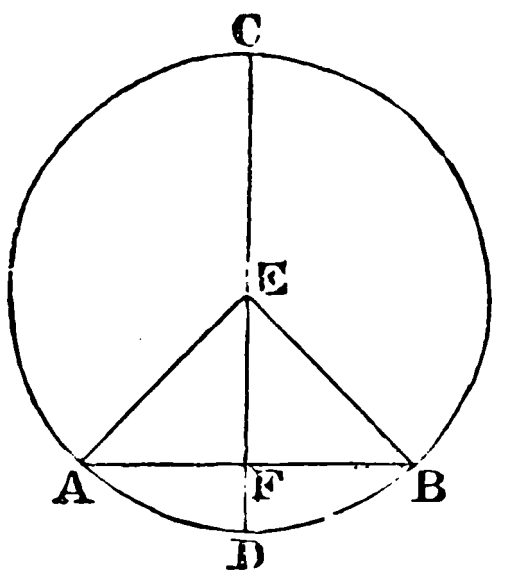
CD shall cut AB at right angles.

CONSTRUCTION.—Take E, the centre of the circle (III. 1), and join EA, EB.

PROOF.—Because AF is equal to FB (Hyp.), and FE common to the two triangles AFE, BFE, and the base EA equal to the base EB (I. Def. 15),

Therefore the angle AFE is equal to the angle BFE (I. 8);

Therefore each of the angles AFE, BFE is a right angle (I. def. 10);



Triangles AFE and BEF are equal in every respect.

Therefore the straight line CD, drawn through the centre, bisecting another, AB, that does not pass through the centre, also cuts it at right angles.

Conversely, let CD cut AB at right angles.

CD shall bisect AB; that is, AF shall be equal to FB —the same construction being made.

PROOF.—Because the radii EA, EB are equal, the angle EAF is equal to the angle EBF (I. 5),

And the angle AFE is equal to the angle BFE (Hyp.),

Therefore, in the two triangles EAF, EBF, there are two angles in the one equal to two angles in the other, each to

each, and the side EF, which is opposite to one of the equal angles in each, is common to both;

Therefore their other sides are equal (I. 26);

Therefore AF is equal to FB.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 4.—Theorem.

If in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect each other.

Let ABCD be a circle, and AC, BD two straight lines in it, which cut one another at the point E, and do not both pass through the centre.

AC, BD shall not bisect one another.

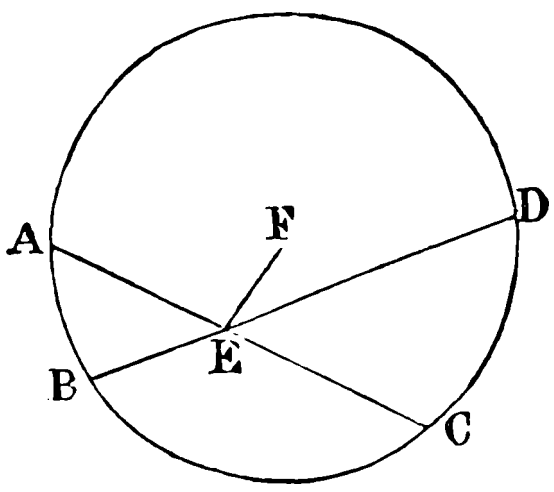
If one of the lines pass through the centre, it is plain that it cannot be bisected by the other, which does not pass through the centre.

If they bisect each other,

But if neither of them pass through the centre, if possible, let AE be equal to EC, and BE to ED.

Construction.—Take F, the centre of the circle (III. 1), and join EF.

EF bisects AC at right angles.



Because FE, a straight line drawn through the centre, bisects another line AC, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEA is a right angle.

And EF bisects BD at right angles.

Again, because the straight line FE bisects the straight line BD, which does not pass through the centre (Hyp.), therefore it cuts it at right angles (III. 3);

Therefore the angle FEB is a right angle.

But the angle FEA was shown to be a right angle;

Therefore the angle FEA is equal to the angle FEB, the less to the greater, which is impossible;

$\therefore \angle FEA = \angle FEB$.

Therefore AC, BD do not bisect each other.

Therefore, if in a circle, &c. *Q.E.D.*

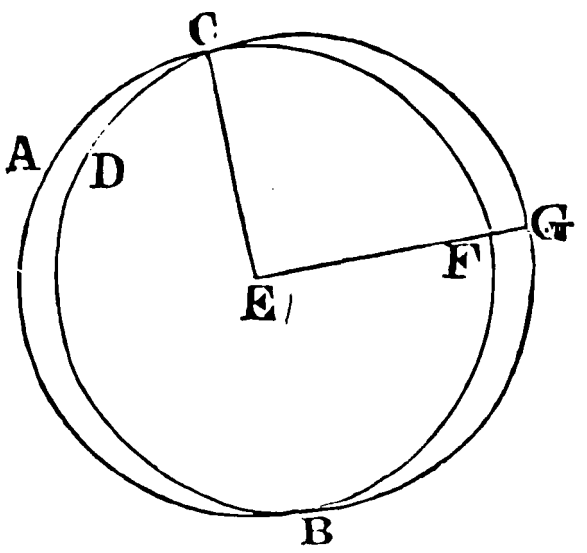
Proposition 5.—Theorem.

If two circles cut one another, they shall not have the same centre.

Let the two circles ABC, CDG cut one another in the points B, C.

They shall not have the same centre.

For, if it be possible, let E be their centre; join EC, and draw any straight line EFG, meeting the circles in F and G.



EC=EF

PROOF.—Because E is the centre of the circle ABC, EC is equal to EF (I. Def. 15).

And because E is the centre of the circle CDG, EC is equal to EG. and =EG.

But EC was shown to be equal to EF;

Therefore EF is equal to EG (Ax. 1), the less to the greater, ∴EF=EG. which is impossible;

Therefore E is not the centre of the circles ABC, CDG.

Therefore, if two circles, &c. *Q.E.D.*

Proposition 6.—Theorem.

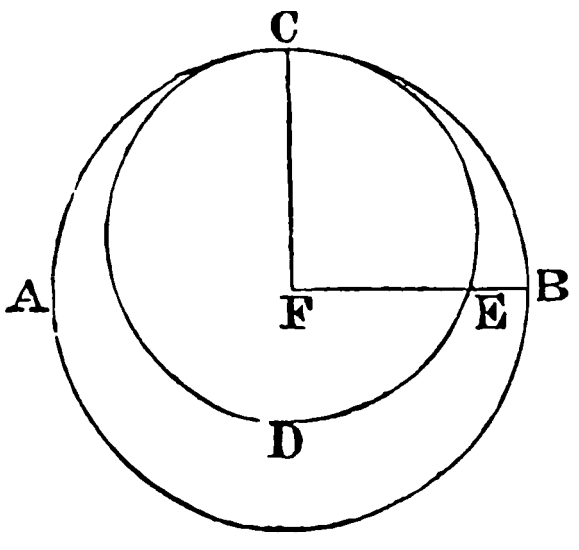
If one circle touch another internally, they shall not have the same centre.

Let the circle CDE touch the circle ABC internally in the point C.

They shall not have the same centre.

CONSTRUCTION.—For, if it be possible, let F be their centre; join FC, and draw any straight line FEB, meeting the circles in E and B.

If they have the same centre F,



PROOF.—Because F is the centre of the circle ABC, FC is equal to FB (I Def. 15).

And because F is the centre of the circle CDE, FC is equal to FE.

FC=FE and=FB.

But FC was shown to be equal to FB;

Therefore FE is equal to FB (Ax. 1), the less to the greater, ∴FE=FB which is impossible;

Therefore F is not the centre of the circles ABC, CDE.

Therefore, if one circle, &c. *Q.E.D.*

Proposition 7.—Theorem.

If any point be taken in the diameter of a circle, which is not the centre of all the straight lines which can be drawn from this point to the circumference, the greatest is that in which the centre is, and the other part of that diameter is the least; and, of any others, that which is nearer to the straight line which passes through the centre is always greater than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the shortest line.

Let ABCD be a circle, AD its diameter, and E its centre, in which let any point F be taken which is not the centre.

FA > FB > FC > FG &c., > FD the least.

Of all the straight lines FB, FC, FG, &c., that can be drawn from F to the circumference, FA, which passes through the centre, shall be the greatest;

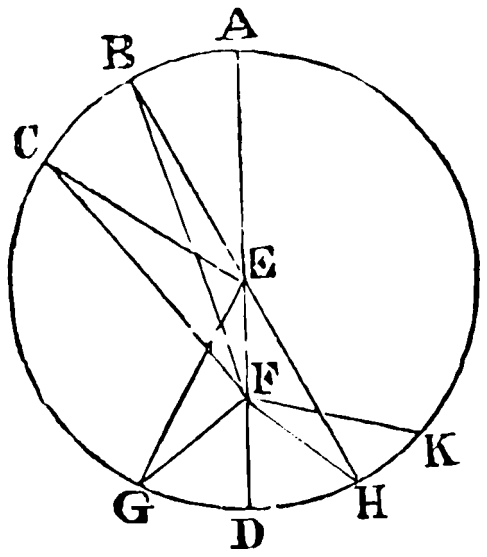
FD, the other part of the diameter AD, shall be the least;

And of the others, FB, the nearer to FA, shall be greater than FC, the more remote; and FC greater than FG.

CONSTRUCTION.—Join BE, CE, GE.

For BE+EF or AF>BF

PROOF.—Because any two sides of a triangle are greater than the third side, BE, EF are greater than BF (I. 20).



But AE is equal to BE; therefore AE, EF, that is, AF is greater than BF.

Again, because BE is equal to CE, and EF common to the two triangles BEF, CEF, the two sides BE, EF are equal to the two CE, EF, each to each.

But the angle BEF is greater than the angle CEF;

And base BF > base FC

Therefore the base FB is greater than the base FC (I. 24).

In the same manner it may be shown that FC is greater than FG.

Again GF+FE >EG and ∴>ED.

Again, because GF, FE are greater than EG (I. 20), and that EG is equal to ED;

∴GF>FD

Therefore GF, FE are greater than ED.

Take away the common part FE, and the remainder GF is greater than the remainder FD (Ax. 5).

Therefore FA is the greatest, and FD the least, of all the straight lines from F to the circumference; and FB is greater than FC, and FC than FG.

Also, there cannot be drawn more than two equal straight lines from the point F to the circumference, one on each side of the shortest line.

CONSTRUCTION.—At the point E, in the straight line EF, make the angle FEH equal to the angle FEG (I. 23), and join FH.

PROOF.—Because EG is equal to EH, and EF common to the two triangles GEF, HEF, the two sides EG, EF are equal to the two sides EH, EF, each to each; Triangles GEF and HEF are equal in every respect.

And the angle GEF is equal to the angle HEF (Const.);

Therefore the base FG is equal to the base FH (I. 4).

But, besides FH, no other straight line can be drawn from F to the circumference equal to FG.

For, if it be possible, let FK be equal to FG;

Then, because FK is equal to FG, and FH is also equal to FG, therefore FH is equal to FK; And if FK = FG, it also = FH which is impossible

That is, a line nearer to that which passes through the centre is equal to a line which is more remote; which is impossible by what has been already shown.

Therefore, if any point, &c. *Q.E.D.*

Proposition 8.—Theorem.

If any point be taken without a circle, and straight lines be drawn from it to the circumference, one of which passes through the centre; of those which fall on the concave circumference, the greatest is that which passes through the centre, and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote; but of those which fall on the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than one more remote; and from the same point there can be drawn to the circumference two straight lines, and only two, which are equal to one another, one on each side of the least line.

Let ABC be a circle, and D any point without it, and from D let the straight lines DA, DE, DF, DC be drawn

to the circumference, of which DA passes through the centre.

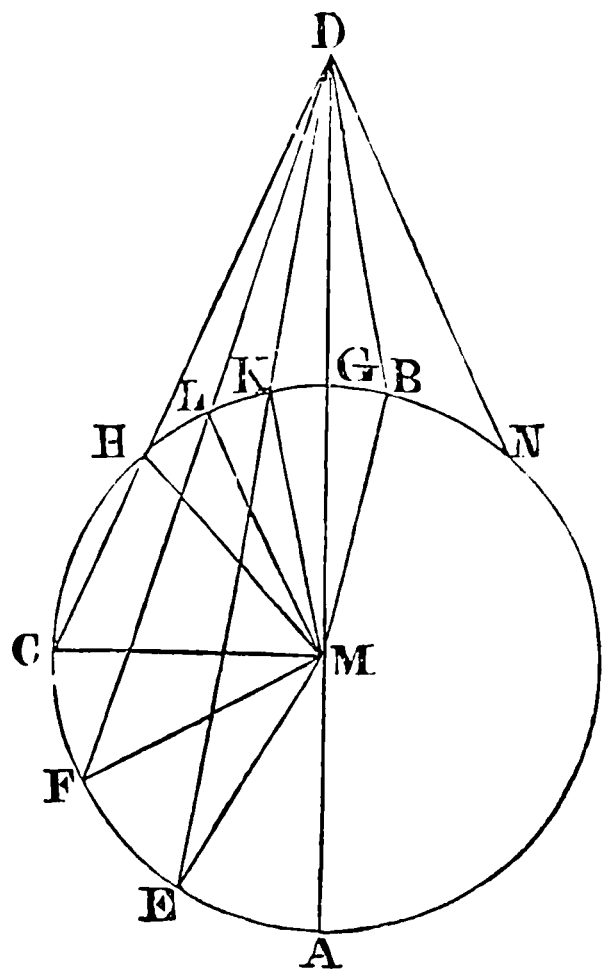
DA>DE >DF >DC,

Of those which fall on the concave part of the circumference AEFC, the greatest shall be DA, which passes through the centre, and the nearer to it shall be greater than the more remote, viz., DE greater than DF, and DF greater than DC.

and DG<DK <DL <DH

But of those which fall on the convex circumference GKLH, the least shall be DG between the point D and the diameter AG, and the nearer to it shall be less than the more remote, viz., DK less than DL, and DL less than DH.

CONSTRUCTION.—Take M, the centre of the circle ABC (III. 1), and join ME, MF, MC, MH, ML, MK.



PROOF.—Because any two sides of a triangle are greater than the third side, EM, MD are greater than ED (I. 20).

But EM is equal to AM; therefore AM, MD are greater than ED—that is, AD is greater than ED.

For EM+MD or AD> ED.

Again, because EM is equal to FM, and MD common to the two triangles EMD, FMD; the two sides EM, MD are equal to the two sides FM, MD, each to each;

But the angle EMD is greater than the angle FMD;

Therefore the base ED is greater than the base FD (I. 24).

Also base ED > base FD, &c.

In like manner it may be shown that FD is greater than CD;

Therefore DA is the greatest, and DE greater than DF and DF greater than DC.

Again, because MK, KD are greater than MD (I. 20), and MK is equal to MG;

Again MK+KD >MD, or >MK+ DG. ∴KD> DG, and ∴DG <KD, &c.

The remainder KD is greater than the remainder GD—that is, GD is less than KD.

And because MK, DK are drawn to the point K within the triangle MLD from M and D, the extremities of its side MD;

Therefore MK,DK are less than ML, LD (I. 21).

But MK is equal to ML; therefore the remainder KD is less than the remainder LD.

In like manner it may be shown that LD is less than HD.

Therefore DG is the least, and KD less than DL, and DL less than DH.

Also, there can be drawn only two equal straight lines from the point D to the circumference, one on each side of the least line.

CONSTRUCTION.—At the point M, in the straight line MD, make the angle DMB equal to the angle DMK (I. 23), and join DB.

PROOF.—Because MK is equal to MB, and MD common to the two triangles KMD, BMD; the two sides KM, MD are equal to the two sides BM, MD, each to each; Triangles KMD and BMD are equal in every respect.

And the angle DMK is equal to the angle DMB (Const);

Therefore the base DK is equal to the base DB (I. 4).

But, besides DB, no other straight line can be drawn from D to the circumference equal to DK.

For, if it be possible, let DN be equal to DK.

Then, because DN is equal to DK, and DB is also equal to DK, therefore DB is equal to DN (Ax. 1); DN=DK and ∴ =DB which is impossible

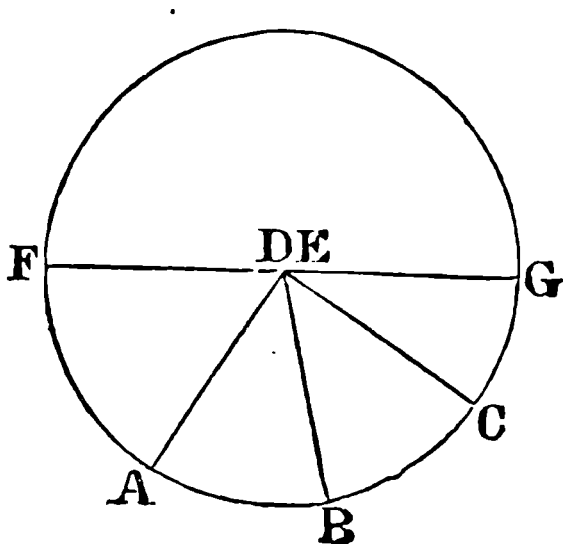
That is, a line nearer to the least is equal to one which is more remote; which is impossible by what has been already shown.

Therefore, if any point, &c. *Q.E.D.*

Proposition 9.—Theorem.

If a point be taken within a circle, from which there can be drawn more than two equal straight lines to the circumference, that point is the centre of the circle.

Let the point D be taken within the circle ABC, from which to the circumference there can be drawn more than two equal straight lines, viz., DA, DB, DC.



The point D shall be the centre of the circle.

CONSTRUCTION.—For if not, let E be the centre; join DE, and produce it to the circumference in F and G.

PROOF.—Then FG is a diameter of the circle ABC (I. Def. 17).

If D be not the centre.

And because in FG, the diameter of the circle ABC, there is taken the point D, not the centre;

DG > DC > DB > DA.

Therefore DG is the greatest straight line from D to the circumference, and DC is greater than DB, and DB greater than DA (III. 7);

But they are also equal.

But these lines are likewise equal, by hypothesis; which is impossible.

Therefore E is not the centre of the circle ABC.

In like manner it may be demonstrated that any other point than D is not the centre;

Therefore D is the centre of the circle ABC.

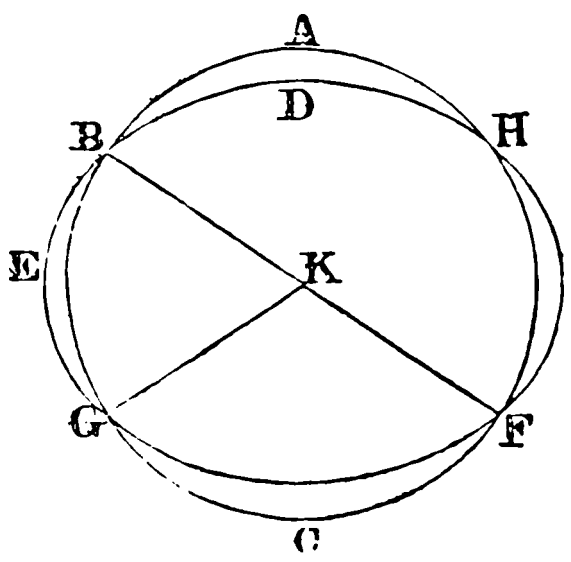
Therefore, if a point, &c. *Q.E.D.*

Proposition 10.—Theorem.

One circumference of a circle cannot cut another in more than two points.

CONSTRUCTION.—If it be possible, let the circumference

If possible,



ABC cut the circumference DEF in more than two points, viz., in the points B, G, F.

Take the centre K of the circle ABC (III. 1), and join KB, KG, KF.

PROOF.—Because K is the centre of the circle ABC, the radii KB, KG, KF are all equal.

the two circles have the same centre.

And because within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal straight lines KB, KG, KF, therefore K is the centre of the circle DEF (III. 9).

But K is also the centre of the circle ABC (Const.);

Therefore the same point is the centre of two circles which cut one another; which is impossible (III. 5).

Therefore, one circumference, &c. *Q.E.D.*

Proposition 11.—Theorem.

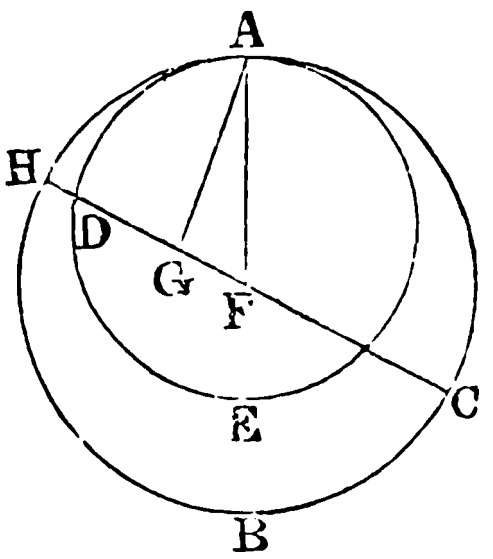
If one circle touch another internally in any point, the straight line which joins their centres, being produced, shall pass through that point.

Let the circle ADE touch the circle ABC internally in the

point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE.

The straight line which joins their centres, being produced, shall pass through the point of contact A.

CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FGDH. Join AF and AG. If not,



PROOF.—Because AG, GF are greater than AF (I. 20), and AF is equal to HF (I. def. 15);

Therefore AG, GF are greater than HF.

Take away the common part GF, and the remainder AG is greater than the remainder HG. AG > HG.

But AG is equal to DG (I. Def. 15); But AG =DG.

Therefore DG is greater than HG, the less than the greater; which is impossible. ∴DG> HG.

Therefore the straight line which joins the centres, being produced, cannot fall otherwise than upon the point A, that is, it must pass through it.

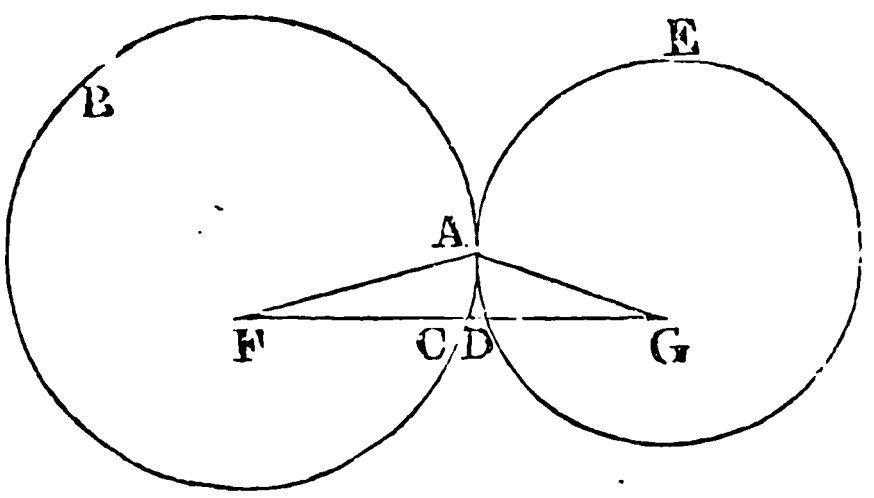
Therefore, if one circle, &c. *Q.E.D.*

Proposition 12.—Theorem.

If two circles touch each other externally in any point, the straight line which joins their centres shall pass through that point.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of the circle ADE;

The straight line which joins their centres shall pass through the point of contact A.



If not,

CONSTRUCTION.—For, if not, let it pass otherwise, if possible, as FCDG. Join FA and AG.

PROOF.—Because F is the centre of the circle ABC, FA is equal to FC (I. Def. 15).

And because G is the centre of the circle ADE, GA is equal to GD;

Therefore FA, AG are equal to FC, DG (Ax. 2).

FG is > FA+AG, but it is also *less*.

Therefore the whole FG is greater than FA, AG.

But FG is also less than FA, AG (I. 20); which is impossible.

Therefore the straight line which joins the centres of the circles shall not pass otherwise than through the point A, that is, it must pass through it.

Therefore, if two circles, &c. *Q.E.D.*

Proposition 13.—Theorem.

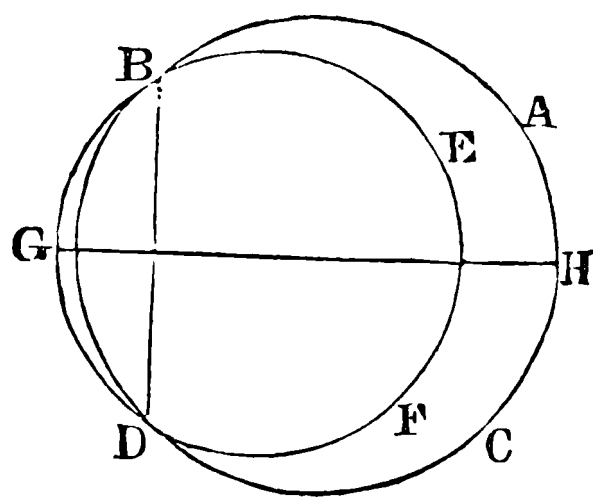
One circle cannot touch another in more points than one, whether it touch it internally or externally.

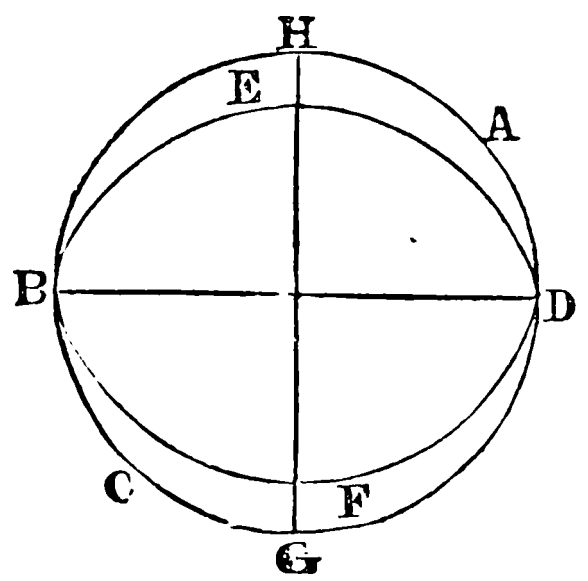
I. First, let the circle EBF touch the circle ABC internally in the point B.

Then EBF cannot touch ABC in any other point.

If possible, let it touch in D also;

CONSTRUCTION.—If it be possible, let EBF touch ABC in another point D; join BD, and draw GH bisecting BD at right angles (I. 10, 11).





PROOF.—Because the two points B, D are in the circumference of each of the circles, the straight line BD falls within each of them (III. 2).

Therefore the centre of each circle is in the straight line GH, which bisects BD at right angles (III. 1 cor.)

then GH passes through the point of contact, which it does not.

Therefore GH passes through the point of contact (III. 11).

But GH does not pass through the point of contact, because the points B, D are out of the line of GH; which is absurd.

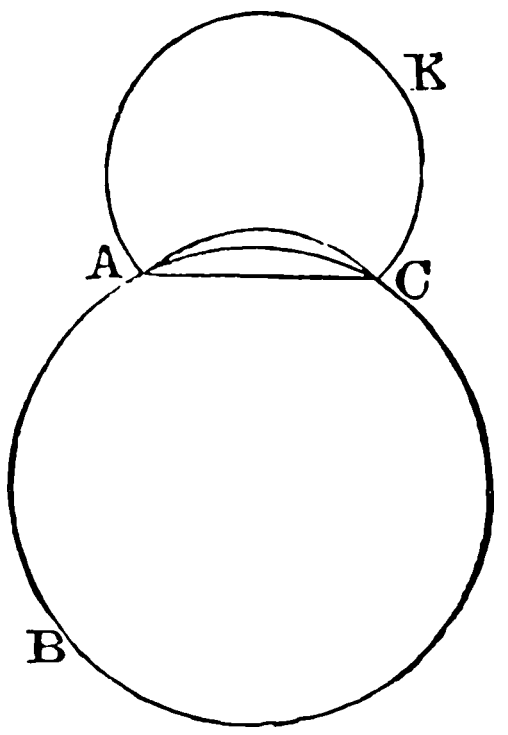
Therefore one circle cannot touch another internally in more points than one.

II. Next, let the circle ACK touch the circle ABC externally in the point A.

Then ACK cannot touch ABC in any other point.

CONSTRUCTION.—If it be possible, let ACK touch ABC in another point C. Join AC.

If possible let it touch in C also;



then AC falls without the circle ABC, which is absurd.

PROOF.—Because the points A, C are in the circumference of the circle ACK, the straight line AC must fall within the circle ACK (III. 2).

But the circle ACK is without the circle ABC (Hyp.);

Therefore the straight line AC is without the circle ABC.

But because the two points A, C are in the circumference of the circle ABC, the straight line AC falls within the circle ABC (III. 2); which is absurd.

Therefore one circle cannot touch another externally in more points than one.

And it has been shown that one circle cannot touch another internally in more points than one.

Therefore, one circle, &c. *Q.E.D.*

Proposition 14.—Theorem.

Equal straight lines in a circle are equally distant from the centre; and, conversely, those which are equally distant from the centre are equal to one another.

Let the straight lines AB, CD, in the circle ABDC, be equal to one another.

Then they shall be equally distant from the centre.

CONSTRUCTION.—Take E, the centre of the circle ABDC (III. 1).

From E draw EF, EG, perpendiculars to AB, CD (I. 12).

Join EA, EC.

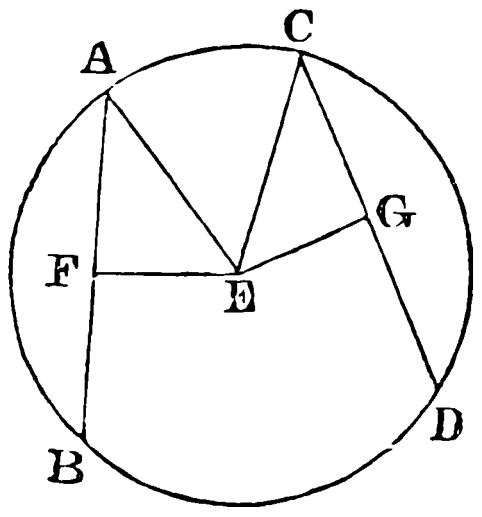
PROOF.—Because the straight line EF, passing through the centre, cuts the straight line AB, which does not pass through the centre, at right angles, it also bisects it (III. 3).

Therefore AF is equal to FB, and AB is double of AF.

For the like reason CD is double of CG.

$AF = CG$, But AB is equal to CD (Hyp.); therefore AF is equal to CG (Ax. 7).

And because AE is equal to CE, the square on AE is equal to the square on CE.



and $AF^2 + FE^2 = CG^2 + EG^2$. But the squares on AF, FE are equal to the square on AE, because the angle AFE is a right angle (I. 47).

For the like reason the squares on CG, GE are equal to the square on CE;

Therefore the squares on AF, FE are equal to the squares on CG, GE (Ax. 1).

But the square on AF is equal to the square on CG, because AF is equal to CG;

Therefore the remaining square on FE is equal to the remaining square on GE (Ax. 3);

$\therefore EF = EG$. And therefore the straight line EF is equal to the straight line EG.

But straight lines in a circle are said to be equally distant from the centre, when the perpendiculars drawn to them from the centre are equal (III. Def. 4);

Therefore AB, CD are equally distant from the centre.

Conversely, let the straight lines AB, CD be equally distant from the centre, that is, let EF be equal to EG;

Then AB shall be equal to CD.

Here $EF = EG$, and $AF^2 + EF^2 = CG^2 + EG^2$. PROOF.—The same construction being made, it may be demonstrated, as before, that AB is double AF, and CD double of CG, and that the squares on EF, FA are equal to the squares on EG, GC.

But the square on EF is equal to the square on EG, because EF is equal to EG (Hyp.);

Therefore the remaining square on FA is equal to the remaining square on GC (Ax. 3),

$\therefore AF = CG$, &c. And therefore the straight line AF is equal to the straight line CG.

But AB was shown to be double of AF, and CD double of CG;

Therefore AB is equal to CD (Ax. 6);

Therefore, equal straight lines, &c. *Q.E.D.*

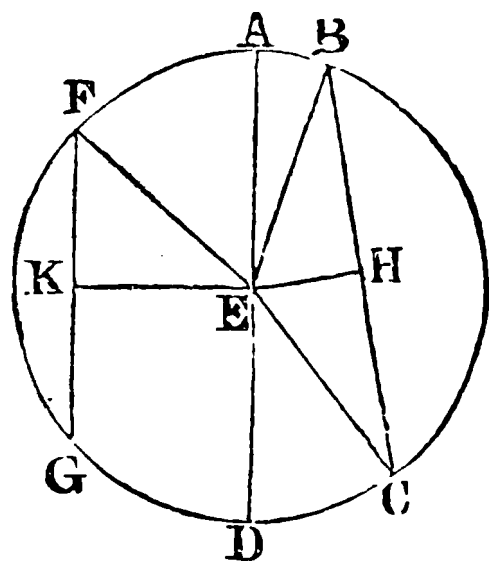
Proposition 15.—Theorem.

The diameter is the greatest straight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the less.

Let ABCD be a circle, of which AD is a diameter, and E the centre; and let BC be nearer to the centre than FG.

Then AD shall be greater than any straight line CB, which is not a diameter; and BC shall be greater than FG.

CONSTRUCTION.—From the centre E draw EH EK perpendiculars to BC, FG (I. 12), and join EB, EC, EF.



PROOF.—Because AE is equal to BE, and ED to EC,

Therefore AD is equal to BE, EC. BE + EC

But BE, EC are greater than BC (I. 20); or AD > BC,

Therefore also AD is greater than BC.

And because BC is nearer to the centre than FG (Hyp.), EH is less than EK (III. Def. 5). and EH < EK.

But, as was demonstrated in the preceding proposition, BC is double of BH, and FG double of FK, and the squares on EH, HB are equal to the squares on EK, KF. ∴ since $EH^2 + HB^2 = EK^2 + KF^2$,

But the square on EH is less than the square on EK, because EH is less than EK;

Therefore the square on HB is greater than the square on KF, and the straight line BH greater than FK; HB > FK.

And therefore BC is greater than FG.

Conversely, let BC be greater than FG.

Then BC shall be nearer to the centre than FG, that is, the same construction being made, EH shall be less than EK.

PROOF.—Because BC is greater than FG, BH is greater than FK.

But the squares on BH, HE are equal to the squares on FK, KE;

And the square on BH is greater than the square on FK, because BH is greater than FK;

Therefore the square on HE is less than the square on KE, and the straight line EH less than EK;

And therefore BC is nearer to the centre than FG (III. def. 5).

Therefore, the diameter, &c. *Q.E.D.*

Proposition 16.—Theorem.

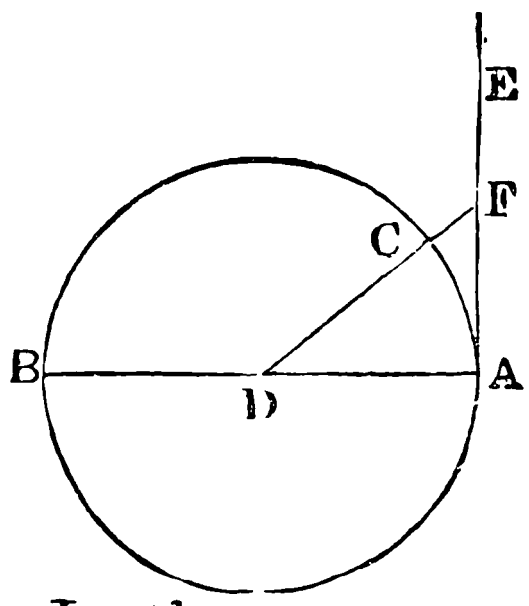
The straight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and a straight line, making an acute angle with the diameter at its extremity, cuts the circle.

Let ABC be a circle, of which D is the centre, and AB a diameter, and AE a line drawn from A perpendicular to AB.

The straight line AE shall fall without the circle.

Take any point F in AE,

CONSTRUCTION.—In AE take any point F; join DF, and let DF meet the circle in C.



then DF > DA and ∴ > DC.

PROOF.—Because DAF is a right angle, it is greater than the angle AFD (I. 17);

Therefore DF is greater than DA (I. 19).

But DA is equal to DC; therefore DF is greater than DC.

Therefore the point F is without the circle.

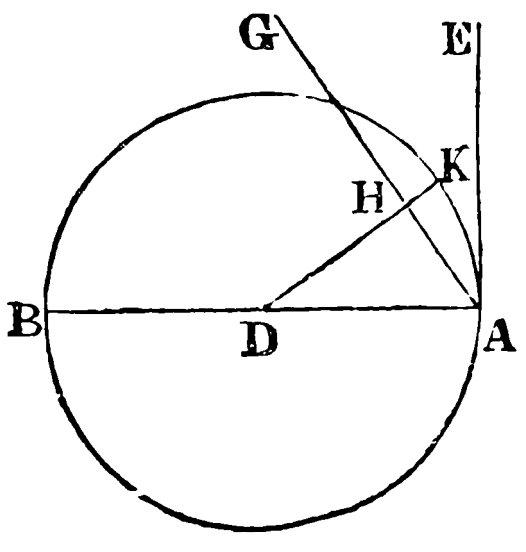
In the same manner it may be shown that any other point in AE, except the point A, is without the circle.

Therefore AE falls without the circle.

Again, let AG make with the diameter the angle DAG less than a right angle.

The line AG shall fall within the circle, and hence cut it.

Draw DH at right angles to HG, then DH < DA, and ∴ < DK.



CONSTRUCTION.—From D draw DH at right angles to AG, and meeting the circumference in K (I. 12).

PROOF.—Because DHA is a right angle, and DAH less than a right angle;

Therefore the side DH is less than the side DA (I. 19).

But DK is equal to DA; therefore DH is less than DK.

Therefore the point H is within the circle.

Therefore the straight line AG cuts the circle.
Therefore, the straight line, &c. *Q.E.D.*

COROLLARY.—From this it is manifest that the straight line which is drawn at right angles to the diameter of a circle, from the extremity of it, touches the circle (III. Def. 2); and that it touches it only in one point, because if it did meet the circle in two points it would fall within it (III. 2). Also it is evident that there can be but one straight line which touches the circle in the same point.

Proposition 17.—Problem.

To draw a straight line from a given point, either without or in the circumference, which shall touch a given circle.

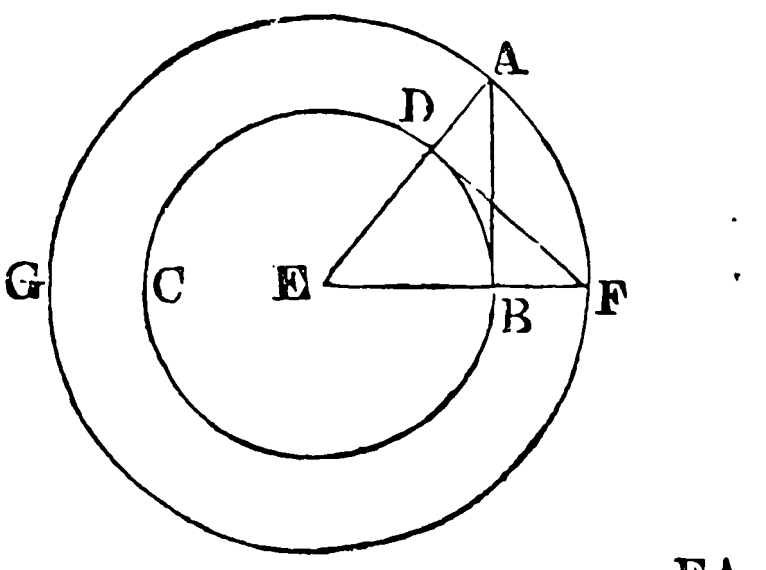
First, let the given point A be without the given circle BCD.

It is required to draw from A a straight line which shall touch the given circle.

CONSTRUCTION.—Find the centre E of the circle (III. 1), and join AE.

From the centre E, at the distance EA, describe the circle AFG.

From the point D draw DF at right angles to EA (I. 11), and join EBF and AB.



Then AB shall touch the circle BCD.

PROOF.—Because E is the centre of the circles AFG, BCD, EA is equal to EF, and ED to EB; EA, ED respectively = EF, EB

Therefore the two sides AE, EB are equal to the two sides FE, ED, each to each; and ∠ E common;

And the angle at E is common to the two triangles AEB, FED;

Therefore the base AB is equal to the base FD, and the triangle AEB to the triangle FED, and the other angles to the other angles, each to each, to which the equal sides are opposite (I. 4);

Therefore the angle ABE is equal to the angle FDE. ∴ ∠ ABE = ∠ FDE

But the angle FDE is a right angle (Const.); a right angle.

Therefore the angle ABE is a right angle (Ax. 1).

And EB is drawn from the centre (Const.)

But the straight line drawn at right angles to a diameter of a circle, from the extremity of it, touches the circle (III. 16, cor.);

∴ AB touches the circle.

Therefore AB touches the circle, and it is drawn from the given point A.

Next, let the given point be in the circumference of the circle, at the point D.

Draw DE to the centre E, and DF at right angles to DE;

Then DF touches the circle (III. 16, cor.)

Therefore, from the given points A and D, straight lines, AB and DF, have been drawn, touching the given circle BCD. *Q.E.F.*

Proposition 18.—Theorem.

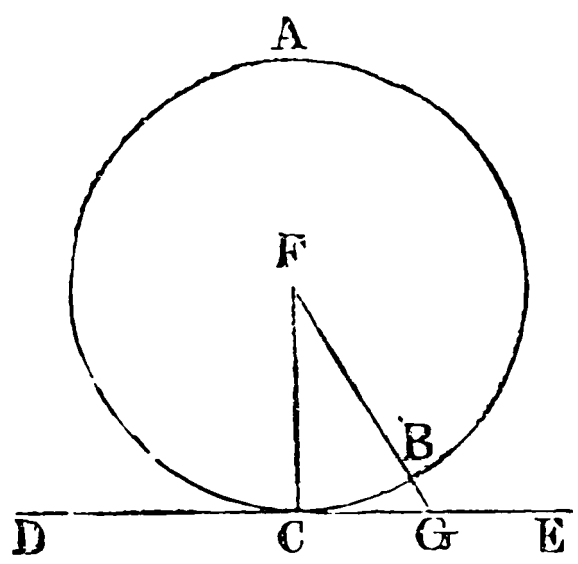
If a straight line touch a circle, the straight line drawn from the centre to the point of contact shall be perpendicular to the line touching the circle.

Let the straight line DE touch the circle ABC in the point C; take the centre F (III. 1), and draw the straight line FC.

FC shall be perpendicular to DE.

If not, suppose FG perpendicular.

CONSTRUCTION.—For, if not, let FG be drawn from the point F perpendicular to DE, meeting the circumference in B.



PROOF.—Because FGC is a right angle (Hyp.), FCG is an acute angle (I. 17), and to the greater angle the greater side is opposite (I. 19);

Therefore FC is greater than FG.

But FC is equal to FB; therefore FB

Then must FB > FG.

is greater than FG; the part greater than the whole, which is impossible.

Therefore FG is not perpendicular to DE.

In the same manner it may be shown that no other straight line from F is perpendicular to DE, but FC; therefore FC is perpendicular to DE.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 19.—Theorem.

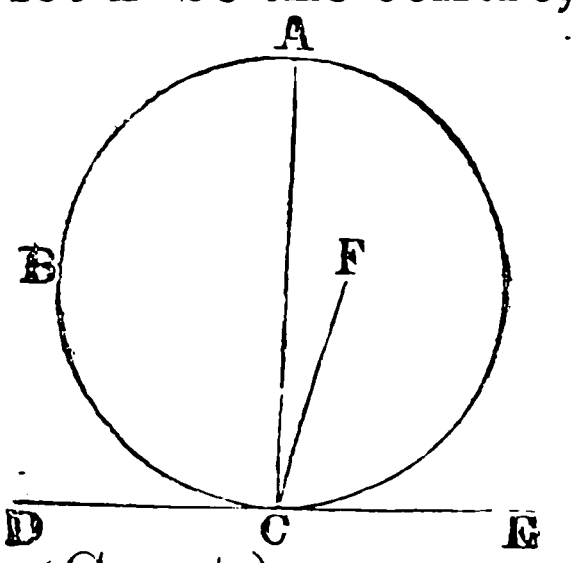
If a straight line touch a circle, and from the point of contact a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Let the straight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE.

The centre of the circle shall be in CA.

CONSTRUCTION.—For, if not, if possible, let F be the centre, and join CF.

If not, take F the centre, out of the line.



PROOF.—Because DE touches the circle ABC, and FC is drawn from the assumed centre to the point of contact,

Therefore FC is perpendicular to DE (III. 18);

Therefore FCE is a right angle.

But the angle ACE is also a right angle (Const.);

Therefore the angle FCE is equal to the angle ACE; the less to the greater, which is impossible.

Then ∠ ACE = ∠ FCE, being right angles.

Therefore F is not the centre of the circle ABC.

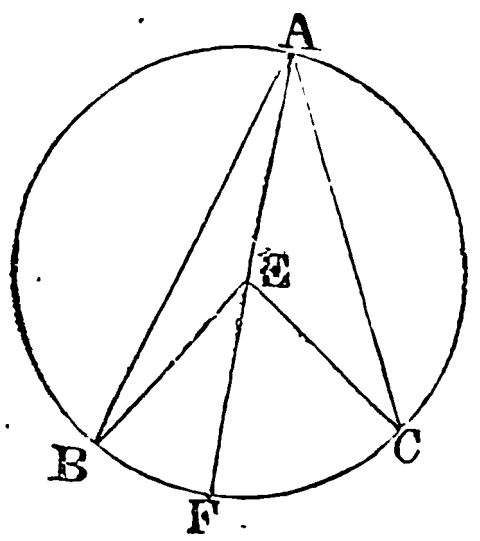
In the same manner it may be shown that no other point which is not in CA is the centre; therefore the centre is in CA.

Therefore, if a straight line, &c. *Q.E.D.*

Proposition 20.—Theorem.

The angle at the centre of a circle is double of the angle at the circumference, upon the same base, that is, upon the same arc.

Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the same arc BC for their base.



The angle BEC shall be double of the angle BAC.

CASE I.—First, let the centre E of the circle be within the angle BAC.

CONSTRUCTION.—Join AE, and produce it to the circumference in F.

PROOF.—Because EA is equal to EB, the angle EAB is equal to the angle EBA (I. 5);

Therefore the angles EAB, EBA are double of the angle EAB.

∠ BEF = ∠ EAB + ∠ EBA = 2 ∠ EAB, and so ∠ FEC = 2 ∠ EAC.

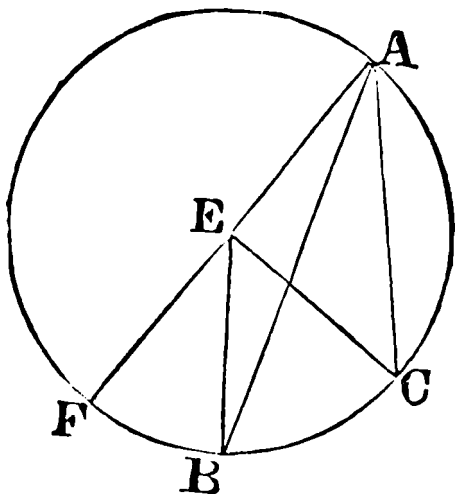
But the angle BEF is equal to the angles EAB, EBA (I. 32);

Therefore the angle BEF is double of the angle EAB.

For the same reason the angle FEC is double of the angle EAC.

∴ ∠ BEC = 2 ∠ BAC.

Therefore the whole angle BEC is double of the whole angle BAC.



∠ FEC = 2 ∠ EAC, and ∠ FEB = 2 ∠ CAB.

CASE II.—Next, let the centre E of the circle be without the angle BAC.

CONSTRUCTION.—Join AE, and produce it to meet the circumference in F.

PROOF.—It may be demonstrated, as in the first case, that the angle FEC is double of the angle FAC, and that FEB, a part of FEC, is double of FAB, a part of FAC;

∴ taking the difference ∠ BEC = 2 ∠ BAC.

Therefore the remaining angle BEC is double of the remaining angle BAC.

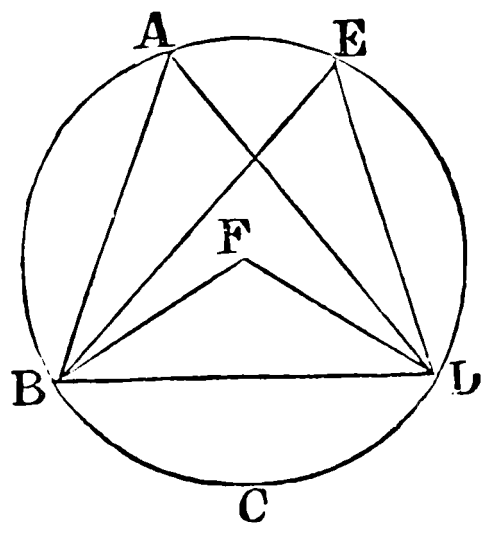
Therefore, the angle at the centre, &c. *Q.E.D.*

Proposition 21.—Theorem.

The angles in the same segment of a circle are equal to one another.

Let ABCD be a circle, and BAD, BED angles in the same segment BAED.

The angles BAD, BED shall be equal to one another.



CASE I.—First, let the segment BAED be greater than a semicircle.

CONSTRUCTION.—Take F, the centre of the circle ABCD (III. 1), and join BF, DF.

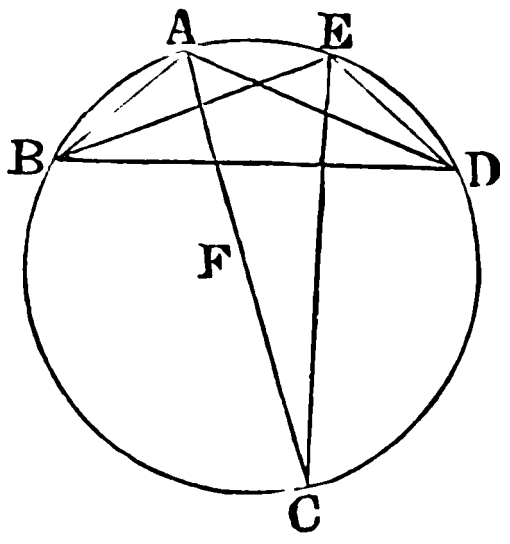
PROOF.—Because the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the same arc for their base, namely, BCD;

∠ BFD = 2 ∠ BAD,

Therefore the angle BFD is double of the angle BAD (III. 20).

For the same reason, the angle BFD is double of the angle BED;

and also = 2 ∠ BED.



Therefore the angle BAD is equal to the angle BED (Ax. 7).

∴ ∠ BAD = ∠ BED.

Case II.—Next, let the segment BAED be not greater than a semicircle.

Construction.—Draw AF to the centre, and produce it to C, and join CE.

Proof.—Then the segment BADC is greater than a semicircle, and therefore the angles BAC, BEC in it are equal by the first case.

∠ BAC = ∠ BEC,

For the same reason, because the segment CBED is greater than a semicircle, the angles CAD, CED are equal.

and ∠ CAD = ∠ CED.

Therefore the whole angle BAD is equal to the whole angle BED (Ax. 2).

∴ ∠ BAD = ∠ BED.

Therefore, the angles in the same segment, &c. *Q.E.D.*

Proposition 22.—Theorem.

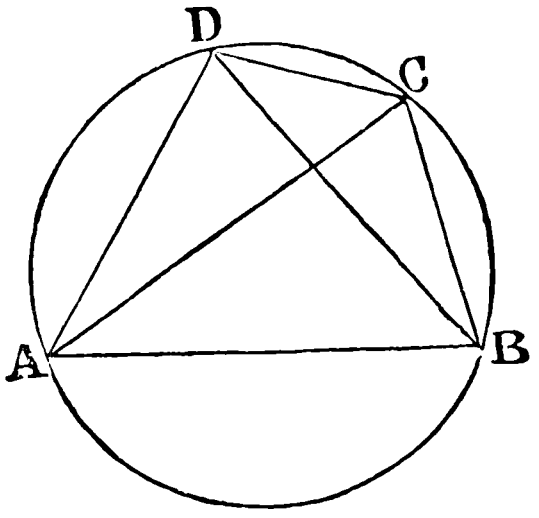
The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in the circle ABCD.

Any two of its opposite angles shall be together equal to two right angles.

Construction.—Join AC, BD.

Proof.—Because the three angles of every triangle are together equal to two right angles (I. 32),



The three angles of the triangle CAB, namely, CAB, ACB, ABC, are together equal to two right angles.

∠ CAB + ∠ ACB + ∠ ABC = 2 right angles.

But the angle CAB is equal to the angle CDB, because they are in the same segment CDAB (III. 21);

The first two together = ∠ CDB +

And the angle ACB is equal to the angle ADB, because they are in the same segment ADCB;

∠ ADB = ∠ ADC.

Therefore the two angles CAB, ACB are together equal to the whole angle ADC (Ax. 2).

To each of these equals add the angle ABC;

Therefore the three angles CAB, ACB, ABC are equal to the two angles ABC, ADC.

But the angles CAB, ACB, ABC are together equal to two right angles (I. 32);

∴ ∠ ADC + ∠ ABC = 2 right angles.

Therefore also the angles ABC, ADC are together equal to two right angles.

In like manner it may be shown that the angles BAD, BCD are together equal to two right angles.

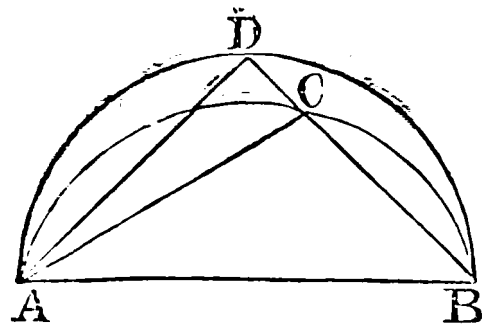
Therefore, the opposite angles, &c. *Q.E.D.*

Proposition 23.—Theorem.

Upon the same straight line, and on the same side of it, there cannot be two similar segments of circles not coinciding with one another.

If possible,

If it be possible, on the same straight line AB, and on the same side of it, let there be two similar segments of circles ACB, ADB not coinciding with one another.



CONSTRUCTION.—Then, because the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point (III. 10);

Therefore one of the segments must fall within the other.

Let ACB fall within ADB; draw the straight line BCD, and join AC, AD.

PROOF.—Because the segment ACB is similar to the segment ADB (Hyp.), and that similar segments of circles contain equal angles (III. Def. 11);

exterior ∠ ACB = interior and opposite ∠ ADC.

Therefore the angle ACB is equal to the angle ADB; that is, the exterior angle of the triangle ACD, equal to the interior and opposite angle; which is impossible (I. 16).

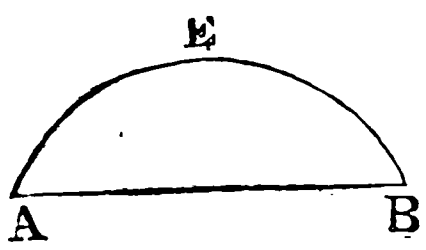
Therefore, there cannot be two similar segments of circles on the same straight line, and on the same side of it, which do not coincide. *Q.E.D.*

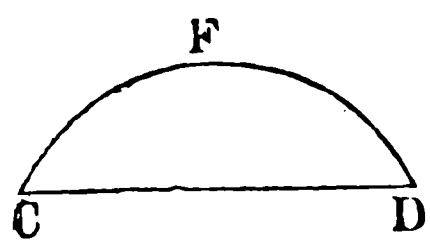
Proposition 24.—Theorem.

Similar segments of circles upon equal straight lines are equal to one another.

Let AEB, CFD be similar segments of circles upon the equal straight lines AB, CD.

The segment AEB shall be equal to the segment CFD.





They are equal, because they must coincide by Prop. 23.

Proof.—For if the segment AEB be applied to the segment CFD, so that the point A may be on the point C, and the straight line AB on the straight line CD,

Then the point B shall coincide with the point D, because AB is equal to CD.

And the straight line AB coinciding with CD, the segment AEB must coincide with the segment CFD (III. 23); and is therefore equal to it.

Therefore, similar segments, &c. *Q.E.D.*

Proposition 25.—Problem.

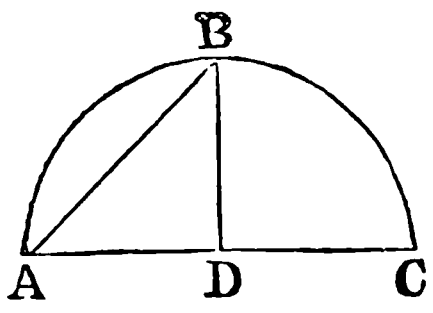
A segment of a circle being given, to describe the circle of which it is the segment.

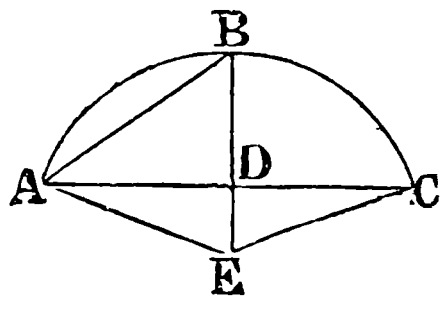
Let ABC be the given segment of a circle.

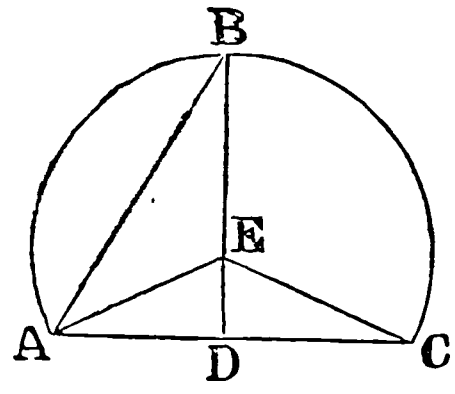
It is required to describe the circle of which ABC is a segment.

Construction.—Bisect AC in D (I. 10).

From the point D draw DB at right angles to AC (I. 11), and join AB.







Case I.—First, let the angles ABD, BAD be equal to one another.

Then D shall be the centre of the circle required.

When ∠ABD = ∠BAD,

PROOF.—Because the angle ABD is equal to the angle BAD (Hyp.);

DA = DB = DC.

Therefore DB is equal to DA (I. 6).

But DA is equal to DC (Const.);

Therefore DB is equal to DC (Ax. 1).

Therefore the three straight lines DA, DB, DC are all equal;

and ∴ D the centre.

And therefore D is the centre of the circle (III. 9).

Hence, if from the centre D, at the distance of any of the three lines, DA, DB, DC a circle be described, it will pass through the other two points, and be the circle required.

CASE II.—Next, let the angles ABD, BAD be not equal to one another.

Make ∠ BAE = ∠ ABD.

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAE equal to the angle ABD (I. 23);

Produce BD, if necessary, to E, and join EC.

Then E shall be the centre of the circle required.

∴ EA = EB

PROOF.—Because the angle BAE is equal to the angle ABE (Const.), EA is equal to EB (I. 6).

And because AD is equal to CD (Const.), and DE is common to the two triangles ADE, CDE,

The two sides AD, DE are equal to the two sides CD, DE, each to each;

And the angle ADE is equal to the angle CDE, for each of them is a right angle (Const.);

and EA = EC.

Therefore the base EA is equal to the base EC (I. 4).

But EA was shown to be equal to EB;

∴ EA = EB = EC,

Therefore EB is equal to EC (Ax. 1).

Therefore the three straight lines EA, EB, EC are all equal;

and therefore E is the centre.

And therefore E is the centre of the circle (III. 9).

Hence, if from the centre E, at the distance of any of the three lines EA, EB, EC, a circle be described, it will pass through the other two points, and be the circle required.

In the *first* case, it is evident that, because the centre D is in AC, the segment ABC is a semicircle.

In the *second* case, if the angle ABD be greater than BAD, the centre E falls without the segment ABC, which is therefore less than a semicircle;

But if the angle ABD be less than the angle BAD, the

centre E falls within the segment ABC, which is therefore greater than a semicircle.

Therefore, a segment of a circle being given, the circle has been described of which it is a segment. *Q.E.F.*

Proposition 26.—Theorem.

In equal circles, equal angles stand upon equal arcs, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, having the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences.

The arc BKC shall be equal to the arc ELF.

CONSTRUCTION.—Join BC, EF.

PROOF.—Because the circles ABC, DEF are equal (Hyp.), the straight lines from their centres are equal (III. def. 1);

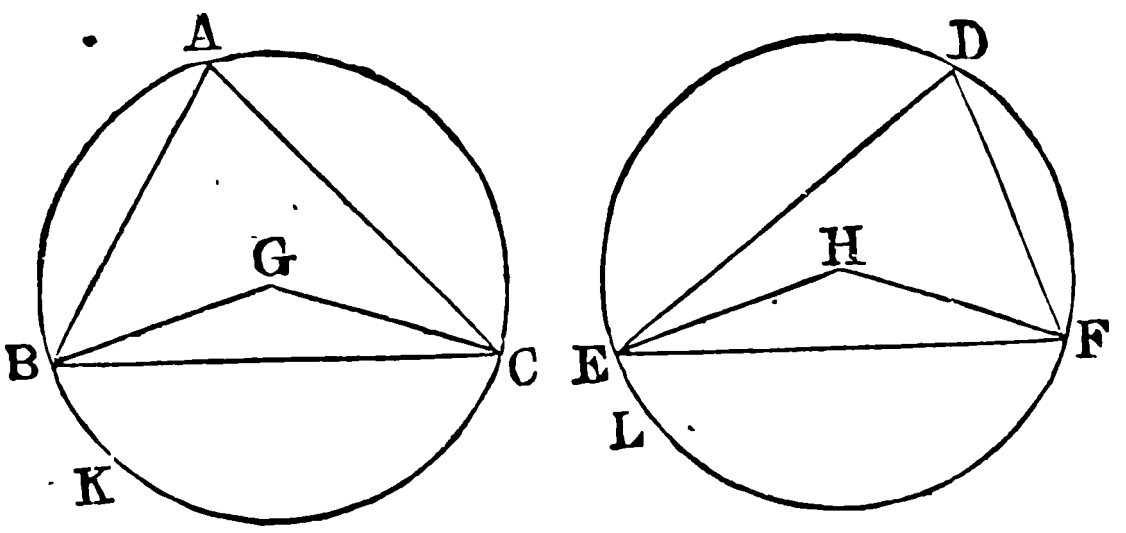
Therefore the two sides BG, GC are equal to the two sides EH, HF, each to each;

Triangles BGC and EHF are equal in every respect.

And the angle at G is equal to the angle at H (Hyp.);

Therefore the base BC is equal to the base EF (I. 4).

And because the angle at A is equal to the angle at D (Hyp.),



The segment BAC is similar to the segment EDF (III. def. 11),

∴ segments BAC and EDF are similar and on equal straight lines.

And they are on equal straight lines BC, EF.

But similar segments of circles on equal straight lines are equal to one another (III. 24);

Therefore the segment BAC is equal to the segment EDF.

∴ are equal.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining segment BKC is equal to the remaining segment ELF (Ax. 3).

∴ arc BKC = arc ELF.

Therefore the arc BKC is equal to the arc ELF.

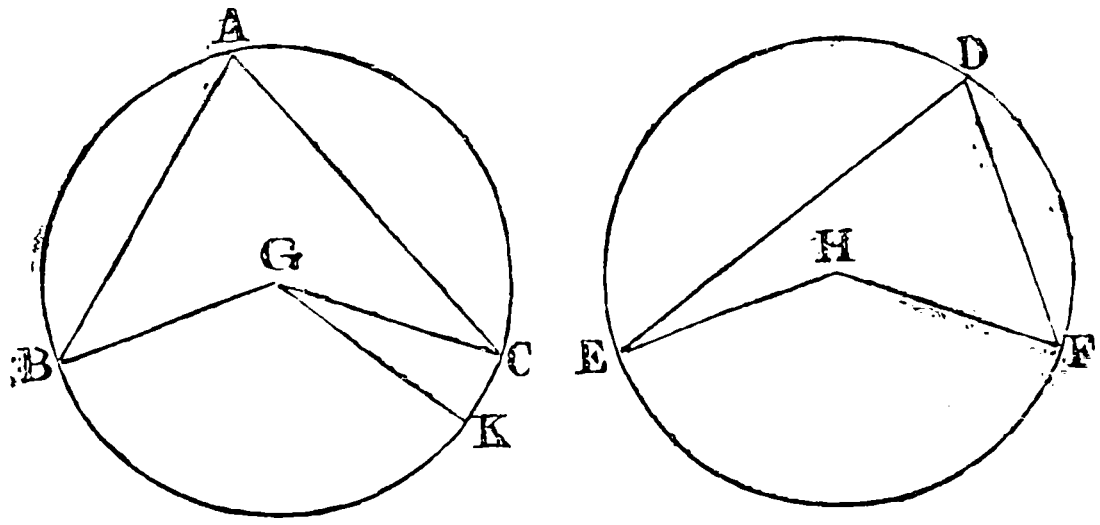
Therefore, in equal circles, &c. *Q.E.D.*

Proposition 27.—Theorem.

In equal circles, the angles which stand upon equal arcs are equal to one another, whether they be at the centres or at the circumferences.

Let ABC, DEF be equal circles, and let the angles BGC, EHF, at their centres, and the angles BAC, EDF, at their circumferences, stand on equal arcs BC, EF.

The angle BGC shall be equal to the angle EHF, and the angle BAC equal to the angle EDF.



Construction.—If the angle BGC be equal to the angle EHF, it is manifest that the angle BAC is also equal to the angle EDF (III. 20, ax. 7).

But, if not, one of them must be the greater. Let BGC

If one ∠ is greater than the other, the corresponding arc is greater.

be the greater, and at the point G, in the straight line BG, make the angle BGK equal to the angle EHF (I. 23).

Proof.—Because the angle BGK is equal to the angle EHF, and that in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BK is equal to the arc EF.

But the arc EF is equal to the arc BC (Hyp.);

Therefore the arc BK is equal to the arc BC (Ax. 1); the less to the greater, which is impossible.

Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it.

And the angle at A is half of the angle BGC, and the angle at D is half of the angle EHF (III. 20);

Therefore the angle at A is equal to the angle at D (Ax. 7).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 28.—Theorem.

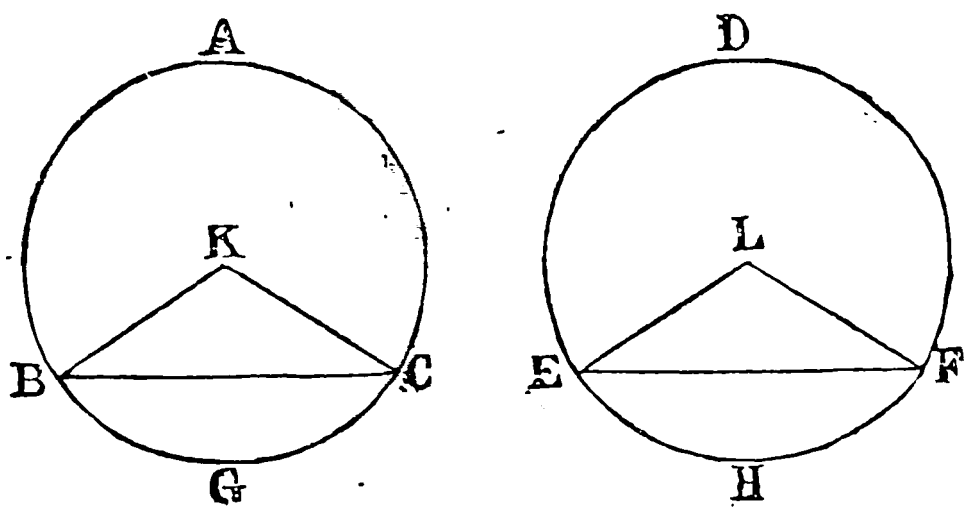
In equal circles, equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less.

Let ABC, DEF be equal circles, and BC, EF equal chords in them, which cut off the two greater arcs BAC, EDF, and the two less arcs BGC, EHF.

The greater arc BAC shall be equal to the greater arc EDF, and the less arc BGC equal to the less arc EHF.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF.

Take K and L the centres.



PROOF.—Because the circles ABC, DEF are equal, their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each;

And the base BC is equal to the base EF (Hyp.);

Therefore the angle BKC is equal to the angle ELF (I. 8).

Triangles BKC and ELF are equal in every respect.

But in equal circles equal angles stand on equal arcs, when they are at the centres (III. 26);

Therefore the arc BGC is equal to the arc EHF.

But the whole circle ABC is equal to the whole circle DEF (Hyp.);

Therefore the remaining arc BAC is equal to the remaining arc EDF (Ax. 3).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 29.—Theorem.

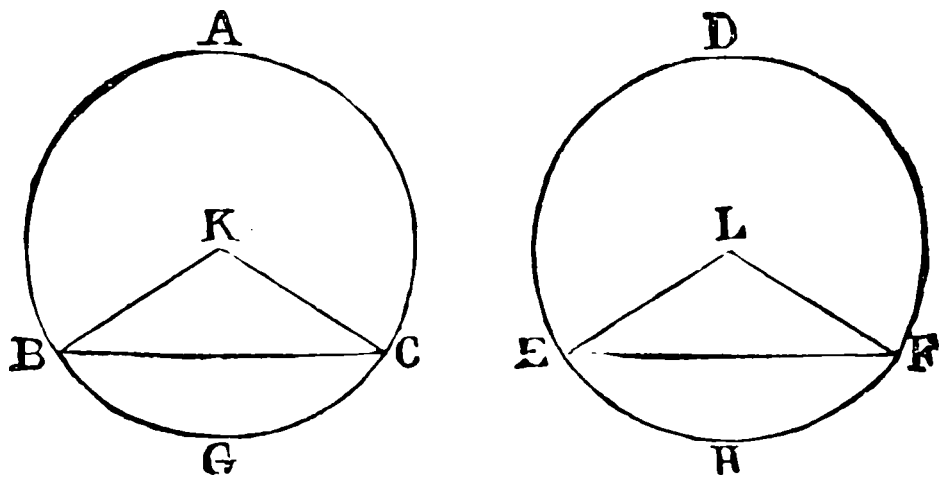
In equal circles equal arcs are subtended by equal chords.

Let ABC, DEF be equal circles, and let BGC, EHF be equal arcs in them, and join BC, EF.

The chord BC shall be equal to the chord EF.

Take K and L the centres.

CONSTRUCTION.—Take K, L, the centres of the circles (III. 1), and join BK, KC, EL, LF.



Then ∠ BKC = ∠ ELF,

PROOF.—Because the arc BGC is equal to the arc EHF (Hyp.), the angle BKC is equal to the angle ELF (III. 27).

And because the circles ABC, DEF are equal (Hyp.), their radii are equal (III. def. 1).

Therefore the two sides BK, KC are equal to the two sides EL, LF, each to each; and they contain equal angles;

and so base BC = base EF.

Therefore the base BC is equal to the base EF (I. 4).

Therefore, in equal circles, &c. *Q.E.D.*

Proposition 30.—Problem.

To bisect a given arc, that is, to divide it into two equal parts.

Let ADB be the given arc.

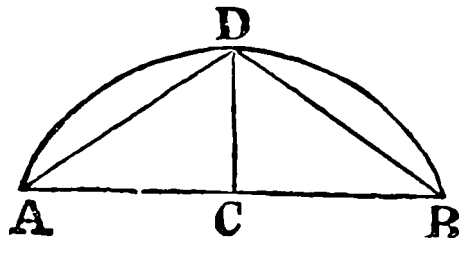
It is required to bisect it.

CONSTRUCTION.—Join AB, and bisect it in C (I. 10).

From the point C draw CD at right angles to AB (I. 11), and join AD and DB.

Then the arc ABD shall be bisected in the point D.

In the triangles ADC and CDB,



PROOF.—Because AC is equal to CB (Const.), and CD is common to the two triangles ACD, BCD;

The two sides AC, CD are equal to the two sides BC, CD, each to each;

And the angle ACD is equal to the angle BCD, because each of them is a right angle (Const.);

Therefore the base AD is equal to the base BD (I. 4). base AD = base BD.

But equal chords cut off equal arcs, the greater equal to the greater, and the less equal to the less (III. 28);

And each of the arcs AD, DB is less than a semicircle, because DC, if produced, is a diameter (III. 1, cor.);

Therefore the arc AD is equal to the arc DB.

Therefore, the given arc is bisected in D. *Q.E.F.*

Proposition 31.—Theorem.

In a circle, the angle in a semicircle is a right angle; but the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

Let ABC be a circle, of which BC is a diameter, and E the centre; and draw CA, dividing the circle into the segments ABC, ADC, and join BA, AD, DC.

The angle in the semicircle BAC shall be a right angle;

The angle in the segment ABC, which is greater than a semicircle, shall be less than a right angle;

The angle in the segment ADC, which is less than a semicircle, shall be greater than a right angle.

CONSTRUCTION.—Join AE, and produce BA to F.

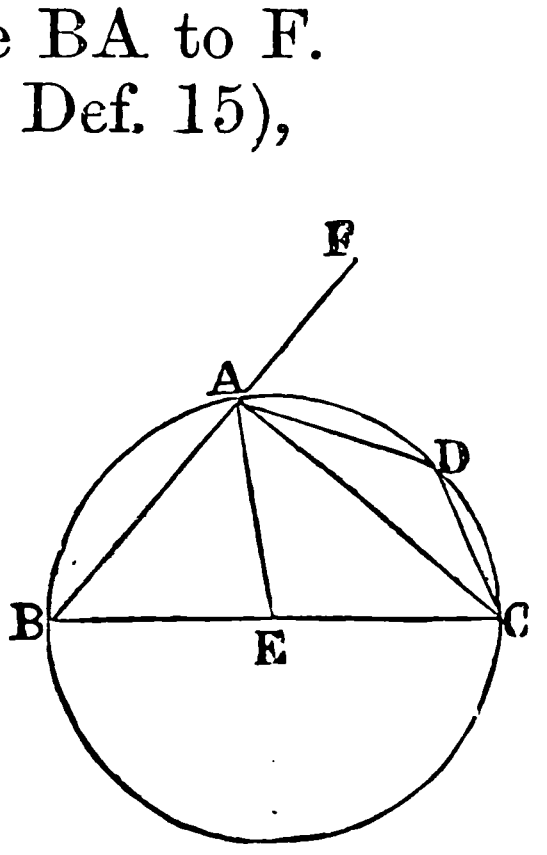
PROOF.—Because EA is equal to EB (I. Def. 15),

The angle EAB is equal to the angle EBA (I. 5);

And, because EA is equal to EC,

The angle EAC is equal to the angle ECA;

Therefore the whole angle BAC is equal to the two angles ABC, ACB (Ax. 2).



∠ BAE + ∠ EAC, or ∠ BAC = ∠ ABC + ∠ ACB = ∠ FAC and ∴ a right angle.

But FAC, the exterior angle of the triangle ABC, is equal to the two angles ABC, ACB (I. 32).

Therefore the angle BAC is equal to the angle FAC (Ax. 1),

And therefore each of them is a right angle (I. Def. 10);
Therefore the angle in a semicircle BAC is a right angle.
And because the two angles ABC, BAC, of the triangle ABC, are together less than two right angles (I. 17), and that BAC has been shown to be a right angle;

∴ ∠ ABC < a right angle.

Therefore the angle ABC is less than a right angle.
Therefore the angle in a segment ABC, greater than a semicircle, is less than a right angle.
And, because ABCD is a quadrilateral figure in a circle, any two of its opposite angles are together equal to two right angles (III. 22);
Therefore the angles ABC, ADC are together equal to two right angles.
But the angle ABC has been shown to be less than a right angle;

Hence ∠ ADC > a right angle, by Prop. 32.

Therefore the angle ADC is greater than a right angle;
Therefore the angle in a segment ADC, less than a semicircle, is greater than a right angle.
Therefore, the angle, &c. *Q.E.D.*

COROLLARY.—From this demonstration it is manifest that, if one angle of a triangle be equal to the other two, it is a right angle.
For the angle adjacent to it is equal to the same two angles (I. 32).
And, when the adjacent angles are equal, they are right angles (I. def. 10).

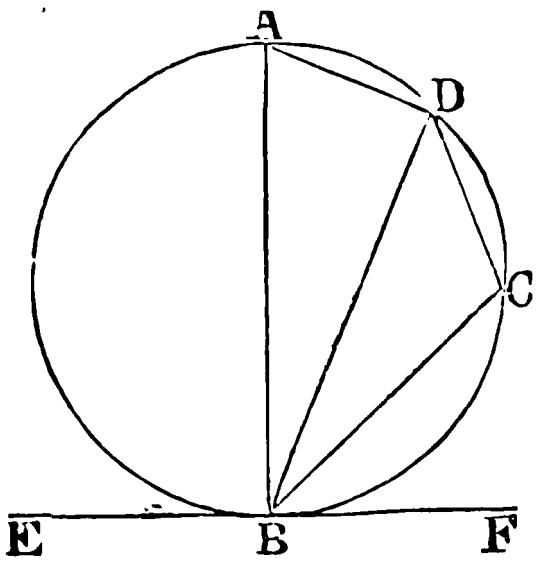
Proposition 32.—Theorem.

The angles contained by a tangent to a circle and a chord drawn from the point of contact are equal to the angles in the alternate segments of the circle.

Let EF be a tangent to the circle ABCD, and BD a chord drawn from the point of contact B, cutting the circle.
The angles which BD makes with the tangent EF shall be equal to the angles in the alternate segments of the circle;
That is, the angle DBF shall be equal to the angle in the segment BAD, and the angle DBE shall be equal to the angle in the segment BCD.

CONSTRUCTION.—From the point B draw BA at right angles to EF (I. 11).

Take any point C in the circumference BD, and join AD, DC, CB.



The centre is in BA.

PROOF.—Because the straight line EF touches the circle ABCD at the point B (Hyp.), and BA is drawn at right angles to the tangent from the point of contact B (Const.),

The centre of the circle is in BA (III. 19).

Therefore the angle ADB, being in a semicircle, is a right angle (III. 31).

∴ ADB is a right angle,

Therefore the other two angles BAD, ABD are equal to a right angle (I. 32).

and ∠ BAD + ∠ ABD = a right angle

But ABF is also a right angle (Const.);

Therefore the angle ABF is equal to the angles BAD, ABD.

= ABF.

From each of these equals take away the common angle ABD;

Therefore the remaining angle DBF is equal to the remaining angle BAD, which is in the alternate segment of the circle (Ax. 3).

∴ ∠ BAD = ∠ DBF.

And because ABCD is a quadrilateral figure in a circle, the opposite angles BAD, BCD are together equal to two right angles (III. 22).

Also, ∠ BCD + ∠ BAD = 2 right angles

But the angles DBF, DBE are together equal to two right angles (I. 13);

Therefore the angles DBF, DBE are together equal to the angles BAD, BCD.

And the angle DBF has been shown equal to the angle BAD;

= ∠ DBF + ∠ DBE.

Therefore the remaining angle DBE is equal to the angle BCD, which is in the alternate segment of the circle (Ax. 3).

∴ ∠ DBE = ∠ BCD.

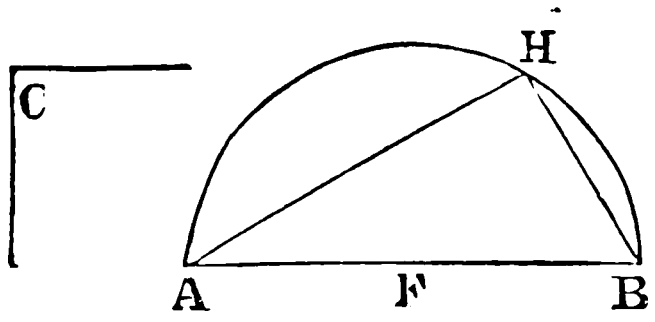
Therefore, the angles, &c. *Q.E.D.*

Proposition 33.—Problem.

Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.

Let AB be the given straight line, and C the given rectilineal angle.

It is required to describe, on the given straight line AB, a segment of a circle, containing an angle equal to the angle C.



CASE I.—Let the angle C be a right angle.

CONSTRUCTION.—Bisect AB in F (I. 10).

From the centre F, at the distance FB, describe the semicircle AHB.

Then AHB shall be the segment required.

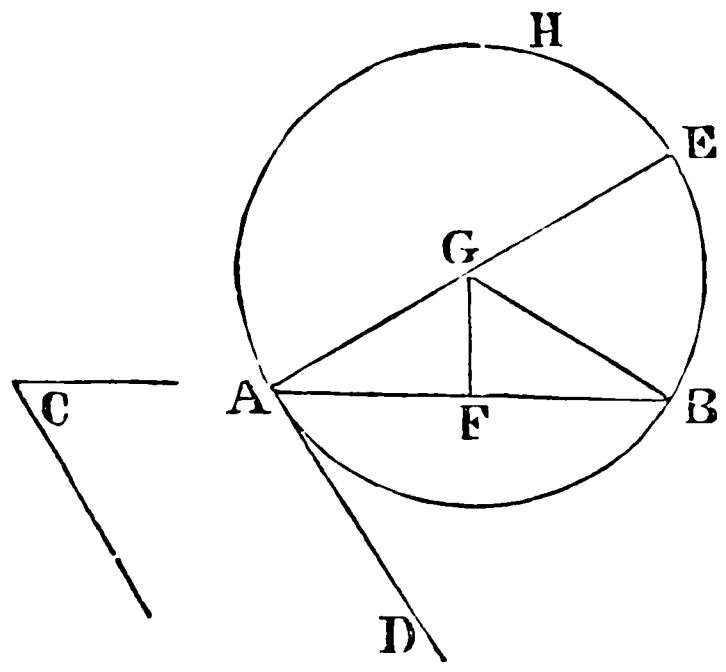
Angle in a semicircle is a right angle.

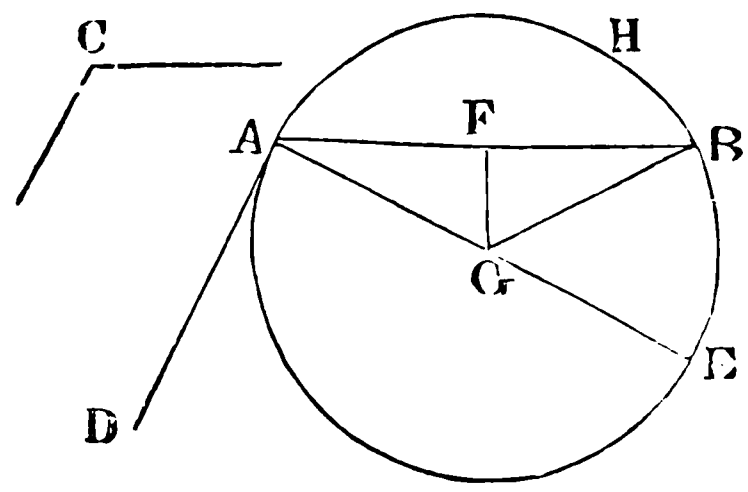
PROOF.—Because AHB is a semicircle, the angle AHB in it is a right angle, and therefore equal to the angle C (III. 31).

CASE II.—Let C be not a right angle.

At point A make ∠ BAD = C;

CONSTRUCTION.—At the point A, in the straight line AB, make the angle BAD equal to the angle C (I. 23).





and draw AE at right angles to AD.

From the point A draw AE at right angles to AD (I. 11).

From F, middle of AB, draw perpendicular, meeting AE in G.

Bisect AB in F (I. 10).

From the point F draw FG at right angles to AB (I. 11), and join GB.

Because AF is equal to BF (Const.), and FG is common to the two triangles AFG, BFG;

The two sides AF, FG are equal to the two sides BF, FG, each to each;

And the angle AFG is equal to the angle BFG (Const.);

Therefore the base AG is equal to the base BG (I. 4).

Then G is the centre of a circle passing through A and B,

And the circle described from the centre G, at the distance GA, will therefore pass through the point B.

Let this circle be described; and let it be AHB.

The segment AHB shall contain an angle equal to the given rectilineal angle C.

PROOF.—Because from the point A, the extremity of the diameter AE, AD is drawn at right angles to AE (Const.); And AD touches the circle,

Therefore AD touches the circle (III. 16, cor.)

Because AB is drawn from the point of contact A, the angle DAB is equal to the angle in the alternate segment AHB (III. 32).

But the angle DAB is equal to the angle C (Const.); and ∴ ∠ in AHB = ∠ DAB or C.

Therefore the angle in the segment AHB is equal to the angle C (Ax. 1).

Therefore, on the given straight line AB, the segment AHB of a circle has been described, containing an angle equal to the given angle C. *Q.E.F.*

Proposition 34.—Problem.

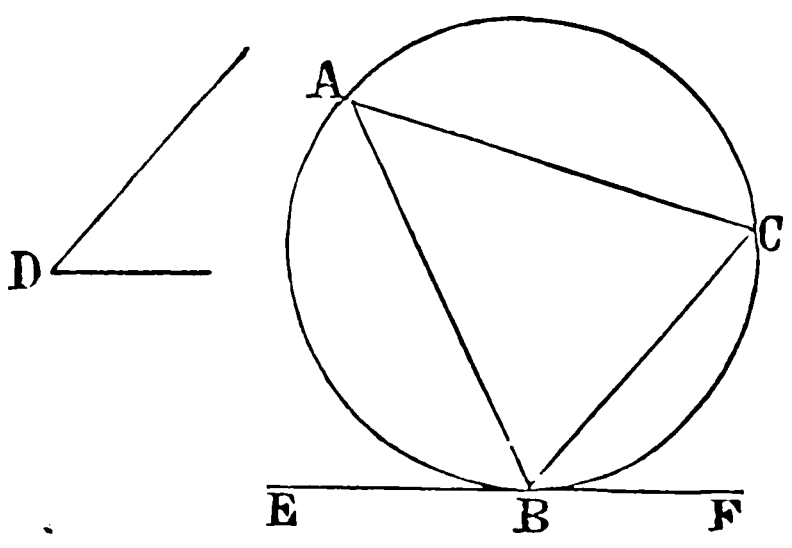
From a given circle to cut off a segment which shall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle.

It is required to cut off from the circle ABC a segment that shall contain an angle equal to the angle D.

CONSTRUCTION.—Draw the straight line EF touching the circle ABC in the point B (III. 17); Draw tangent EBF,

And at the point B, in the straight line BF, make the angle FBC equal to the angle D (I. 23). and make ∠ FBC = given ∠.



Then the segment BAC shall contain an angle equal to the given angle D.

PROOF.—Because the straight line EF touches the circle ABC, and BC is drawn from the point of contact B (Const.);

Therefore the angle FBC is equal to the angle in the alternate segment BAC of the circle (III. 32).

But the angle FBC is equal to the angle D (Const.);

Therefore the angle in the segment BAC is equal to the angle D (Ax. 1). ∴ ∠ BAC = ∠ FBC = ∠ D.

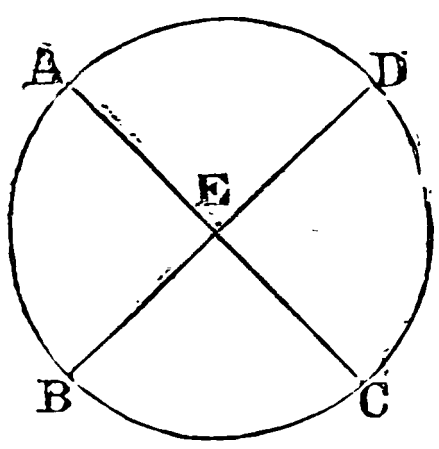
Therefore, from the given circle ABC, the segment BAC has been cut off, containing an angle equal to the given angle D. *Q.E.F.*

Proposition 35.—Theorem.

If two straight lines within a circle cut one another, the rectangle contained by the segments of one of them shall be equal to the rectangle contained by the segments of the other.

Let the two straight lines AC, BD cut one another in the point E, within the circle ABCD.

The rectangle contained by AE and EC shall be equal to the rectangle contained by BE and ED.

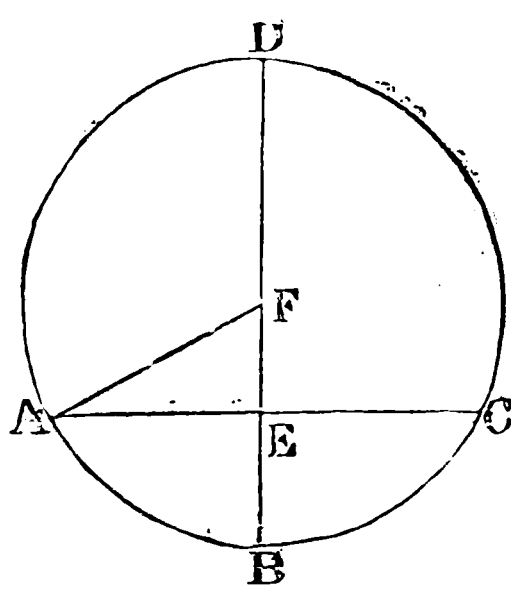


Case I.—Let AC, BD pass each of them through the centre.

Proof.—Because E is the centre, EA, EB, EC, ED are all equal (I. def. 15);

Therefore the rectangle AE, EC is equal to the rectangle BE, ED.

Case II.—Let one of them, BD, pass through the centre, and cut the other, AC, which does not pass through the centre, at right angles, in the point E.



Construction.—Bisect BD in F, then F is the centre of the circle; join AF.

Proof.—Because BD, which passes through the centre, cuts AC, which does not pass through the centre, at right angles in E (Hyp.);

Therefore AE is equal to EC (III. 3).

AE = EC.

And because BD is cut into two equal parts in the point F, and into two unequal parts in the point E,

BE·ED + EF² = FB² = AF² = AE² + EF².

The rectangle BE, ED, together with the square on EF, is equal to the square on FB (II. 5); that is, the square on AF.

But the square on AF is equal to the squares on AE, EF (I. 47);

Therefore the rectangle BE, ED, together with the square on EF, is equal to the squares on AE, EF (Ax. 1).

Take away the common square on EF;

∴ BE·ED = AE² = AE·EC.

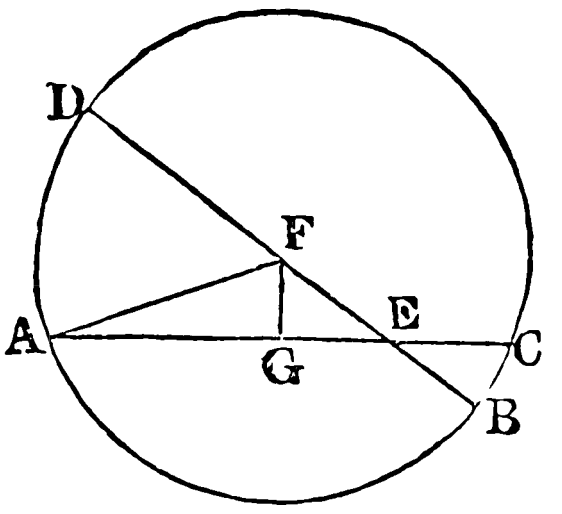
Then the remaining rectangle BE, ED is equal to the remaining square on AE; that is, to the rectangle AE, EC, since AE is equal to EC.

CASE III.—Let BD, which passes through the centre, cut the other AC, which does not pass through the centre, in the point E, but not at right angles.

CONSTRUCTION.—Bisect BD in F, then F is the centre of the circle.

Join AF, and from F draw FG perpendicular to AC (I. 12).

PROOF.—Then AG is equal to GC (III. 3).



Bisect BD in F the centre. Draw FG at right angles to AC. ∴ AG = GC.

Therefore the rectangle AE, EC, together with the square on EG, is equal to the square on AG (II. 5).

To each of these equals add the square on GF;

Then the rectangle AE, EC, together with the squares on EG, GF, is equal to the squares on AG, GF (Ax. 2).

Now, $AE \cdot EC + EG^2 = AG^2$.

But the squares on EG, GF are equal to the square on EF;

And the squares on AG, GF are equal to the square on AF (I. 47).

Therefore the rectangle AE, EC, together with the square on EF, is equal to the square on AF; that is, the square on FB.

But the square on FB is equal to the rectangle BE, ED, together with the square on EF (II. 5);

∴ $AE \cdot EC + EF^2 = AF^2 = FB^2 = BE \cdot ED + EF^2$.

Therefore the rectangle AE, EC, together with the square on EF, is equal to the rectangle BE, ED, together with the square on EF.

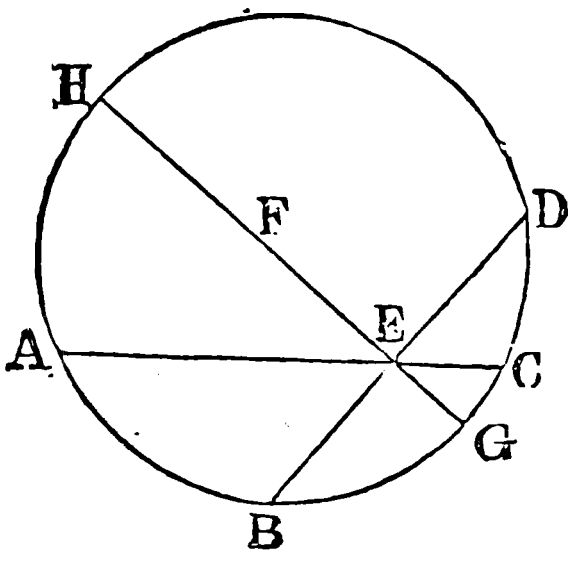
Take away the common square on EF;

And the remaining rectangle AE, EC is equal to the remaining rectangle BE, ED (Ax. 3).

∴ $AE \cdot EC = BE \cdot ED$.

CASE IV.—Let neither of the straight lines AC, BD pass through the centre.

CONSTRUCTION.—Take the centre F (III. 1), and through E, the intersection of the lines AC, BD, draw the diameter GEFH.



Again, $GE \cdot EH = AE \cdot EC$ and $= BE \cdot ED$, as just shown.

PROOF.—Because the rectangle GE, EH is equal, as has been shown, to the rectangle AE, EC, and also to the rectangle BE, ED;

Therefore the rectangle AE, EC is equal to the rectangle BE, ED (Ax. 1).

∴ $AE \cdot EC = BE \cdot ED$.

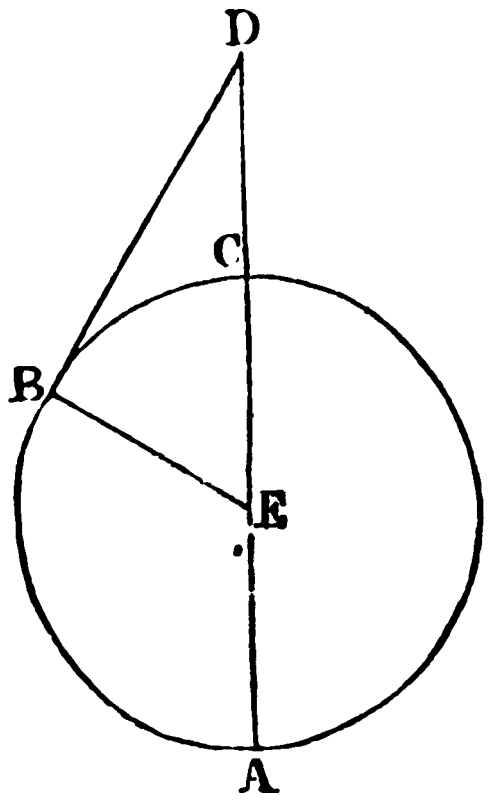
Therefore, if two straight lines, &c. *Q.E.D.*

Proposition 36.—Theorem.

If from a point without a circle two straight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, shall be equal to the square on the line which touches it.

Let D be any point without the circle ABC, and let DCA, DB be two straight lines drawn from it, of which DCA cuts the circle, and DB touches it.

The rectangle AD, DC shall be equal to the square on DB.



$AD \cdot DC + EC^2 = ED^2$.

Case I.—Let DCA pass through the centre E, and join EB.

Proof.—Then EBD is a right angle (III. 18).

And because the straight line AC is bisected in E, and produced to D, the rectangle AD, DC, together with the square on EC, is equal to the square on ED (II. 6).

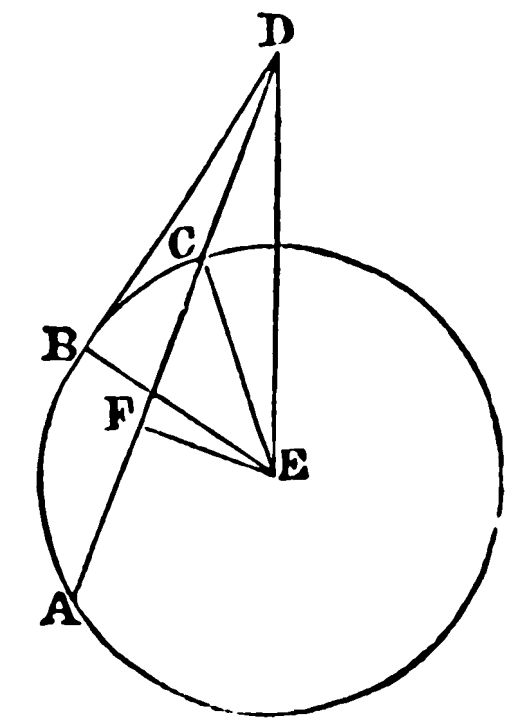
But EC is equal to EB;

Therefore the rectangle AD, DC, together with the square on EB, is equal to the square on ED.

But the square on ED is equal to the squares on EB, BD, because EBD is a right angle (I. 47);

$\therefore AD \cdot DC + EB^2 = BD^2 + EB^2$.

Therefore the rectangle AD, DC, together with the square on EB, is equal to the squares on EB, BD.



$\therefore AD \cdot DC = BD^2$.

Take away the common square on EB;

Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3).

Case II.—Let DCA not pass through the centre of the circle ABC.

Draw EF perpendicular to AC.

Construction.—Take the centre E (III. 1), and draw EF perpendicular to AC (I. 12), and join EB, EC, ED.

Proof.—Because the straight line EF, which passes through the centre, cuts the straight line AC, which does not pass through the centre, at right angles, it also bisects it (III. 3);

Therefore AF is equal to FC. ∴ AF = FC.

And because the straight line AC is bisected in F and produced to D, the rectangle AD, DC, together with the square on FC, is equal to the square on FD (II. 6). ∴ AD·DC + FC² = FD².

To each of these equals add the square on FE;

Therefore the rectangle AD, DC, together with the squares on CF, FE, is equal to the squares on DF, FE (Ax. 2).

But the squares on CF, FE are equal to the square on CE, because CFE is a right angle (I. 47);

And the squares on DF, FE are equal to the square on DE;

Therefore the rectangle AD, DC, together with the square on CE, is equal to the square on DE. ∴ AD·DC + EC² = DE².

But CE is equal to BE;

Therefore the rectangle AD, DC, together with the square on BE, is equal to the square on DE.

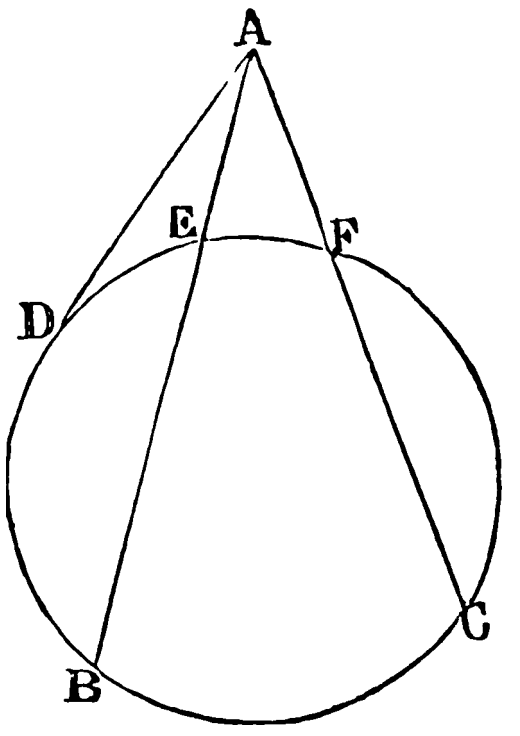
But the square on DE is equal to the squares on DB, BE, because EBD is a right angle (I. 47);

Therefore the rectangle AD, DC, together with the square on BE, is equal to the squares on DB, BE. ∴ AD·DC + BE² = DB²+BE².

Take away the common square on BE;

Then the remaining rectangle AD, DC is equal to the square on DB (Ax. 3). ∴ AD·DC = DB².

Therefore, if from any point, &c. *Q. E. D.*



COROLLARY.—If from any point without a circle there be drawn two straight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another; namely, the rectangle BA, AE is equal to the rectangle CA, AF; for each of them is equal to the square on the straight line AD, which touches the circle.

Proposition 37.—Theorem.

If from a point without a circle there be drawn two straight lines, one of which cuts the circle, and the other meets it, and if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, be equal to the square on

the line which meets the circle, the line which meets the circle shall touch it.

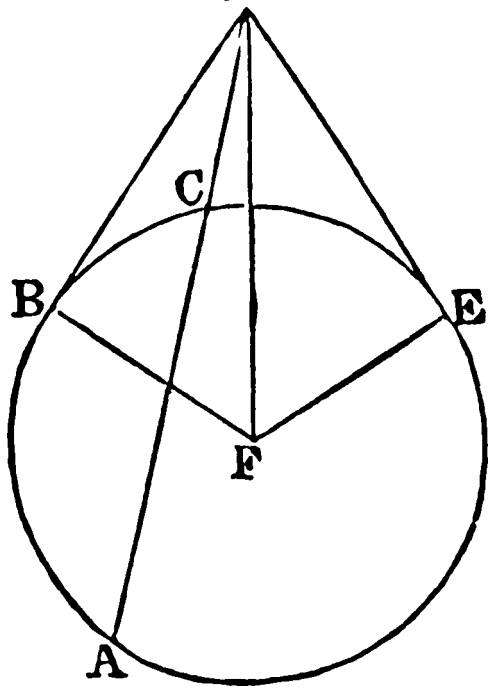
Let any point D be taken without the circle ABC, and from it let two straight lines, DCA, DB, be drawn, of which DCA cuts the circle, and DB meets it; and let the rectangle AD, DC be equal to the square on DB.

Then DB shall touch the circle.

Draw DE touching the circle.

CONSTRUCTION.—Draw the straight line DE, touching the circle ABC (III. 17);

Find F the centre (III. 1) and join FB, FD, FE.



PROOF.—Then FED is a right angle (III. 18).

And because DE touches the circle ABC, and DCA cuts it, the rectangle AD, DC is equal to the square on DE (III. 36).

But the rectangle AD, DC is equal to the square on DB (Hyp.);

Therefore the square on DE is equal to the square on DB (Ax. 1);

Then DE = DB.

Therefore the straight line DE is equal to the straight line DB.

And EF is equal to BF (I. Def. 15);

Therefore the two sides DE, EF are equal to the two sides DB, BF, each to each;

And triangles DBF and DEF are equal in every respect.

And the base DF is common to the two triangles DEF, DBF;

Therefore the angle DEF is equal to the angle DBF (I. 8).

But DEF is a right angle (Const.);

Therefore also DBF is a right angle (Ax. 1).

∴ DBF is a right angle;

And BF, if produced, is a diameter; and the straight line which is drawn at right angles to a diameter, from the extremity of it, touches the circle (III. 16, Cor.);

and therefore DB touches the circle.

Therefore DB touches the circle ABC.

Therefore, if from a point, &c. *Q.E.D.*

EXERCISES ON BOOK III.

Prop. 1—15.

1. Two straight lines intersect. Describe a circle passing through the point of intersection and two other points, one in each straight line.

2. If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circumferences are equal.

3. Show that the centre of a circle may be found by drawing perpendiculars from the middle points of any two chords.

4. Through a given point, which is not the centre, draw the least line to meet the circumference of a given circle, whether the given point be within or without the circle.

5. The sum of the squares of any two chords in a circle, together with four times the sum of the squares of the perpendiculars on them from the centre, is equal to twice the square of the diameter.

6. With a given radius, describe a circle passing through the centre of a given circle and a point in its circumference.

7. If two chords of a circle are given in magnitude and position, describe the circle.

8. Describe a circle which shall touch a given circle in a given point, and shall also touch another given circle.

9. If, from any point in the diameter of a circle, straight lines be drawn to the extremities of a parallel chord, the squares of these lines are together equal to the squares of the segments into which the diameter is divided.

10. If two circles touch each other externally, and parallel diameters be drawn, the straight line joining extremities of these diameters will pass through the point of contact.

11. Draw three circles of given radii touching each other.

12. If a circle of constant radius touch a given circle, it will always touch the same concentric circle.

13. If a chord of constant length be inscribed in a circle, it will always touch the same concentric circle.

14. The locus of the middle points of chords parallel to a given straight line is a line drawn through the centre perpendicular to the parallel chords.

Prop. 16—30.

15. Show that the two tangents from an external point are equal in length.

16. Draw a tangent to a given circle, making a given angle with a given straight line.

17. If a polygon having an even number of sides be inscribed in a circle, the sums of the alternate angles are equal.

18. If such a polygon be described about a circle, the sums of the alternate sides are each equal to half the perimeter of the polygon.

19. If a polygon be inscribed in a circle, the sum of the angles in the segments exterior to the polygon, together with two right angles, is equal to twice as many right angles as the polygon has sides.

20. Draw the common tangents to two given circles.

21. From a given point draw a straight line cutting a given circle, so that the intercepted segment of the line may have a given length.

22. The straight line which joins the extremities of equal arcs towards the same parts are parallel.

23. Any parallelogram described about a circle is equilateral, and any parallelogram inscribed in a circle is rectangular.

24. Two opposite sides of a quadrilateral circumscribing a circle touch the circle at extremities of a diameter. Show that the area of the quadrilateral is equal to one-half the rectangle contained by the diameter, and the sum of the other sides.

Prop. 31—37.

25. A tangent is drawn to a circle of 21 inches diameter from a point 17·5 inches from the centre. Find the length of the tangent.

26. Show that a man 6 feet high, standing at the sea level, has a view of 3 miles (approximately) in every direction, along a horizontal plane passing through his eye.

27. The angle between a tangent to a circle and the chord through the point of contact is equal to half the angle which the chord subtends at the centre.

28. From a given point P, within or without a circle, draw a straight line cutting the circle in A and B such that PA shall be three-fourths of PB.

Ex. Let the circle be of 1·5 inches radius, and point P 3·5 inches from its centre. Prove your construction by scale.

29. The greatest rectangle which can be inscribed in a circle is a square whose area is equal to half that of the square described upon the diameter as side.

30. If the base and vertical angle of a triangle remain constant in magnitude while the sides vary, show that the locus of the middle point of the base is a circle.

31. Given the vertical angle, the difference of the two sides containing it, and the difference of the segments of the base made by a perpendicular from the vertex, to construct the triangle.

32. Show that the locus of the middle point of a straight line, which moves with its extremities upon two straight lines at right angles to each other, is a circle.

33. Show how to produce a straight line, that the rectangle contained by the given line, and the whole line thus produced, may be equal to the square of the part produced.

Ex. If the length of the given line be 2 inches, show geometrically that the length of the part produced is $(\sqrt{5} + 1)$ inches.

34. Given the height and chord of a segment of a circle to find the radius of the circle.

Ex. If the chord be 24 inches, and the height of the segment be 4 inches, show that the radius of the circle is 20 inches.

35. Show that the locus of the middle points of chords which pass through a fixed point is the circle described as diameter upon the line joining the fixed point and the centre of the given circle.

36. Let ACDB be a semicircle whose diameter is AB, and AD, BC any two chords intersecting in P; prove that

$$AB^2 = DA \cdot AP + CB \cdot BP.$$

MATHEMATICS.

SECOND STAGE.

SECTION II.

ALGEBRA.

CHAPTER I.

QUADRATIC EQUATIONS.

1. Equations of this class, when reduced to a rational integral form, contain the *square* of the unknown quantity, but no higher powers.

When the equation contains the *square only* of the unknown quantity, and *not the first power*, it is called a *pure quadratic.*

Thus, $x^2 - 25 = 0$, $4x^2 + 10 = 19$, $5x^2 = 180$, are *pure quadratics.* When the equation contains the *square* of the unknown quantity, *as well as the first power*, it is called an *adfected quadratic.*

Thus, $x^2 - 5x = 6$, $x^2 - x - 30 = 0$, $2x^2 + x + 3 = 6$, are *adfected quadratics.*

Pure Quadratics.

2. To solve these, we treat them exactly as we do simple equations, until we obtain the value of the *square* of the unknown quantity; then, taking the square root of each side, we obtain the *value* of the unknown quantity. It will be

seen (Stage I., Alg.; Art. 35) that the unknown quantity in a pure quadratic has always two values, *equal* in magnitude, but of *opposite* sign.

Ex. 1. Given $3x^2 + 12 = 687$, find x.

We have $3x^2 = 687 - 12 = 675$, or $x^2 = 225$.

$$\therefore x = \pm 15.$$

Ex. 2. Given $\dfrac{2x + 7}{2x^2 - 7x} - \dfrac{2x - 7}{2x^2 + 7x} = \dfrac{56}{6x^2 - 99}$, find x.

Bringing the fractions on the first side to a common denominator, we have—

$$\frac{(2x + 7)^2 - (2x - 7)^2}{x(4x^2 - 49)} = \frac{56}{6x^2 - 99},$$

$$\text{or, } \frac{56x}{x(4x^2 - 49)} = \frac{56}{6x^2 - 99},$$

$$\text{or, } \frac{1}{4x^2 - 49} = \frac{1}{6x^2 - 99}, \text{ clearing of fractions;}$$

$$\text{then, } 6x^2 - 99 = 4x^2 - 49, \text{ from which}$$

$$x^2 = 25$$

$$x = \pm 5.$$

Adfected Quadratics.

3. *Solution by completing the square.*

Suppose we have given the equation $x^2 + 2ax = 3a^2$, to find x.

It will be remembered that $(x^2 + 2ax + a^2)$ is a *perfect square*, viz., the square of $(x + a)$. It is evident, then, that the first side of our equation will become a perfect square by the addition of a^2 as a third term.

Adding, then, a^2 to each side of the given equation, we have—

$$x^2 + 2ax + a^2 = 4a^2, \text{ or}$$

$$(x + a)^2 = 4a^2.$$

Taking now the square root of each side, we have—

$$x + a = \pm 2a.$$

$$\therefore x = \pm 2a - a.$$

$$= a \text{ or } -3a.$$

We may remark that the quantity a^2, added to the expression $x^2 + 2ax$ in order to make it a *perfect square*, is the square of half the coefficient of x. The operation itself is called *completing the square.*

An adfected quadratic may therefore be solved as follows:—

1. *Reduce and arrange it until all the terms involving* x *are on the first side, and the coefficient of* x^2 *is unity.*

2. COMPLETE THE SQUARE *by adding to each side the* SQUARE OF HALF *the coefficient of* x.

3. *Take the square root of each side, put a double sign to the second side, and transpose the term of the first side not involving* x.

Ex. 1. Solve the equation $3x^2 + 18x + 4 = 52$.

We have $3x^2 + 18x = 52 - 4 = 48$; or, dividing each side by 3, $x^2 + 6x = 16$.

Here, the coefficient of x is 6, the half of which is 3. Adding then the square of 3 to each side, to *complete the square*, we have—

$$x^2 + 6x + 3^2 = 16 + 9 = 25.$$

Taking the square root of each side, we have—

$$x + 3 = \pm 5.$$

$$\therefore x = \pm 5 - 3 = 2 \text{ or } -8.$$

Ex. 2. Given $\frac{x+3}{x+4} - \frac{x+1}{x+2} = \frac{4x+9}{2x+7} - \frac{12x+17}{6x+16}$, find x.

We have $\left(1 - \frac{1}{x+4}\right) - \left(1 - \frac{1}{x+2}\right) = \left(2 - \frac{5}{2x+7}\right) - \left(2 - \frac{15}{6x+16}\right)$;

or, simplifying, $\frac{1}{x+2} - \frac{1}{x+4} = \frac{15}{6x+16} - \frac{5}{2x+7}$;

or, $\frac{(x+4) - (x+2)}{(x+2)(x+4)} = \frac{15(2x+7) - 5(6x+16)}{(6x+16)(2x+7)}$;

or, simplifying, $\frac{2}{x^2+6x+8} = \frac{25}{12x^2+74x+112}$;

or, clearing of fractions—

$$24x^2 + 148x + 224 = 25x^2 + 150x + 200;$$

or, transposing and reducing, $-x^2 - 2x = -24$;
or, changing the sign of each side, then—

$$x^2 + 2x = 24;$$

or, completing the square, $x^2 + 2x + 1^2 = 24 + 1 = 25$.

Taking the square root of each side, we have—

$$x + 1 = \pm 5$$
$$x = \pm 5 - 1 = 4 \text{ or } -6.$$

Ex. 3. Solve the equation $x^2 + 6x + 25 = 0$.

We have $x^2 + 6x = -25$;

or, completing the square—

$$x^2 + 6x + 3^2 = -25 + 9 = -16;$$

or, extracting the square root of each side—

$$x + 3 = \pm\sqrt{-16}$$
$$\therefore x = -3 \pm \sqrt{-16}.$$

As the quantity $\sqrt{-16}$ has no exact or approximate value, the given equation has no real roots. The roots are therefore said to be imaginary.

4. *Solution by breaking into factors.*

We have seen (Stage I., Alg., Art. 30) that it is often easy to find by inspection the factors of quadratic expressions. We may make use of this knowledge to solve quadratic equations.

Ex. 1. Solve the equation $x^2 + 5x = 66$.

Transposing all to the first side, we have—

$$x^2 + 5x - 66 = 0.$$

And, resolving the first side into its elementary factors, we get—

$$(x - 6)(x + 11) = 0.$$

Now, if either of these factors is put equal to 0, the equation is satisfied.

Hence, we have, $x - 6 = 0$, and $x + 11 = 0$;
or, $x = 6$, and $x = -11$.

$\therefore$ 6 and -11 are the roots required.

Ex. 2. Given $x^2 - (a + b)x + ab = 0$, find x.

We have $(x - a)(x - b) = 0$.

Now this equation is satisfied by making either of the factors $= 0$.

$$\left.\begin{aligned} \text{Hence, } x - a &= 0, \text{ or } x = a; \\ \text{and, } x - b &= 0, \text{ or } x = b; \end{aligned}\right\}$$

$\therefore a, b$ are the roots required.

Ex. I.

1. $4x^2 - 7 = 29$.

2. $5x^2 + 6 = 86$.

3. $\dfrac{3x^2 - 7}{2} = 34$.

4. $\frac{5}{2}x^2 + \frac{3}{8} = 6$.

5. $3x - \dfrac{12}{x} = 0$.

6. $2(a^2 + b^2) - x^2 = (a - b)^2$.

7. $^*x\sqrt{24 + x^2} = 4 + x^2$.

8. $^*\sqrt{x - a} = \sqrt{x - \sqrt{b^2 + x^2}}$.

9. $\dfrac{ab}{x} + \sqrt{\dfrac{a^2b^2}{x^2} - b^2} = x$.

10. $\dfrac{1}{ax - \sqrt{a^2x^2 - 1}} - \dfrac{1}{ax + \sqrt{a^2x^2 - 1}} = ax$

11. $\dfrac{\sqrt[3]{a^2 + x^2} + \sqrt[3]{a^2 - x^2}}{\sqrt[3]{a^2 + x^2} - \sqrt[3]{a^2 - x^2}} = b$.

12. $\dfrac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = a$.

13. $x^2 - 5x + 6 = 0$.

14. $x^2 - x = 72$.

15. $3x^2 - x = 2$.

16. $4x^2 - 9x = 28$.

17. $6x^2 + x - 35 = 0$.

18. $x^2 + 6\frac{3}{5}x + 8 = 0$.

19. $ax^2 + bx + c = 0$.

20. $3x - \dfrac{14}{2x - 3} = 4x - 7$.

21. $x - \dfrac{42}{x} = 1$.

22. $2x + 1 - \dfrac{26}{2x + 1} = 11$.

* See Remark, page 304.

23. $(a + bx)(cx + d) = (a + b)(c + dx)$.

24. $(a - b)(x^2 - b^2) = (a + b)(x - b)^2$.

25. $(a + b)(x - a)(x - b) = abx$.

26. $\frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}$.

27. $ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}$.

28. $a^3x^2 - (a - b)^2 x + a^3 = b^3x^2 + (a^2 + ab + b^2)^2x + b^3$.

29. $\frac{23}{x + 2} - \frac{2}{x} = 3\frac{1}{3}$.

30. $\frac{x + 1}{2x + 3} + \frac{7}{13} = \frac{2x + 5}{4x - 5}$.

31. $\frac{2x + 9}{x} + \frac{x}{2x + 9} = 5\frac{1}{5}$.

32. $\frac{x + 2}{x + 3} + \frac{x - 10}{x + 8} = 0$.

33. $\frac{3x - 7}{4x - 2} + \frac{5x + 3}{7x + 4} = \frac{5}{9}$.

34. $\frac{6x + 5}{2x - 7} + \frac{4x - 1}{x - 2} = \frac{7x + 1}{x - 3}$.

35. $\frac{2x + 1}{x + 2} - \frac{x + 2}{4x + 4} = \frac{7x + 8}{4x + 13}$.

36. $\frac{3x}{x - 2} - \frac{2x + 9}{x + 2} = \frac{2x + 24}{2x - 1}$.

37. $\frac{3x^2 - 2x + 7}{6x^2 - 4x + 11} = \frac{x^2 + 7x - 3}{2x^2 + 14x - 9}$.

38. $\frac{3x^2 + 10}{x} - \frac{x^2 + 2x + 3}{x + 2} = \frac{2x^2 + 2x + 10}{x + 1}$.

39. * $\sqrt{5x + 6} + \sqrt{x + 10} = 4$.

40. $\sqrt{3x - 4} - \sqrt{2x - 4} = \sqrt{x}$.

41. * $\sqrt{a^2 + b^2 + x} + \sqrt{a^2 + b^2 - x} = \sqrt{\frac{2ax}{b}}$.

* See Remark, page 304.

42. $* \sqrt{2\,a + x} + \sqrt{4\,a - x} = 2\sqrt{x - a}.$

43. $* \sqrt{ax + b} + \sqrt{bx + a} = \sqrt{(a - b)x + a + b + 2\sqrt{ab}}.$

44. $* \sqrt{x + \frac{b}{a}} + \sqrt{x + \frac{a}{b}} = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{x}{b}}\right)\sqrt{a + b}.$

45. $\left(\frac{a}{b} - 1\right).\frac{x - c}{b} = \left(\frac{a^2}{b^2} + 1\right).\frac{a + b + c}{x} - 2.$

46. $(a - b)(x + b - c)(x + c - a) + a^2(x + b - c) + b^2(x + c - a) + c^2(x + a - b = 0).$

47. Show that, if x be real, the value of $x + \frac{1}{x}$ cannot lie between 2 and $-$ 2.

48. If, in the equation $x - \frac{x^2}{20} = a$, the quantity x be real, show that a cannot be greater than 5.

Equations which may be Solved like Quadratics.

5. Certain equations of a degree higher than the second may be solved like quadratics. It will be seen that, although it is impossible to lay down definite rules for the treatment of every such equation, the object to be attained is either (1.) To throw the equation into the form—

$$X^2 + pX = q,$$

when X is an expression containing the unknown quantity, and solving when possible this equation for X; or (2.), To strike out from each side a factor containing the unknown quantity, thus reducing the dimensions of the equation, and obtaining a value or values of the unknown quantity by equating this factor to zero.

Ex. 1. Solve the equation $x^4 + 144 = 25\,x^2$.

We have, transposing, $x^4 - 25\,x^2 = -144$;

or, $(x^2)^2 - 25\,(x^2) = -144$;

* See Remark, page 304.

or, completing the square—

$$(x^2)^2 - 25\,(x^2) + (\tfrac{25}{2})^2 = \tfrac{625}{4} - 144 = \tfrac{49}{4}.$$

$$\therefore x^2 - \tfrac{25}{2} = \pm \tfrac{7}{2}.$$

$$\therefore x^2 = \tfrac{25}{2} \pm \tfrac{7}{2} = 16 \text{ or } 9.$$

$$\text{Hence, } x = \pm 4 \text{ or } \pm 3.$$

The given equation has therefore *four* roots, being as many roots as the *degreé* of the equation.

Ex. 2. Solve $x^6 + 3\,x^3 = 88$.

We have, $(x^3)^2 + 3\,(x^3) = 88$; or, transposing,

$$(x^3)^2 + 3\,(x^3) - 88 = 0;$$

$$\text{or, } (x^3 - 8)\,(x^3 + 11) = 0.$$

Hence, the given equation is satisfied by—

$$x^3 - 8 = 0, \text{ and also by } x^3 + 11 = 0.$$

We have then $x^3 - 2^3 = 0$, and $x^3 + (\sqrt{11})^3 = 0$;

or, breaking into factors, we have—

$$(x - 2)\,(x^2 + x \cdot 2 + 2^2) = 0, \text{ and}$$

$$(x + \sqrt[3]{11})\left\{x^2 - x \cdot \sqrt[3]{11} + (\sqrt[3]{11})^2\right\} = 0.$$

From the *first* of these equations, we get—

$$x - 2 = 0 \text{ or } x = 2,$$

and $x^2 + 2\,x + 4 = 0$, from which two other roots may be found.

And from the *second* equation, we get—

$$x + \sqrt[3]{11} = 0, \text{ or } x = -\sqrt[3]{11},$$

and $x^2 - x\sqrt[3]{11} + \sqrt[3]{121} = 0$, which gives two other roots.

We have therefore shown how to obtain the six roots of the given equation.

Ex. 3. Solve—

$$23\,x^2 - 75\,x - 6\,x\sqrt{4\,x^2 - 9\,x + 9} + 40 = 0.$$

Changing the sign of each side, and transposing, we have—

$$6\,x\sqrt{4\,x^2 - 9\,x + 9} = 23\,x^2 - 75\,x + 40;$$

adding to each side $13\,x^2 - 9\,x + 9$, then—

$$13\,x^2 - 9\,x + 9 + 6\,x\sqrt{4\,x^2 - 9\,x + 9} = 36\,x^2 - 84\,x + 49;$$

$$\text{or, } (4\,x^2 - 9\,x + 9) + 2\,(3\,x)\sqrt{4\,x^2 - 9\,x + 9} + (3\,x)^2 = (6\,x)^2 - 2\,(6\,x) \cdot 7 + 7^2;$$

or, $(\sqrt{4x^2 - 9x + 9} + 3x)^2 = (6x - 7)^2$.

$\therefore \sqrt{4x^2 - 9x + 9} + 3x = \pm (6x - 7)$,.....(1.);

or, taking the upper sign—

$$\sqrt{4x^2 - 9x + 9} = (6x - 7) - 3x = 3 - 7.$$

Hence, squaring each side—

$$4x^2 - 9x + 9 = 9x^2 - 42x + 49;$$

or, $5x^2 - 33x + 40 = 0$;

or, $(x - 5)(5x - 8) = 0$.

$$\therefore x = 5 \text{ or } \tfrac{8}{5}.$$

Again, taking the lower sign of (1), we have—

$\sqrt{4x^2 - 9x + 9} = -(6x - 7) - 3x = -9x + 7$.....(2.);

or, squaring—

$$4x^2 - 9x + 9 = 81x^2 - 126x + 49;$$

from which $77x^2 - 117x + 40 = 0$;

or, $(x - 1)(77x - 40) = 0$.

$$\therefore x = 1 \text{ or } \tfrac{40}{77}.$$

Hence, the roots of the given equation are $5, \tfrac{8}{5}, 1, \tfrac{40}{77}$.

Remark.—If we proceed to verify these values of x, we shall find that the last two values—viz., 1 and $\tfrac{40}{77}$—will not satisfy the given equation unless we obtain the value of $\sqrt{4x^2 - 9x + 9}$ by means of the equation from which these last roots were found.

Thus from (2) we find, on putting 1 and $\tfrac{40}{77}$ successively for x—

$$\sqrt{4x^2 - 9x + 9} = -9(1) + 7 = -2,$$

$$\text{and } \sqrt{4x^2 - 9x + 9} = -9\left(\tfrac{40}{77}\right) + 7 = 2\tfrac{25}{77};$$

and on substituting either of these values of $\sqrt{4x^2 - 9x + 9}$ along with the corresponding value of x, the given equation is satisfied.

Ex. 4. Solve—

$$(x + b + c)(x + b - c)(b + c - x) = (a + b + c)(a + b - c)(b + c - a).$$

By inspection we see that a is one of the roots. We shall therefore so arrange the equation as to be able to strike out $x - a$ as a factor of each side (Art. 5).

We have—

$$\frac{(x + b + c)(x + b - c)}{(a + b + c)(a + b - c)} = \frac{b + c - a}{b + c - x};$$

$$\text{or, } \frac{(x + b)^2 - c^2}{(a + b)^2 - c^2} = \frac{b + c - a}{b + c - x};$$

or, taking the *difference* of numerator and denominator—

$$\frac{(x + b)^2 - (a + b)^2}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x};$$

$$\text{or, } \frac{(x - a)(x + 2b - a)}{(a + b)^2 - c^2} = \frac{x - a}{b + c - x}.$$

Dividing each side by $x - a$, we have—

$$\frac{x + 2b - a}{(a + b)^2 - c^2} = \frac{1}{b + c - x}, \text{ and } x - a = 0, \text{ or } x = a;$$

Hence also $(x + 2b - a)(b + c - x) = (a + b)^2 - c^2$; an ordinary quadratic from which two other roots may be determined.

Ex. 5. * Solve the equation—

$$a^2b^2x^{\frac{1}{q}} - 4\,a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p+q}{2pq}} = (a - b)^2x^{\frac{1}{p}}.$$

Dividing each side by $x^{\frac{1}{p}}$, we have—

$$a^2b^2x^{\frac{p-q}{pq}} - 4\,a^{\frac{3}{2}}b^{\frac{3}{2}}x^{\frac{p-q}{2pq}} = (a - b)^2;$$

$$\text{or, } \left(abx^{\frac{p-q}{2pq}}\right)^2 - 4\,a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) = (a - b)^2;$$

or, completing the square—

$$^2\left(abx^{\frac{p-q}{2pq}}\right) - 4\,a^{\frac{1}{2}}b^{\frac{1}{2}}\left(abx^{\frac{p-q}{2pq}}\right) + \left(2\,a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^2$$

$$= (a - b)^2 + 4ab = (a + b)^2.$$

* We shall *assume* in this example that the laws of multiplication and division proved in Algebra, Stage I., Arts. 25, 27, hold for fractional indices.

Hence, extracting the square root—

$$abx^{\frac{p-q}{2pq}} - 2\,a^{\frac{1}{2}}b^{\frac{1}{2}} = \pm\,(a + b).$$

$$\therefore\ abx^{\frac{p-q}{2pq}} = \pm\,(a + b) + 2\,a^{\frac{1}{2}}b^{\frac{1}{2}} = (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 \text{ or } -(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2;$$

$$\text{or, } x^{\frac{p-q}{2pq}} = \left(\frac{a^{\frac{1}{2}} + b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\right)^2;$$

$$= \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^2 \text{ or } -\left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^2.$$

Then, raising each side to the $(2\,pq)$th power, we have—

$$x^{p-q} = \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^{4pq} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^{4pq}.$$

Hence, taking the $(p - q)$th root, we get—

$$x = \left(\frac{1}{b^{\frac{1}{2}}} + \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}} \text{ or } \left(\frac{1}{b^{\frac{1}{2}}} - \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}$$

$$= \left(\frac{1}{b^{\frac{1}{2}}} \pm \frac{1}{a^{\frac{1}{2}}}\right)^{\frac{4pq}{p-q}}$$

We have also, since the factor $x^{\frac{1}{p}}$ has been struck out—

$$x^{\frac{1}{p}} = 0, \text{ and } \therefore\ x = 0 \text{ as another solution.}$$

Ex. II.

1. $4\,x^4 - 11\,x^2 = 225.$
2. $5\,x^6 - 17\,x^3 = 184.$
3. $x^2 + 5\,x^{-2} = 25\frac{1}{5}.$
4. $(x^2 + 3\,x + 3)^2 + 2\,x^2 = 189 - 6\,x.$
5. $3\,x^{-4} - 17\,x^{-2} = 1450.$
6. $x^n + \frac{6}{x^n} = 5.$
7. $\sqrt{x + 12} + \sqrt{x + 12} = 6.$
8. $\sqrt{x^2 + 9} + 20 = x^2 + 9.$
9. $\left(\frac{12}{x} - x\right)^2 + \frac{60}{x} = 6 + 5\,x.$

10. $(3x + 4) + \sqrt{3x + 4} - 12 = 0.$

11. $x^2 + 2x = \dfrac{2}{x + 4} - 1.$

12. $\dfrac{\sqrt[n]{a - x}}{x} - \dfrac{\sqrt[n]{a - x}}{a} = \dfrac{\sqrt[n]{x}}{a}.$

13. $9x + 24\sqrt{x} - \dfrac{4}{\sqrt{x}}\left(\dfrac{1}{\sqrt{x}} + 9\right) = 65.$

14. $x = \sqrt{a} \cdot \sqrt{b - x} - \sqrt{b} \cdot \sqrt{x + a}.$

15. $\dfrac{1}{(a - \sqrt{a^2 - x^2})} + \dfrac{1}{(a + \sqrt{a^2 - x^2})^2} = \dfrac{a^2 + 2x^2}{x^4}.$

16. $\dfrac{x^2}{b} - 2\sqrt{\dfrac{a^2 - ax + x^2}{b}} = \dfrac{ax}{b} - 1.$

17. $x^{\frac{p+q}{2pq}} - \dfrac{1}{2} \cdot \dfrac{a^2 - b^2}{a^2 + b^2}\left(x^{\frac{1}{p}} + x^{\frac{1}{q}}\right) = 0.$

18. $\sqrt{\dfrac{a^2 + x^2}{x^2}} - \sqrt{\dfrac{x^2}{a^2 + x^2}} = 1.$

19. $(a^{4m} + 1)\,(x^{\frac{1}{2}} - 1)^2 = 2\,(x + 1).$

20. $\dfrac{x^2}{x^2 - 9} - \dfrac{975}{256} \cdot \dfrac{x^2 - 9}{x^2} + \dfrac{7}{8} = 0.$

21. $\dfrac{12}{x + 1}\left\{\dfrac{3}{x + 1} + \dfrac{2}{x}\right\} + \dfrac{4}{x^2} = 9.$

22. $6x^2 - 5x - 8\sqrt{3x^2 + 5x - 4} = 12.$

23. $2x^2 + 5x - 2x\sqrt{2x^2 - 7x + 1} = 35.$

24. $20x^2 - 9ax - 8x\sqrt{5x^2 - 3ax + 2a^2} = 7a^2.$

25. $x^2(x - 2)^2 + 6x^2(x - 2) = 24x + 36 - 5x^2.$

26. $2x^2 + 20x - \sqrt{3x^2 - x + 5} = 105.$

27. $4x^2 + 12x - 2x\sqrt{4x^2 - 2x + 19} = 30.$

28. $x^4 + x^2 + 1 = a(x^2 + x + 1).$

29. $x^6 - 1 = 0.$

30. $(x^2 - a)^3 + (x^2 - b)^3 + (x^2 - c)^3$
$= 3(x^2 - a)(x^2 - b)(x^2 - c) + 9ax^4 - a(a + b + c)^2$.

31. $(x - b)(x^2 - c^2) = (a - b)(a^2 - c^2)$.

32. $(x - a)(x - b)(x - c) = (m - a)(m - b)(m - c)$.

33. $(x - a)(x - b)(x - c) = (a + b)(a + c)(b + c)$.

34. $(n - 1)x(x^2 + ax + a^2) = a^3 - x^3$.

35. $(ax - b)^3 + (cx - d)^3 = (a + c)^3x^3 - (b + d)^3$.

36. $x^{2^n} + x^{2^{n-1}} + 1 = a(x^2 + x + 1)$.

Simultaneous Quadratic Equations.

6. The following worked examples are given as specimens of the methods to be employed, but it must be understood that practice alone will give the student complete mastery over equations of this class.

Ex. 1. Solve $x^2 + y^2 = 20$.....(1.)
$x + y = 6$.....(2.)

As we have given the *sum* of the unknown quantities, we shall work for the *difference*.

From (2), multiplying each side by 2, we have—

$$2x^2 + 2y^2 = 40$$

and from (1), squaring, $x^2 + 2xy + y^2 = 36$

Then, subtracting, $x^2 - 2xy + y^2 = 4$;

and, taking the square root, we have $x - y = \pm 2$.....(3).

(2) + (3), then $2x = 8$ or 4, and $\therefore x = 4$ or 2.
(2) − (3), then $2y = 4$ or 8, and $\therefore y = 2$ or 4.

NOTE.—Having found that $x = 4$, $y = 2$, we might have told by inspection that the values $x = 2$, $y = 4$, would also satisfy the given equations, for x and y are similarly involved in both equations.

Ex. 2. Solve $x^2 - y^2 = 5$.....(1.)
$xy = 6$.....(2.)

As we have given here the *difference of the squares* of the unknown quantities, it will be convenient to work for the *sum of the squares*.

From (1.), squaring, $x^4 - 2x^2y^2 + y^4 = 25$,
and from (2.), squaring, &c., $4x^2y^2 = 144$
Then, adding, $x^4 + 2x^2y^2 + y^4 = 169$
and taking the square root, $x^2 + y^2 = \pm 13$.....(3).

(3.) + (1.), then, $2x^2 = 18$ or -8, or $x^2 = 9$ or -4,
and $\therefore x = \pm 3$ or $\pm 2\sqrt{-1}$.

(3.) − (1.), then, $2y^2 = 8$ or -18, or $y^2 = 4$ or -9,
and $\therefore y = \pm 2$ or $\pm 3\sqrt{-1}$.

NOTE.—The student will see that the pairs of values which satisfy the given equations are, $x = 3, y = 2$; $x = -3, y = -2$; $x = 2\sqrt{-1}$, $y = 3\sqrt{-1}$; $x = -2\sqrt{-1}$, $y = -3\sqrt{-1}$.

Ex. 3. Solve
$$\left.\begin{aligned} x^2 + y &= 11x \text{.....(1.)} \\ y^2 + x &= 11y \text{.....(2.)} \end{aligned}\right\}$$

Subtracting, then, $x^2 - y^2 - x + y = 11x - 11y$;
or, $x^2 - y^2 = 12(x - y)$.

Now $(x - y)$ is a factor of each side, and hence, striking it out, we have—

$$x + y = 12 \text{.....(3.)},$$
and also $x - y = 0$.....(4.).

Equations (3.) and (4.) may not be used as simultaneous equations, but *each of them* may be used in turn with either of the given equations.

Thus, taking equations (3.) and (1.), we have—

(1.) − (3.), $x^2 - x = 11x - 12$,
from which $x = 6 \pm 2\sqrt{6}$;
and hence from (3.), by substitution, we easily get-

$$y = 6 \mp 2\sqrt{6}.$$

Again, taking equations (4.) and (1.)—

we have, from (4.) $x = y$,
and $\therefore$ from (1.), $x^2 + x = 11x$ or $x^2 = 10x$,
from which $x = 10$ or 0;
and so, from (4.), $y = 10$ or 0.

Hence, the pairs of values satisfying the given equations are—

$$x = 10, y = 10\,; x = 0, y = 0\,;$$
$$x = 6 + 2\sqrt{6}, y = 6 - 2\sqrt{6}\,;$$
$$x = 6 - 2\sqrt{6}, y = 6 + 2\sqrt{6}.$$

NOTE.—It is worth while remarking that when *each of the terms* of the given equations contain at least one of the unknown quantities, the values $x = 0$, $y = 0$ will always satisfy.

Ex. 4. Solve $3x^2 - 2xy = 55$ ..... (1.)
$x^2 - 5xy + 8y^2 = 7$ ..... (2.)

Multiplying the equations together *crosswise*, we get—

$$55x^2 - 275xy + 440y^2 = 21x^2 - 14xy\,;$$
or, transposing, $34x^2 - 261xy + 440y^2 = 0\,;$
or, $(2x - 5y)(17x - 88y) = 0,$
from which $2x = 5y$, and $17x = 88y$.

Each of these equations taken *in turn* with either (1.) or (2) will easily give the required values of x and y.

Ex. 5. $x^4 + y^4 = 337$ .....(1.),
$x + y = 7$ .....(2.)

From (2), raising each side to the *fourth power*, we have—

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 2401\,;\text{.....(3.)}$$
(3) − (1), then $4x^3y + 6x^2y^2 + 4xy^3 = 2064\,;$
or, $2x^3y + 3x^2y^2 + 2xy^3 = 1032\,;$
or, arranging, $2xy(x + y)^2 - x^2y^2 = 1032\,;$
but from (2), $(x + y)^2 = 49,$
and hence, $2xy(49) - x^2y^2 = 1032\,;$
or, $x^2y^2 - 98xy + 1032 = 0,$
from which $xy = 12$ or 86,.....(4.)

From (2) and (4), $x - y$ may now be easily obtained, and hence also the required values of x and y.

Ex. III.

1. $x + y = 5$, $xy = 6$.
2. $x - y = 2$, $xy = 15$.
3. $x^2 + y^2 = 25$, $xy = 12$.
4. $x^2 + y^2 = 20$, $x + y = 6$.

5. $x^2 + y^2 = 29, x - y = 3.$

6. $x^2 - y^2 = 13, (x - y)^2 = 1.$

7. $x^2 - y^2 = 27, xy = 18.$

8. $x^2 - y^2 = 12, x + y = 6.$

9. $x^2 + y^2 = 53, x^2 - y^2 = -45.$

10. $x^2 + xy = 28, y^2 + xy = 21.$

11. $x^2 + xy + y^2 = 19, xy - x^2 = -3.$

12. $x + y = 13, \sqrt{x} + \sqrt{y} = 5.$

13. $x^2 + xy + y^2 = 84, x + \sqrt{xy} + y = 14.$

14. $x^3 + y^3 = 35, x^2y + xy^2 = 30.$

15. $\frac{1}{x} + \frac{1}{y} = a, \frac{1}{x^2} + \frac{1}{y^2} = b.$

16. $x + y = a(x - y), x^2 + y^2 = b^2.$

17. $x^4 - y^4 = a, x^2 - y^2 = b.$

18. $x + y = 5, x^3 + y^3 = 35.$

19. $x + y = 5, x^5 + y^5 = 275.$

20. $x^2 + y^2 = \frac{3}{5}(x + y), xy = 6.$

21. $x - y = 2, x^3 - y^3 = 98.$

22. $xy(x + y) = a, x^2y^2(x^2 + y^2) = b.$

23. $xy(x + y) = 30, x^2y^2(x^5 + y^5) = 9900.$

24. $4x^2 - 3xy = 18, 5y^2 - 2xy = 8.$

25. $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 35, x^{\frac{1}{2}} + y^{\frac{1}{2}} = \frac{30}{x^{\frac{1}{2}}y^{\frac{1}{2}}}.$

26. $x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x, x^{\frac{1}{2}} + y^{\frac{1}{3}} = x.$

27. $xy + 6 = \frac{216}{xy}, x + y + 4 = \frac{77}{x + y}.$

28. $(x + y)^2 + 2(x - y)^2 = 3(x + y)(x - y), x^2 + y^2 = 10.$

29. $x^2 + 10xy + y^2 = \frac{7}{5}(x^2 - y^2), x^2 + 5y^2 = x + 13y.$

30. $x^4 - 2x^2y^2 + y^4 = 1 + 4xy, x^2(x + 1) + y^2(y + 1) = xy.$

31. $3x + y - 9 = \frac{9x^2 - y^2 - 117}{3x - y - 6} = \frac{8x^2 - y^2 + 1}{4x + y + 1}.$

32. $\dfrac{x+2}{y-2}+\dfrac{y+2}{x-2}=5\frac{1}{2}, \dfrac{x+2}{y+2}+\dfrac{y-2}{x-2}=1\frac{5}{6}.$

33. $\dfrac{\sqrt{x^2-y^2}+(x-y)}{\sqrt{x^2-y^2}-(x-y)}=3+2\sqrt{2}, \dfrac{1}{y}-\dfrac{1}{x}=\dfrac{2}{xy}.$

34. $y-2\sqrt{5x^2+y+3}=32-5x^2,$

$$\left(\frac{x}{y}+\frac{y}{x}\right)^2+\left(\frac{x}{y}+\frac{y}{x}\right)=14\frac{4}{9}.$$

35. $\dfrac{x}{y}-\dfrac{y}{x}-3=-\frac{9}{4}\cdot\dfrac{xy}{x^2-y^2}, 3x+y=7.$

36. $(x+y)xy=c(bx+ay),$
$xy(bx+ay)=x^2y^2+abc(x+y-c).$

37. $y^4=x^2(ay-bx), x^2=ax-by.$

38. $\dfrac{\sqrt{y^2+1}+1}{y}=\dfrac{\sqrt{x+9}+3}{\sqrt{x}}, x(y+1)^2=36(y^3+\frac{6}{9}).$

39. $x=\dfrac{a}{y+z}, y=\dfrac{b}{x+z}, z=\dfrac{c}{x+y}.$

40. $x+y+z=6, xy+xz+yz=11, xyz=6.$

41. $x^2-yz=1, y^2-xz=0, z^2-xy=0.$

42. $xyz=a^2(x+y)=b^2(y+z)=c^2(x+z).$

43. $x^2+y^2+z^2=\dfrac{a^2}{x^2}=\dfrac{b^2}{y^2}=\dfrac{c^2}{z^2}.$

44. $x+y+z+u=4a+4b,$
$xy+xz+xu+yz+yu+zu=6a^2+12ab+6b^2,$
$xyz+xyu+xzu+yzu=4a^3+12a^2b+12ab^2+4b^3,$
$xyzu=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$

45. $x^2y^2+xy^2z+x^2yz=a,$
$y^2z^2+xy^2z+xyz^2=b,$
$x^2z^2+x^2yz+xyz^2=c.$

46. $(x+y)^3+z^3=1125,$
$x+y+z=15,$
$xy=24.$

47. If $ax^3=by^3=cz^3$, and $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=a$, show that $ax^2+by^2+cz^2=(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3a^2.$

48. Given $R = 1 + r$, $P = \frac{A}{r}(1 - R^{-n})$, $M = PR^n$, show that $R = A\left(\frac{1}{P} - \frac{1}{M}\right) + 1$.

CHAPTER II.

Problems Producing Quadratic Equations.

7. We shall now discuss one or two problems whose solutions depend upon quadratic equations.

Ex. 1. A person raised his goods a certain rate per cent., and found that to bring them back to the original price he must lower them $3\frac{1}{3}$ less per cent. than he had raised them. Find the original rise per cent.

Let x = the original rise per cent.,

then $\frac{x}{100 + x}.100$ = the *fall per cent.* to bring them to the original price.

Hence, by problem—

$$x - \frac{100\,x}{100 + x} = 3\tfrac{1}{3}, \text{ which solved, gives}$$

$$x = 20 \text{ or } -16\tfrac{2}{3}.$$

The value $x = 20$ is alone applicable to the problem. Remembering, however, the algebraical meaning of the *negative* sign, it is easy to see that the second value, $x = -16\frac{2}{3}$, gives us the solution of the following problem:—

A person *lowered* his goods a certain rate per cent., and found that to bring them back to the original price he must *raise* them $3\frac{1}{3}$ more per cent. than he had *lowered* them. Find the original fall per cent.

The above solution tells us that the fall required is $16\frac{2}{3}$ per cent.

Had we worked the latter problem first, we should have obtained $x = 16\frac{2}{3}$ or -20, the value $x = -20$ indicating the solution of the former problem.

Ex. 2. Find a number such that when multiplied by its deficiency from 100 the product is 196.

Let x = the number,
then $100 - x$ = its deficiency from 100.
Hence, by problem—

$$x\,(100 - x) = 196, \text{ or } x^2 - 100\,x + 196 = 0;$$

from which, $x = 2$ or 98.

Both these values will be found to be consistent with the conditions of the problem.

Ex. 3. The number of men required to build a house is such that, when four times the number is subtracted from three times the square of the number, the result is 160. Find the number of men.

Let x = the number of men,
then, by problem—

$$3\,x^2 - 4\,x = 160, \text{ from which}$$
$$x = 8 \text{ or } -6\tfrac{2}{3}.$$

The value $x = 8$ is alone applicable to the problem as it stands. If, however, we may conceive of a *fractional* number of men—and this we may easily do here by supposing a *boy's* work to be equal to $\frac{2}{3}$ of a man's—we find that the second result gives us the solution of the following problem:—

The number of men required to build a house is such that when four times the number is *added to* three times the square of the number, the result is 160. Find the number.

The answer, as above indicated, is 6 men and 1 boy, where a boy is worth $\frac{2}{3}$ of a man.

The student will find, however, that in some cases there is no *obvious* interpretation to the second result, owing occasionally to the fact that certain *terms* are used in the problem to which the results will not apply, and indeed that the algebraical expression of the conditions of the problem is more general than the language of the problem itself.

Ex. IV.

1. Find a number whose square is equal to the product of two other numbers, one of which is less by 6 than the required number, and the other greater by 9 than twice that number.

2. When the numerator and denominator of a certain fraction are each increased by unity the fraction is increased by $\frac{1}{1\,9\,0}$, and when they are each diminished by unity the fraction is diminished by $\frac{1}{9\,5}$. Find the fraction.

3. The mean proportional between the excess of a certain number above 21, and its defect from 37, is 28. Find the number.

4. A number of articles, which were bought for £4, cost each 3 shillings more than half as many shillings as there were articles. Find the number of articles.

5. There is a square court-yard, such that if its length be increased by 10 feet, and its breadth diminished by 20 feet, its area would be 5,104 feet. Required the side of the square.

6. If the number of shillings given for an article be added to the number of articles which can be bought at the same price for 18 shillings, the result is 11. Find the price..

7. Two travellers set out to meet each other from two places 180 miles distant; the first goes 3 miles an hour, and the second goes 1 mile more per hour than one-fourth of the number of hours before they meet. When will they meet?

8. A farmer bought a number of calves, sheep, and pigs, the number of calves being equal to that of the sheep and pigs together. For the calves he gave 64s. a head, and for the sheep twice as many shillings as there were sheep. He paid £153. 12s. for the calves and sheep together, and £36. 12s. for the pigs—a pig costing as much as a sheep and calf together. Find the cost of the sheep per head.

9. There are two squares, and an oblong whose sides are equal to those of the squares, and it is noted that three times the area of the first square exceeds four times the area of the oblong by 3 square feet, while twice the area of the square, together with three times the area of the rectangle, is 36 square feet. Required the sides of the squares.

10. The sum of two quantities is equal to 6 times the square of their product, and the sum of their cubes is equal to 36 times the product of their fifth powers. Find the quantities.

11. The solid content of a rectangular parallelopiped is 60 cubic feet, and the total area of the side is 98 square feet, while the sum of the edges is 48 feet. Required the dimensions.

12. The products of the number of units of length in the sides of a polygon of n sides, when taken $n - 1$, together are respectively a_1, a_2, a_3, &c., a_n. Required the lengths of the sides.

13. A, B, and C can together do a piece of work in a day, and C's rates of work is the product of the rates of A and B. Moreover, C is one-fifth as good a workman as A and B together. Find the respective times required for A, B, C to do a piece of work.

14. The compound interest of a certain sum of money for 3 years is a, and the third year's interest is b. Find the principal and the rate per cent.

15. A owes B £a due m months hence, and also £b due n months hence. Find the equated time, reckoning interest at 5 per cent. per annum.

16. Find three quantities such that the sum of any two is equal to the reciprocal of the third.

17. Find three magnitudes, when the quotients arising from dividing the products of every two by the other are respectively a, b, c.

18. The sum of three quantities is 9, the sum of their products, taken two and two together, is 23, and their continued product is 15. Show that the three quantities are the roots of the equation $a^3 - 9\,a^2 + 23\,a - 15 = 0$.

CHAPTER III.

INDICES.

8. We shall reserve the discussion of the complete theory of Indices for Vol. II., confining ourselves here to a few simple cases, and giving a few examples involving fractional and negative exponents.

9. *Fractional exponents.*

DEF.—The numerator of a fractional exponent indicates the power to which the quantity must be raised, and the

denominator the root which must be taken of the power so obtained.

Thus, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$, $a^{\frac{5}{7}} = \sqrt[7]{a^5}$, $a^{\frac{6}{3}} = \sqrt[3]{a^6} = a^2$;

and generally $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

The above definition is that which follows at once if we *assume* the law proved in Art. 24, page 159, viz., $a^m \times a^n = a^{m+n}$ to be true, whatever be the value of m and n.

Thus we have—

$$\left(a^{\frac{m}{n}}\right)^n = a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots\ldots\ldots\ldots\text{to } n \text{ factors}$$

$$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}} \text{ to } n \text{ terms} = a^{\frac{m}{n}\cdot n} = a^m.$$

Hence, taking the nth root,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

10. *To show that* $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let $x = \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \sqrt[s]{\left(a^{\frac{p}{q}}\right)^r}$, by Def., Art. 9.

$$\therefore x^s = \left(a^{\frac{p}{q}}\right)^r = \left(\sqrt[q]{a^p}\right)^r,$$

or, raising each side to the qth power, we have—

$$x^{qs} = (a^p)^r = a^{pr}.$$

Hence, taking the (qs)th root, we have $x = a^{\frac{pr}{qs}}$.

11. *To show that* $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$.

Now, $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = \sqrt[n]{a^m} \times \sqrt[n]{b^m} = \sqrt[n]{a^m b^m} = \sqrt[n]{(ab)^m}$

$$= (ab)^{\frac{m}{n}}, \text{ by Def., Art. 9.}$$

And so, $a^{\frac{m}{n}} \div b^{\frac{m}{n}} = \left(\frac{a}{b}\right)^{\frac{m}{n}}$.

Ex. 1. Multiply together $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$ by $a^{\frac{1}{3}}b^{\frac{1}{5}}c^{\frac{1}{7}}$.

Adding together the indices of like letters, we have—

$$\tfrac{1}{2}+\tfrac{1}{3} = \tfrac{5}{6},\ \tfrac{1}{3}+\tfrac{1}{5} = \tfrac{8}{15},\ \tfrac{1}{4}+\tfrac{1}{7} = \tfrac{11}{28}.$$

Hence, the required product is $a^{\frac{5}{6}}b^{\frac{8}{15}}c^{\frac{11}{28}}$.

Ex. 2. Multiply $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

$$\begin{array}{r}
x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y \\
x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y \\
\hline
x^2 + x^{\frac{3}{2}}y^{\frac{1}{2}} + xy \\
- x^{\frac{3}{2}}y^{\frac{1}{2}} - xy - x^{\frac{1}{2}}y^{\frac{3}{2}} \\
xy + x^{\frac{1}{2}}y^{\frac{3}{2}} + y^2 \\
\hline
x^2 + xy + y^2
\end{array}$$

Ex. 3. Divide $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

$$\begin{array}{l}
x^{\frac{1}{3}} - y^{\frac{1}{3}})x - y(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}} \\
\underline{x - x^{\frac{2}{3}}y^{\frac{1}{3}}} \\
x^{\frac{2}{3}}y^{\frac{1}{3}} \\
\underline{x^{\frac{2}{3}}y^{\frac{1}{3}} - x^{\frac{1}{3}}y^2} \\
x^{\frac{1}{3}}y^{\frac{2}{3}} - y \\
\underline{x^{\frac{1}{3}}y^{\frac{2}{3}} - y}
\end{array}$$

Ex. V.

Find the value of—

1. $(a^6)^{\frac{1}{3}}$, $a^{\frac{8}{4}}$, $(a^{-2})^{-\frac{1}{4}}$, $(a^{\frac{3}{5}})^{\frac{5}{3}}$.

2. $(a + x)^{\frac{1}{3}}(a^2 + 2ax + x^2)^{\frac{1}{3}}$, $(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{1}{2}}$.

Multiply together—

3. $a^{\frac{1}{2}} - a^{\frac{1}{4}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$ and $a^{\frac{1}{2}} + a^{\frac{1}{4}}x^{\frac{1}{3}} + x^{\frac{2}{3}}$.

4. $x^{\frac{1}{2}} - y^{\frac{1}{2}}$ and $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$.

5. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ and $x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1}$.

Divide—

6. $a^3 - b$ by $a - b^{\frac{1}{3}}$, $a^{\frac{3}{4}} - b^{\frac{3}{4}}$ by $a^{\frac{3}{8}} - b^{\frac{3}{8}}$.

7. $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} + x^{-\frac{1}{2}}y$.

8. $a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ by $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}}$.

9. $a^{\frac{7}{3}} - a^2b^{-\frac{2}{3}} - a^{\frac{1}{3}}b + b^{\frac{1}{3}}$ by $a^{\frac{1}{3}}b^{\frac{1}{3}} - b^{-\frac{1}{3}}$.

Show that—

10. $(x^{2m} + x^{2n})^{\frac{1}{mn}} = x^{\frac{1}{m}} + \frac{1}{n}\left\{x^{m-n} + x^{n-m}\right\}^{\frac{1}{mn}}$.

11. $\dfrac{a^2 - b^2}{a^2 + b^2}\left\{x^{\frac{1}{p}} + x^{\frac{1}{q}}\right\} - 2x^{\frac{p+q}{2pq}}$

$$= \frac{a^2 - b^2}{a^2 + b^2}\left\{x^{\frac{1}{2p}} - x^{\frac{1}{2q}}\right\} - \frac{4b^2}{a^2 + b^2}x^{\frac{p+q}{2pq}}.$$

12. $(x^m - a^m) \div (x^{\frac{m}{2^n}} - a^{\frac{m}{2^n}})$

$$= (x^{\frac{m}{2}} + a^{\frac{m}{2}})(x^{\frac{m}{4}} + a^{\frac{m}{4}})(x^{\frac{m}{8}} + a^{\frac{m}{8}}).....(x^{\frac{m}{2^n}} + a^{\frac{m}{2^n}}).$$

Find the square roots of—

13. $a + b + c + 2(a^{\frac{1}{2}}b^{\frac{1}{2}} + a^{\frac{1}{2}}c^{\frac{1}{2}} + b^{\frac{1}{2}}c^{\frac{1}{2}})$.

14. $4xy^{\frac{4}{3}} - 12x^{\frac{3}{2}}y + 17x^2y^{\frac{2}{3}} - 12x^{\frac{5}{2}}y^{\frac{1}{3}} + 4x^3$.

15. $a^2b^{-1} - 4ab^{-\frac{1}{2}} - 8a^{-1}b^{\frac{1}{2}} + 4a^{-2}b + 8$.

Find the cube roots of—

16. $x^4 + 9x^{\frac{10}{3}} + 6x^{\frac{8}{3}} - 99x^2 - 42x^{\frac{4}{3}} + 441x^{\frac{2}{3}} - 343$.

17. $x^6y^{-1} + 3x^4y^{-\frac{2}{3}} + 3x^2y^{-\frac{1}{3}} + 1$.

18. $ab(1 + 3a^{-\frac{1}{3}}b^{\frac{1}{3}} + 3a^{-\frac{2}{3}}b^{\frac{2}{3}} + a^{-1}b)(ab^{-1} - 3a^{\frac{2}{3}}b^{-\frac{2}{3}} + 3a^{\frac{1}{3}}b^{-\frac{1}{3}} + 1)$.

CHAPTER IV.

SURDS.

12. A **surd** quantity is one in which the root indicated cannot be denoted without the use of a fractional index.

Thus, the following quantities are surds :—

$$\sqrt{a},\ \sqrt[4]{a^2 + x^2},\ \sqrt[n]{a^2 + b^2 + c^2},\ \sqrt{\frac{x + y}{a}},\ \frac{\sqrt[4]{a + x}}{\sqrt[3]{x + y}}.$$

Since, from what has been explained in the last chapter, these quantities may be written thus—

$$a^{\frac{1}{2}},\ (a^2 + x^2)^{\frac{1}{4}},\ (a^2 + b^2 + c^2)^{\frac{1}{n}},\ \left(\frac{x + y}{a}\right)^{\frac{1}{2}},\ \frac{(a + x)^{\frac{4}{4}}}{(x + y)^{\frac{1}{3}}},$$

it follows that surds may be dealt with exactly as we deal with their equivalent expressions with fractional indices.

It is evident that rational quantities may be put in the *form* of surds, and conversely, expressions which have the *form* of surds may sometimes be rational quantities.

Thus, $a^2 = \sqrt[3]{(a^2)^3} = \sqrt[3]{a^6}$;

and $\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3} = \sqrt[3]{(a + b)^3} = a + b.$

13. *A mixed quantity may be expressed as a surd.*

Thus, $3\sqrt[3]{5} = \sqrt[3]{3^3}.\ \sqrt[3]{5} = \sqrt[3]{3^3 \times 5} = \sqrt[3]{135},$

and so, $x\sqrt[n]{y} = \sqrt[n]{x^n}.\ \sqrt[n]{y} = \sqrt[n]{x^n y}.$

14. Conversely, *a surd may be expressed as a mixed quantity*, when the root of any factor can be obtained.

Thus, $\sqrt{18\,a^3b^2} = \sqrt{9\,a^2b^2 \times 2\,a}$

$= \sqrt{9\,a^2b^2}.\ \sqrt{2\,a} = 3ab\sqrt{2\,a}.$

And $\sqrt[3]{(a^2 + b^2)^6 x^4 y^5} = \sqrt[3]{(a^2 + b^2)^6 x^3 y^3 \times xy^2}$

$= \sqrt[3]{(a^2 + b^2)^6 x^3 y^3}.\ \sqrt[3]{xy^2} = (a^2 + b^2)^2 xy\sqrt[3]{xy^2}.$

15. *Fractional surd expressions may be so expressed that the surd portion may be integral.*

The process is called *rationalizing the denominator*, and is worth special notice.

Ex. 1. $\sqrt{\frac{3}{7}} = \sqrt{\frac{3 \times 7}{7^2}} = \frac{\sqrt{21}}{\sqrt{7^2}} = \frac{\sqrt{21}}{7}.$

It is much easier to find approximately the value of $\sqrt{21}$, and divide the result by 7, than to find the values of $\sqrt{3}$ and $\sqrt{7}$, and divide the former by the latter.

Ex. 2. Reduce to its simplest form $\sqrt{\frac{xy}{b - c}}$.

$$\sqrt{\frac{xy}{b - c}} = \sqrt{\frac{xy(b - c)}{(b - c)^2}} = \frac{\sqrt{xy(b - c)}}{b - c}.$$

Ex. 3. Find the arithmetical value of $\frac{4}{2 - \sqrt{3}}$.

The denominator is the *difference* of two quantities, one of hich is a *quadratic* surd.

Now, we know that $(2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$, and hence we see that by multiplying numerator and denominator by the *sum* of the quantities in the denominator we can obtain the denominator in a *rational form.*

$$\text{Thus, } \frac{4}{2 - \sqrt{3}} = \frac{4(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})} = \frac{4(2 + \sqrt{3})}{2^2 - (\sqrt{3})^2}$$

$$\frac{4(2 + \sqrt{3})}{4 - 3} = \frac{4(2 + \sqrt{3})}{1} = 4(2 + \sqrt{3})$$

$$= 4(2 + 1{\cdot}73205) = 4(3{\cdot}73205.)$$

$$= 14{\cdot}92820.$$

$$\text{na so, } \frac{4}{4\sqrt{2} + 3\sqrt{3}} = \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2} + 3\sqrt{3})(4\sqrt{2} - 3\sqrt{3})}$$

$$= \frac{4(4\sqrt{2} - 3\sqrt{3})}{(4\sqrt{2})^2 - (3\sqrt{3})^2} = \frac{4(4\sqrt{2} - 3\sqrt{3})}{32 - 27} = \tfrac{4}{5}(4\sqrt{2} - 3\sqrt{3}).$$

We shall now give an example when the surds are not quadratic.

Ex 4. Rationalize the denominator of $\frac{a}{x^{\frac{1}{4}} - y^{\frac{1}{3}}}$.

Since $(x^{\frac{1}{4}})^{12} - (y^{\frac{1}{3}})^{12}$ is (Art. 29, page 175) divisible by $x^{\frac{1}{4}} - y^{\frac{1}{3}}$, it follows that the rationalizing factor is their quotient, which is easily found.

16. *Surds may be reduced to a common index.*

Ex. 1. Express $\sqrt[m]{a}$ and $\sqrt[n]{b}$ as surds having a common index.

Since $\sqrt[m]{a} = a^{\frac{1}{m}}$, and $\sqrt[n]{b} = b^{\frac{1}{n}}$, it follows that, by reducing the fractional indices to a common denominator, the given surds become respectively $a^{\frac{n}{mn}}$, $b^{\frac{m}{mn}}$, or $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.

Ex. 2. Reduce $\sqrt[4]{a^3b}$ and $\sqrt[3]{x^5y^2}$ to a common index.

The least common denominator of the fractional indices of the given surds is 4×3 or 12. Hence we proceed as follows:—

$$\sqrt[4]{a^3b} = (a^3b)^{\frac{1}{4}} = (a^3b)^{\frac{3}{12}} = \sqrt[12]{(a^3b)^3} = \sqrt[12]{a^9b^3},$$
$$\sqrt[3]{x^5y^2} = (x^5y^2)^{\frac{1}{3}} = (x^5y^2)^{\frac{4}{12}} = \sqrt[12]{(x^5y^2)^4} = \sqrt[12]{x^{20}y^8}.$$

When the student has had a little practice, the first two steps of each of the operations may be omitted.

17. *Addition and subtraction of similar surds.*

Def. Similar surds are those which have the same irrational factors.

Ex. 1. Find the sum of $\sqrt{12}$, $5\sqrt{27}$, $-2\sqrt{75}$.

We have—

$$\sqrt{12} + 5\sqrt{27} - 2\sqrt{75}$$
$$= \sqrt{2^2 \times 3} + 5\sqrt{3^2 \times 3} - 2\sqrt{5^2 \times 3}$$
$$= 2\sqrt{3} + 5 \times 3\sqrt{3} - 2 \times 5\sqrt{3}$$
$$= 2\sqrt{3} + 15\sqrt{3} - 10\sqrt{3}$$
$$= (2 + 15 - 10)\sqrt{3} = 7\sqrt{3}.$$

Ex. 2. Simplify—

$$\sqrt{\frac{a^2b + 2ab^2 + b^3}{a^2 - 2ab + b^2}} - \sqrt{\frac{a^2b - 2ab^2 + b^3}{a^2 + 2ab + b^2}}.$$

The given expression—

$$= \sqrt{\frac{(a + b)^2b}{(a - b)^2}} - \sqrt{\frac{(a - b)^2b}{(a + b)^2}}$$
$$= \frac{a + b}{a - b}\sqrt{b} - \frac{a - b}{a + b}\sqrt{b} = \left(\frac{a + b}{a - b} - \frac{a - b}{a + b}\right)\sqrt{b}$$
$$= \frac{(a + b)^2 - (a - b)^2}{(a - b)(a + b)}\sqrt{b} = \frac{4ab}{a^2 - b^2}\sqrt{b}.$$

18. *Multiplication and division of surds.*

The following examples will best illustrate these operations:—

Ex. 1. Multiply $a\sqrt{x^3yz}$ by $b\sqrt{xy^3u}$.

We have, $a\sqrt{x^3yz} \times b\sqrt{xy^3u} = ab\sqrt{x^3yz \times xy^3u}$

$= ab\sqrt{x^4y^4uz} = abx^2y^2\sqrt{uz}$.

Ex. 2. Multiply $a\sqrt{b} + c\sqrt{d}$ by $a - \sqrt{bd}$.

Arranging as in the case of rational quantities, we have—

$$\begin{array}{r} a\sqrt{b} + c\sqrt{d} \\ a - \sqrt{bd} \\ \hline a^2\sqrt{b} + ac\sqrt{d} \\ - ab\sqrt{d} - cd\sqrt{b} \\ \hline (a^2 - cd)\sqrt{b} + a(c - b)\sqrt{d} \end{array}$$

Ex. 3. Divide $a\sqrt{b}$ by $b\sqrt{a}$.

We have, $\dfrac{a\sqrt{b}}{b\sqrt{a}} = \dfrac{a\sqrt{b}.\sqrt{a}}{b\sqrt{a}.\sqrt{a}} = \dfrac{a\sqrt{ab}}{b.a} = \dfrac{1}{b}\sqrt{ab}$.

When the divisor is a compound quantity it will *generally* be the best to express the surds as quantities with fractional indices, and proceed as in ordinary division.

19. *The square root of a rational quantity cannot be partly rational and partly irrational.*

If possible, let $\sqrt{a} = m + \sqrt{b}$;

then, squaring, $a = m^2 + 2m\sqrt{b} + b$;

or, $2m\sqrt{b} = a - (m^2 + b)$;

$$\text{or, } \sqrt{b} = \frac{a - (m^2 + b)}{2m};$$

that is, an irrational quantity is equal to a rational quantity, which is absurd.

20. *To find the square root of a binomial, one of whose terms is a quadratic surd.*

Let $a + \sqrt{b}$ be the binomial.

Assume $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,.....(1);

then, squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$,.....(2).

Equating the rational and irrational parts (Art. 19), we have—

$$x + y = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.),$$

$$\text{and } 2\sqrt{xy} = \sqrt{b} \text{ or } 4xy = b \ldots\ldots\ldots\ldots\ldots (4.)$$

From (3) and (4) we easily find $x = \frac{1}{2}(a + \sqrt{a^2 - b})$,

$$\text{and } y = \frac{1}{2}(a - \sqrt{a^2 - b}).$$

Hence, from (1), the square root required is—

$$\sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} + \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})}.$$

NOTE.—It is evident that, unless $(a^2 - b)$ is a perfect square, our result is more complicated than the original expression, and therefore the above method fails in that case.

Ex. 1. Find the square root of $14 + 6\sqrt{5}$.

Let $\sqrt{14 + 6\sqrt{5}} = \sqrt{x} + \sqrt{y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$

Squaring, then, $14 + 6\sqrt{5} = x + y + 2\sqrt{xy}$.

Hence, equating the rational and irrational parts—

$$x + y = 14 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.),$$

$$2\sqrt{xy} = 6\sqrt{5} \text{ or } 4xy = 180 \ldots\ldots\ldots\ldots\ldots (3).$$

From (2) and (3) we easily find $x = 9$, $y = 5$.

Hence the square root required is $\sqrt{9} + \sqrt{5}$ or $3 + \sqrt{5}$.

Ex. 2. Find the square root of $39 + \sqrt{1496}$.

Let $\sqrt{39 + \sqrt{1496}} = \sqrt{x} + \sqrt{y}$.

Squaring, &c., we have, $x + y = 39$;

$$\text{and } 4xy = 1496.$$

From these equations we easily find $x = 22$, $y = 17$.

Hence, the square root required is $\sqrt{22} + \sqrt{17}$.

21. *The square roots of quantities of this kind may often be found by inspection.*

Ex. 1. Find the square root of $19 + 8\sqrt{3}$.

We shall throw this expression into the form $a^2 + 2ab + b^2$, which we know is a perfect square.

Dividing the irrational term by 2, we have $4\sqrt{3}$. Now all we have to do is to break this up into two such factors that the sum of their squares shall be 19. The factors are evidently 4 and $\sqrt{3}$.

Thus, we have $19 + 8\sqrt{3} = (4)^2 + 2(4)\sqrt{3} + (\sqrt{3})^2$
$= (4 + \sqrt{3})^2$.

The square root is therefore $4 + \sqrt{3}$.

Ex. 2. Find the square root of $29 + 12\sqrt{5}$.

We have $29 + 12\sqrt{5} = (3)^2 + 2(3)2\sqrt{5} + (2\sqrt{5})^2$
$= (3 + 2\sqrt{5})^2$.

The square root is therefore $3 + 2\sqrt{5}$.

Ex. VI.

Express with fractional indices—

1. $\sqrt{x^5}$, $\sqrt[3]{a^5b^2}$, $\sqrt[4]{x^2y^3}$, $\sqrt[8]{a^2b^4}$.

2. $\frac{xy}{\sqrt{ab}}$, $\frac{\sqrt{c^3d}}{\sqrt{xy}}$, $\frac{\sqrt[n]{x^m}}{\sqrt[m]{a^n}}$.

Reduce to entire surds—

3. $3\sqrt{3}$, $4\sqrt{\frac{3}{2}}$, $\frac{3}{5}\sqrt{15}$, $3\sqrt[3]{\frac{1}{3}}$.

4. $4 . 2^{\frac{1}{2}}$, $9 . 3^{-\frac{1}{2}}$, $4 . 2^{-\frac{3}{4}}$, $\frac{3}{2}\left(\frac{2}{3}\right)^{-\frac{1}{2}}$.

5. $3\sqrt{ab}$, $a\sqrt{\frac{bc}{a}}$, $(a + x)\sqrt[3]{\frac{a - x}{(a + x)^2}}$.

Reduce to a common index—

6. $\sqrt[3]{2}$, $\sqrt{3}$.

7. $\sqrt[4]{2}$, $\sqrt[3]{3}$.

8. $2\sqrt{2}$, $3\sqrt[3]{5}$.

9. $\sqrt[m]{a}$, $\sqrt[n]{b}$.

10. $(a + x)^{\frac{1}{2}}$, $\sqrt[3]{a - x}$.

11. $a^{\frac{1}{p} + \frac{1}{q}}$, $b^{\frac{1}{p} - \frac{1}{q}}$.

Simplify—

12. $\sqrt{12}$, $\sqrt[3]{48}$, $3\sqrt{28}$, $\frac{1}{2}\sqrt[3]{648}$.

13. $\sqrt{4a^3 + 4a^2b}$, $\sqrt[3]{a^3b^3 + b^6}$, $\sqrt[3]{\frac{a + b}{64\,a}}$.

14. $\sqrt{\frac{1}{x^2} - \frac{2a}{x} + a^2}$, $\sqrt[3]{\frac{a^2}{x^2} - \frac{3\,a^2}{x} + 3\,a^2 - a^2x}$.

15. $\sqrt[3]{\frac{x^3 + 1}{9\,x^2} + \frac{x + 1}{3\,x}}$, $\sqrt[3]{\frac{(x + a)^2(x^2 - a^2)}{(x - a)^2(x + a)}}$.

Find the value of—

16. $\sqrt{12} + \sqrt{48} - 2\sqrt{3}, \sqrt{56} + \sqrt[3]{189}$.

17. $\sqrt[3]{\frac{a^4x^2y^3}{b}} - \sqrt[3]{a^4b^2x^2} + \sqrt[3]{\frac{ax^2c^3}{b}}$.

18. $\sqrt[3]{27\ a^{m+6}\ b^3} - \sqrt[3]{8a^{4m+9}} + 3\sqrt[3]{64\ a^m}$.

Multiply—

19. $a + \sqrt{ab} + b$ by $\sqrt{a} - \sqrt{b}$, $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $\sqrt{a} - \sqrt{b}$.

20. $(x+y)^{\frac{1}{2}}$ by $(x+y)^{\frac{1}{3}}$, $a + b\sqrt{d}$ by $a^2 - ab\sqrt{d} + b^2d$.

21. $c\sqrt{\frac{a}{a+b}} + \sqrt{\frac{1}{b}}$ by $\frac{c^2}{d}\sqrt{a\ (a+b)} - \sqrt{b^3}$.

22. $a^{\frac{1}{2}} + b^{\frac{1}{3}} + c^{\frac{1}{4}} + d^{\frac{1}{5}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{3}} + c^{\frac{1}{4}} - d^{\frac{1}{5}}$.

Divide—

23. $x^2 + xy + y^2$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

24. $x^3 - y^2$ by $x^{\frac{1}{2}} + y^{\frac{1}{3}}$.

Rationalize the denominators of—

25. $\sqrt{\frac{5}{3}}, \frac{4}{\sqrt{7}}, \sqrt[3]{\frac{7}{5}}$.

26. $\frac{3}{2+\sqrt{3}}, \frac{4}{3\sqrt{2}-2\sqrt{3}}, \frac{1}{\sqrt{5}-\sqrt{3}}$.

27. $\frac{3}{1+\sqrt{2}+\sqrt{3}}, \frac{2}{\sqrt{2}+\sqrt{3}+\sqrt{5}}, \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

28. $\frac{a}{x^{\frac{1}{2}}-y^{\frac{1}{3}}}, \frac{1}{\sqrt[3]{2}+\sqrt{3}}, \frac{b}{x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y}$.

Find the square roots of—

29. $11 + 4\sqrt{7}$, $8 + 2\sqrt{15}$, $30 - 10\sqrt{5}$.

30. $8 + 2\sqrt{12}$, $9 - 6\sqrt{2}$, $20 - 10\sqrt{3}$.

CHAPTER V.

RATIO AND PROPORTION.

Ratio.

22. The student is referred to Chapter II. of the Arithmetic section of this work for definitions and observations which need not be repeated here.

23. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by increasing the terms of the ratio by the same quantity.*

Let $a : b$ or $\frac{a}{b}$ be the ratio, and let each of its terms be increased by m. It will then become $\frac{a + m}{b + m}$.

Now, $\frac{a + m}{b + m} \gtreqless \frac{a}{b}$, as $(a + m)\, b \gtreqless (b + m)\, a$,

or, as $ab + bm \gtreqless ab + am$; or, as $bm \gtreqless am$, or as $b \gtreqless a$.

Hence the ratio $\frac{a}{b}$ is *increased* when $b > a$, that is, when it is a ratio of less inequality; and is diminished when $b < a$, that is, when it is a ratio of greater inequality.

Cor. It may be shown in the same way that—

A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by diminishing the terms of the ratio by the same quantity.

24. When the difference between the antecedent and consequent is small compared with either, the ratio of the higher powers of the terms is found by doubling, trebling, &c., their difference.

Let $a + x : a$ or $\frac{a + x}{a}$ be the ratio, where x is small compared with a.

Then $\frac{(a + x)^2}{a^2} = \frac{a^2 + 2ax + x^2}{a^2} = 1 + \frac{2x}{a}$ nearly $= \frac{a + 2x}{a}$ nearly.

$$\frac{(a+x)^3}{a^3} = \frac{a^3 + 3a^2x + 3ax^2 + x^3}{a^3} = 1 + \frac{3x}{a} \text{ nearly} = \frac{a+3x}{a} \text{ nearly; and so on.}$$

Ex. $(1002)^2 : (1000)^2 = 1004 : 1000$ nearly.
$(1002)^3 : (1000)^3 = 1006 : 1000$ nearly.

Proportion.

25. Proportion, as has been already said, is the relation of equality expressed between ratios.

Thus, the expression $a : b = c : d$,
or $a : b :: c : d$,
or $\frac{a}{b} = \frac{c}{d}$,
is called a proportion.

26. The following results are easily obtained :—

(1.) Since $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$ or $\frac{a}{c} = \frac{b}{d}$,

$\therefore a : c :: b : d$ (*alternando*).

(2.) $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$ or $\frac{b}{a} = \frac{d}{c}$,

$\therefore b : a :: d : c$ (*invertendo*).

Also, by Art. 64, page 214, we have—

(3.) $a + b : b :: c + d : d$ (*componendo*).

(4.) $a - b : b :: c - d : d$ (*dividendo*).

(5.) $a - b : a :: c - d : c$ (*convertendo*).

(6.) $a + b : a - b :: c + d : c - d$ (*componendo* and *dividendo*).

27. If $a : b :: c : d$ and $e : f :: g : h$, we may compound the proportions.

Thus we have $\frac{a}{b} = \frac{c}{d}$ .....(1), and $\frac{e}{f} = \frac{g}{h}$ .....(2).

(1) $\times$ (2), then, $\frac{ae}{bf} = \frac{cg}{dh}$,

or $ae : bf :: cg : dh$.

And in the same way we may show that, if the corresponding terms of any number of proportions be multiplied together, the products will be proportional.

28. *If three quantities are in continued proportion, the first has to the third the duplicate ratio of what it has to the second.*

Let a, b, c be the given quantities in continued proportion; then—

$$\frac{a}{b} = \frac{b}{c}$$

Hence, $\frac{a}{c}$ or $\frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$;

$$\therefore a : c :: a^2 : b^2.$$

And, similarly, if a, b, c, d are four quantities *in continued proportion*, $a : d :: a^3 : b^3$, that is—

The first has to the fourth the triplicate ratio of what it has to the second; and so on, for any number of quantities.

29. We shall now give one or two examples of problems in Proportion.

Ex. 1. If $a : b :: c : d$, prove that $\frac{a^3 + c^3}{b^3 + d^3} = \left(\frac{a + c}{b + d}\right)^3$.

Let $\frac{a}{b} = \frac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

Hence, $\frac{a^3 + c^3}{(a + c)^3} = \frac{(bx)^3 + (dx)^3}{(bx + dx)^3} = \frac{b^3 + d^3}{(b + d)^3}$, then, *alternando*, $\frac{a^3 + c^3}{b^3 + d^3} = \left(\frac{a + c}{b + d}\right)^3$.

Ex. 2. If $a : b :: c : d$, prove that $\frac{a + b}{a - b} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}$.

Let $\frac{a}{b} = \frac{c}{d} = x$; $\therefore a = bx$, and $c = dx$.

Hence, $\frac{a + b}{a - b} = \frac{bx + b}{bx - b} = \frac{x + 1}{x - 1} = \frac{\sqrt{bd}.x + \sqrt{bd}}{\sqrt{bd}.x - \sqrt{bd}}$

$$= \frac{\sqrt{bx.dx} + \sqrt{bd}}{\sqrt{bx.dx} - \sqrt{bd}} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}.$$

Or, it may be worked thus—

Since $\frac{a}{b} = \frac{c}{d}$, we have $\sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}}$,

Or, $\sqrt{\frac{a}{b}} \times \sqrt{\frac{a}{b}} = \sqrt{\frac{c}{d}} \times \sqrt{\frac{a}{b}}$, or $\frac{a}{b} = \frac{\sqrt{ac}}{\sqrt{bd}}$,

Hence, by Art. 26—

$$\frac{a + b}{a - b} = \frac{\sqrt{ac} + \sqrt{bd}}{\sqrt{ac} - \sqrt{bd}}.$$

Ex. 3. If $a : b :: c : d :: e : f$, show that

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}}$$

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = x$.....(1).

$$\therefore \frac{a^r}{b^r} = \frac{c^r}{d^r} = \frac{e^r}{f^r} = x^r.$$

Hence, $\left.\begin{aligned} a^r &= b^r x^r, \ \therefore \ ma^r = mb^r x^r \\ c^r &= d^r x^r, \ \therefore \ nc^r = nd^r x^r \\ e^r &= f^r x^r, \ \therefore \ pe^r = pf^r x^r \end{aligned}\right\}$, and $\therefore$ by addition,

$$ma^r + nc^r + pe^r = (mb^r + nd^r + pf^r)x^r.$$

$$\therefore \frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r} = x^r, \text{ or } \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}} = x \ldots (2).$$

$\therefore$ Equating (1) and (2), we have—

$$\frac{a}{b} = \left(\frac{ma^r + nc^r + pe^r}{mb^r + nd^r + pf^r}\right)^{\frac{1}{r}}. \quad Q.E.D.$$

Ex. VII.

1. Compare the ratios $a + b : a - b$, and $a^2 + b^2 : a^2 - b^2$.

2. Which is the greater of the ratios $a + b : 2a$, and $2b : a + b$?

3. What quantity must be subtracted from the consequent of the ratio $a : b$ in order to make it equal to the ratio $c : d$?

4. Compound the ratios $1 - x^2 : 1 + y$, $x - xy^2 : 1 + x^2$, and $1 : x - x^2$.

5. There are two numbers in the ratio of 6 : 7, but if 10 be added to each they are in the ratio of 8 : 9. Find the numbers.

6. In what cases is $x + \frac{6}{x} >$ or < 5 ?

7. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{z-x}$, show that $a + b + c = 0$.

8. Find the value of x when the ratio $x + 2a : x + 2b$ is the duplicate ratio of $2x + a + c : 2x + b + c$.

9. Find x when the ratio $x - b : x + 2a - b$ is the triplicate ratio of $x - a : x + a - b$.

10. If $\frac{x+y}{a+b} = \frac{y+z}{b+c} = \frac{x+z}{c+a}$, show that each of the fractions is equal to $\frac{x+y+z}{a+b+c}$, and that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

11. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each is equivalent to $\frac{la + mc + nc}{lb + md + nf}$;

hence, show that—

$$\frac{a}{2z + 2x - y} = \frac{b}{2x + 2y - z} = \frac{c}{2y + 2z - x},$$

when $\frac{x}{2a + 2b - c} = \frac{y}{2b + 2c - a} = \frac{z}{2c + 2a - b}$.

12. If $a : b :: c : d$, then

$$a + b : c + d :: \sqrt{a^2 + ab + b^2} : \sqrt{c^2 + cd + d^2}.$$

13. Find a fourth proportional to the quantities—

$$\frac{x+1}{x-1}, \quad \frac{x^2 + x + 1}{x^2 - x + 1}, \quad \frac{x^3 + 1}{x^3 - 1}.$$

14. Find c in terms of a and b when—

(1.) $a : a :: a - b : b - c$.

(2.) $a : b :: a - b : b - c$.

(3.) $a : c :: a - b : b - c$.

15. If a, b, c are in continued proportion, show that $\frac{a + b}{b + c}$, b, bc are also in continued proportion.

16. If $a : b :: c : d$, then—

$$\sqrt{a^{2n} + b^{2n}} : \sqrt{c^{2n} + d^{2n}} :: (a - b)^n : (c - d)^n.$$

17. From a vessel containing a cubic inches of hydrogen gas, b cubic inches are withdrawn, the vessel being filled up with oxygen at the same pressure. Show that if this operation be repeated n times successively, the quantity of hydrogen remaining in the vessel is $\frac{(a - b)^n}{a^{n-1}}$ cubic inches, when reduced to the original pressure.

18. If, in Ex. 34, page 225, (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3) are corresponding terms respectively, show that $a_1b_2b_3(c_2 - c_3) + a_2b_3b_1(c_3 - c_1) + a_3b_1b_2(c_1 - c_2) = 0$.

SECTION III.

PLANE TRIGONOMETRY.

CHAPTER I.

MODES OF MEASURING ANGLES BY DEGREES AND GRADES.

1. We are able to determine geometrically a right angle, and it might therefore be taken as the unit of angular measurement. Practically, however, it is too large, and so we take a determinate part of a right angle as a standard.

In England we divide a right angle into 90 equal parts, called *degrees*, and we further subdivide a *degree* into 60 equal parts, called *minutes*, and again a *minute* into 60 equal parts, called *seconds*. This is the English or sexagesimal method.

In France the right angle is divided into 100 equal parts, called *grades*, a *grade* into a hundred equal parts, called *minutes*, and a *minute* into 100 equal parts, called *seconds*. This is the French or centesimal method, and its advantages are those of the metric system generally.

The symbols °, ′, ″, are used to express English *degrees, minutes, seconds* respectively, and the symbols ᵍ, ˈ, ˝, to express French *grades, minutes, seconds* respectively.

Conversion of English and French Units.

2. Let D = the number of degrees in an angle,
and G = the number of grades in the same angle;
then $\frac{D}{90}$ expresses the angle in terms of a right angle;
and so also does $\frac{G}{100}$.

Hence, $\frac{D}{90} = \frac{G}{100}$ or $\frac{D}{9} = \frac{G}{10}$.

$$\therefore D = \frac{9}{10}G = G - \frac{1}{10}G \quad \ldots\ldots(1).$$

$$\text{and } G = \frac{10}{9}D = D + \frac{1}{9}D \quad \ldots\ldots(2).$$

Hence the following rules:—

1. *To convert grades into degrees.*

From the number of grades SUBTRACT $\frac{1}{10}$, and the remainder is the number of degrees.

2. *To convert degrees into grades.*

To the number of degrees ADD $\frac{1}{9}$, and the sum is the number of grades.

Ex. 1. Convert $13^g\ 18'\ 75''$ into English measure.

No. of grades = 13·1875
Subtract $\frac{1}{10}$ of this = 1·31875
∴ No of degrees = 11·86875
60
52′·12500
60
7″·500

Ans. 11° 52′ 7″·5.

Ex. 2. Convert 18° 7′ 30″ into French measure.

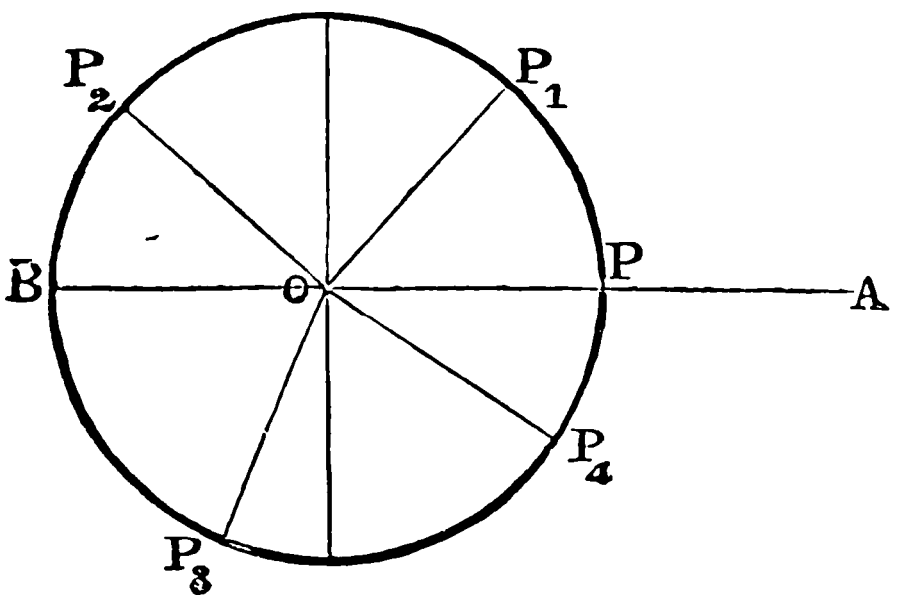
No. of degrees = 18·125
Add $\frac{1}{9}$ of this = 2·013$\dot{8}$
∴ No. of grades = 20·138$\dot{8}$

Ans. $20^g\ 13'\ 88''\cdot\dot{8}$.

3. An angle may be conceived to be generated by the revolution of a line about a fixed point. Thus—

Let OA be an initial line, and let a line, OP, starting from OA, revolve with O as centre, and take up successively the positions OP_1, OP_2, OP_3, OP_4.

Now the magnitude of an angle may be measured by *the amount of turning required to generate it.* When, therefore, the revolving line reaches the position OB, we may conceive an angle to have been generated whose magnitude is *two right angles.* And, further, when the revolving line assumes the positions OP_3, OP_4, the angles AOP_3, AOP_4 (the letters being read in the direction of revolution) are angles whose magnitudes are each *greater than two right angles.* Indeed, when the revolving line again reaches the position OP, we may conceive an angle to have been generated whose magnitude is four right angles. Lastly, if the revolution of the line OP be continued, we may conceive of angles being generated to whose magnitude there is no limit.



Ex. I.

1. Express $39° 22' 30''$ in French measure, and $13^g 15' 75''$ in English measure.

2. One of the angles at the base of an isosceles triangle is $50°$. Express the vertical angle in grades.

3. Divide an angle of n degrees into two such parts that the number of degrees in one part may be twice the number of grades in the other.

4. Two angles of a triangle are respectively $a°$, b^g, express the other angle in degrees and grades.

5. If $\frac{3}{4}$ of a right angle be the unit of measurement, express an angle which contains $22{\cdot}5$ degrees.

6. Show how to reduce English seconds to French seconds.

7. If the unit of measurement be $8°$, what is the value of 10^g.

8. If two of the angles of a triangle be expressed in grades, and the third in degrees, they are respectively as the numbers 5, 15, 18. Find the angles.

9. What is the value in degrees and grades of an angle

which is the result of the revolution of a line $3\frac{1}{5}$ times round.

10. In what quadrants are the following angles found:— 145°, 96^g, 327°, 272^g, 272°.

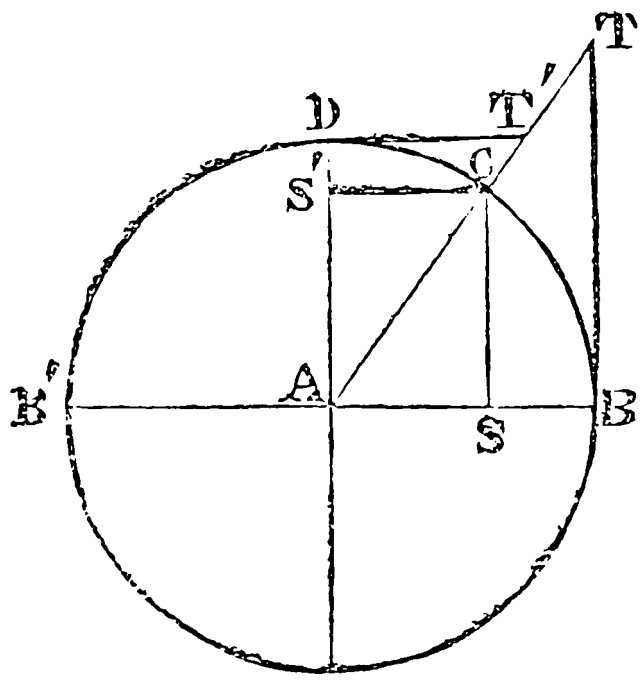
11. If $a°$ be taken as the unit of angular measurement, express an angle containing b^g.

12. What is the unit of measurement when a expresses $\frac{p}{q}$ of a right angle?

CHAPTER II.

THE GONIOMETRIC FUNCTIONS.

4. It was formerly usual in works on Trigonometry to give the following definitions:—



Let a circle be described from centre A, with radius AB supposed to be unity, then—

(1.) The *sine* of an arc BC is the perpendicular from one extremity, C, of the arc upon the diameter passing through the other extremity B.

Thus CS *is the* SINE *of the arc BC.*

(2.) The *cosine* of an arc is the *sine of the complement* of the arc.

Thus, since DC is the complement of BC,

S'C *is the* COSINE *of the arc BC.*

(3.) The *tangent* of an arc BC is a line drawn from one extremity, B, of the arc touching the circle, and terminated in the diameter which passes through the other extremity, C, of the arc.

Thus, BT *is the* TANGENT *of the arc BC.*

(4.) The *cotangent* of an arc is the *tangent of the complement* of the arc.

Thus, DT′ *is the* COTANGENT *of the arc BC.*

(5.) The *secant* of an arc BC is a line drawn from the centre through one extremity, C, of the arc, and terminated in the tangent at the other extremity.

Thus, AT *is the* SECANT *of the arc BC.*

(6.) The *cosecant* of an arc is the *secant of the complement* of the arc.

Thus, AT′ *is the* COSECANT *of the arc BC.*

(7.) The *versed sine* is that portion of the radius upon which the *sine* falls, which is included between the sine and the extremity of the arc.

Thus, SB *is the* VERSED SINE *of the arc BC.*

(8.) The *coversed sine* is the *versed sine of the complement* of the arc.

Thus, S′D *is the* COVERSED-SINE *of the arc BC.*

(9.) The *suversed sine* is the *versed sine of the supplement* of the arc.

Thus, B′S *is the* SUVERSED-SINE *of the arc BC.*

Representing the arc BC by A, it is usual to write the above functions thus :—Sin A, cos A, tan A, cot A, sec A, cosec A, vers A, covers A, suvers A.

By mere inspection, the student will see that the following relations hold :—

(1.) $\text{Sin A} = \text{CS} = \frac{\text{CS}}{1} = \frac{\text{CS}}{\text{AS}} = \frac{\text{AD}}{\text{AT}'} = \frac{1}{\text{cosec A}}$.

(2.) $\text{Cos A} = \text{S}'\text{C} = \frac{\text{AS}}{1} = \frac{\text{AS}}{\text{AC}} = \frac{\text{AB}}{\text{AT}} = \frac{1}{\text{sec A}}$.

(3.) $\text{Tan A} = \text{BT} = \frac{\text{BT}}{1} = \frac{\text{BT}}{\text{AB}} = \frac{\text{AD}}{\text{DT}'} = \frac{1}{\text{cot A}}$.

(4.) $\text{Sin}^2 \text{A} + \cos^2 \text{A} = \text{CS}^2 + \text{S}'\text{C}^2 = \text{CS}^2 + \text{AS}^2 = \text{AC}^2 = 1$.

(5.) $\text{Sec}^2 \text{A} = \text{AT}^2 = \text{AB}^2 + \text{BT}^2 = 1 + \tan^2 \text{A}$.

(6.) $\text{Cosec}^2 \text{A} = \text{AT}'^2 = \text{AD}^2 + \text{DT}'^2 = 1 + \cot^2 \text{A}$.

(7.) $\text{Tan A} = \text{BT} = \frac{\text{BT}}{\text{AB}} = \frac{\text{CS}}{\text{AS}} = \frac{\text{CS}}{\text{S}'\text{C}} = \frac{\sin \text{A}}{\cos \text{A}}$.

(8.) $\text{Vers A} = \text{SB} = \text{AB} - \text{AS} = \text{AB} - \text{S}'\text{C} = 1 - \cos \text{A}$.

(9.) $\text{Covers A} = \text{S}'\text{D} = \text{AD} - \text{AS}' = \text{AD} - \text{CS} = 1 - \sin \text{A}$.

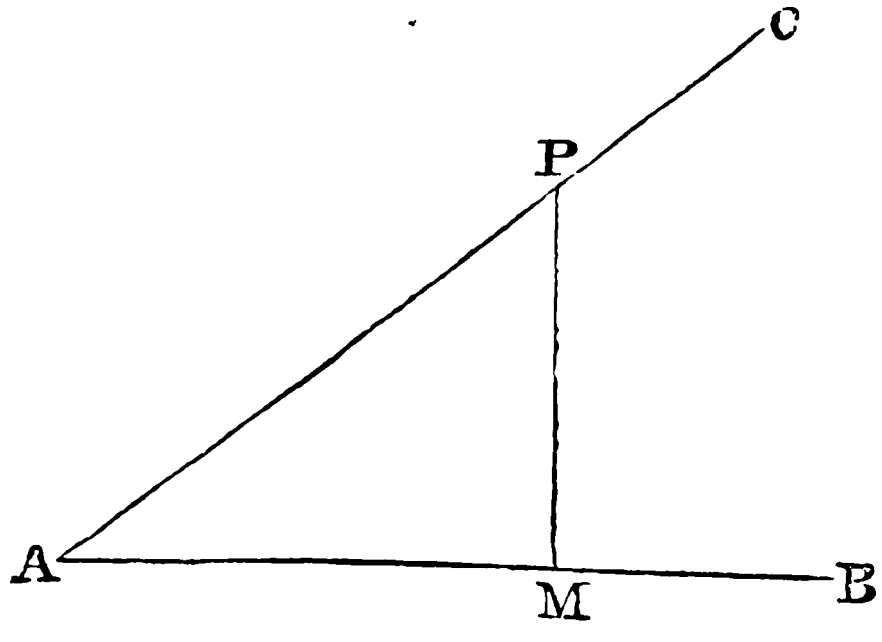
(10.) $\text{Suvers A} = \text{B}'\text{S} = \text{B}'\text{A} + \text{AS} = 1 + \cos \text{A}$.

It is more convenient, however, to define the *sine, cosine,* &c., as in the next article, according to which definitions they are commonly called TRIGONOMETRICAL RATIOS. The student will see that if the above definitions be so far modified that, instead of the *lines* themselves, the goniometric functions be taken as *the ratios* which the lines respectively bear to the radius, they are included in the definitions of the next article.

Trigonometrical Ratios.

5. Let BAC be any angle, which we may denote by A, and P any point in the line AC. Draw PM perpendicular to AB.

Then (1.) Sin A $= \frac{\text{perpendicular}}{\text{hyp.}} = \frac{\text{PM}}{\text{AP}}$.



(2.) Cos A $= \frac{\text{base}}{\text{hyp.}} = \frac{\text{AM}}{\text{AP}}$.

(3.) Tan A $= \frac{\text{perpendicular}}{\text{base}} = \frac{\text{PM}}{\text{AM}}$.

(4.) Cot A $= \frac{\text{base}}{\text{perpendicular}} = \frac{\text{AM}}{\text{PM}}$.

(5.) Sec A $= \frac{\text{hyp.}}{\text{base}} = \frac{\text{AP}}{\text{AM}}$.

(6.) Cosec A $= \frac{\text{hyp.}}{\text{perpendicular}} = \frac{\text{AP}}{\text{PM}}$.

(7.) The *versed sine* is the remainder after subtracting the *cosine* from unity, or—

$$\text{vers A} = 1 - \cos \text{A}.$$

(8.) The *coversed sine* is the remainder after subtracting the *sine* from unity, or—

$$\text{covers A} = 1 - \sin \text{A}.$$

(9.) The *suversed sine* is the sum of the *cosine* and unity, or—

$$\text{suvers A} = 1 + \cos \text{A}.$$

The last three are not much used in practice.

6. Comparing (1.) and (6.), (2.) and (5.), (3.) and (4.), of the last article, we see at once that the *sine* and *cosecant*, the *cosine* and *secant*, and the *tangent* and *cotangent*, are respectively each the reciprocal of the other.

We therefore have—

(1.) $\text{Sin } A = \frac{1}{\text{cosec } A}$, $\text{cosec } A = \frac{1}{\sin A}$.

(2.) $\text{Cos } A = \frac{1}{\sec A}$, $\sec A = \frac{1}{\cos A}$.

(3.) $\text{Tan } A = \frac{1}{\cot A}$, $\cot A = \frac{1}{\tan A}$.

Further—

(4.) $\text{Sin}^2 A + \cos^2 A = \frac{PM^2}{AP^2} + \frac{AM^2}{AP^2} = \frac{PM^2 + AM^2}{AP^2}$

$= \frac{AP^2}{AP^2} = 1.$

Hence also, transposing and taking the square root—

(5.) $\text{Sin } A = \sqrt{1 - \cos^2 A}.$

(6.) $\text{Cos } A = \sqrt{1 - \sin^2 A}.$

And again—

(7.) $\text{Sec}^2 A = \frac{AP^2}{AM^2} = \frac{AM^2 + PM^2}{AM^2} = 1 + \frac{PM^2}{AM^2} = 1 + \tan^2 A.$

(8.) $\text{Cosec}^2 A = \frac{AP^2}{PM^2} = \frac{PM^2 + AM^2}{PM^2} = 1 + \frac{AM^2}{PM^2}$

$= 1 + \cot^2 A.$

(9.) $\text{Tan } A = \frac{PM}{AM} = \frac{PM}{AP} \div \frac{AM}{AP} = \sin A \div \cos A$

$= \frac{\sin A}{\cos A}.$

(10.) $\text{Cot } A = \frac{AM}{PM} = \frac{AM}{AP} \div \frac{PM}{AP} = \cos A \div \sin A$

$= \frac{\cos A}{\sin A}.$

The student must make himself thoroughly master of the results in this article.

7. To express the trigonometrical ratios in terms of the *sine.*

(1.) $\text{Cos } A = \sqrt{1 - \sin^2 A}$, by Art. 6 (6.)

(2.) $\text{Tan } A = \dfrac{\sin A}{\cos A}$, by Art. 6 (9.),

$$= \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

(3.) $\text{Cot } A = \dfrac{\cos A}{\sin A}$, by Art. 6 (10.),

$$= \frac{\sqrt{1 - \sin^2 A}}{\sin A}.$$

(4.) $\text{Sec } A = \dfrac{1}{\cos A}$, by Art. 6 (2.),

$$= \frac{1}{\sqrt{1 - \sin^2 A}}.$$

(5.) $\text{Cosec } A = \dfrac{1}{\sin A}$, by Art. 6 (1.)

Ex. If $\sin A = \frac{3}{5}$, find the other trigonometrical ratios.

We have, $\cos A = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{5} \div \frac{4}{5} = \frac{3}{4}.$$

8. To express the trigonometrical ratios in terms of the *cotangent.*

(1.) $\text{Sin } A = \dfrac{1}{\text{cosec } A} = \dfrac{1}{\sqrt{1 + \cot^2 A}}$, by Art. 6 (8.)

(2.) $\text{Cos } A = \dfrac{1}{\sec A} = \dfrac{1}{\sqrt{1 + \tan^2 A}}$, by Art. 6 (7.)

$$= \frac{1}{\sqrt{1 + \dfrac{1}{\cot^2 A}}} = \frac{\cot A}{\sqrt{\cot^2 A + 1}}.$$

(3.) $\text{Tan } A = \dfrac{1}{\cot A}$.

(4.) $\text{Sec A} = \dfrac{1}{\cos \text{A}} = 1 \div \dfrac{\cot \text{A}}{\sqrt{\cot^2 \text{A} + 1}}$, by (2.) above,

$$= \frac{\sqrt{\cot^2 \text{A} + 1}}{\cot \text{A}}.$$

(5.) $\text{Cosec A} = \sqrt{1 + \cot^2 \text{A}}$, by Art. 6 (8.)

And in the same way the trigonometrical ratios may be expressed in terms of any one of them.

Ex. II.

1. Given $\sin \text{A} = \frac{1}{1}\frac{2}{3}$, find the other trigonometrical ratios.

2. Given $\tan \text{A} = \frac{2}{2}\frac{4}{5}$, find the remaining trigonometrical ratios.

3. If $\cot \text{A} = a$, show that $\sin \text{A} = \dfrac{1}{\sqrt{1 + a^2}}$.

4. If $\text{vers A} = b$, then $\tan \text{A} = \dfrac{\sqrt{2b - b^2}}{1 - b}$.

5. Construct by scale and compass an angle (1.) whose cosine is $\frac{3}{7}$; (2.) whose tangent is $\frac{5}{6}$; (3.) whose secant is $\sqrt{2}$; (4.) whose cotangent is $2 + \sqrt{3}$.

Prove the following identities:—

6. $(\text{Sin A} + \cos \text{A})^2 + (\sin \text{A} - \cos \text{A})^2 = 2$.

7. $\text{Sec}^2 \text{A} + \cos^2 \text{B} \cdot \text{cosec}^2 \text{B} = \text{cosec}^2 \text{B} + \sin^2 \text{A} \cdot \sec^2 \text{A}$.

8. $\text{Sec}^2 \text{A} \cdot \text{cosec}^2 \text{A} = \sec^2 \text{A} + \text{cosec}^2 \text{A}$.

9. $\text{Sec A} \cdot \text{cosec A} = \tan \text{A} + \cot \text{A}$.

10. $\text{Sin}^8 \text{A} - \cos^8 \text{A} = (\sin^2 \text{A} - \cos^2 \text{A})(\sin^4 \text{A} + \cos^4 \text{A})$.

11. $\dfrac{\text{Sec A} + \tan \text{A}}{\text{Cosec B} + \cot \text{B}} = \dfrac{\text{cosec B} - \cot \text{B}}{\sec \text{A} - \tan \text{A}}$.

12. $1 + \sin \text{A} \cos \text{A} = \dfrac{\sin^4 \text{A} + \sin^2 \text{A} \cos^2 \text{A} + \cos^4 \text{A}}{1 - \sin \text{A} \cos \text{A}}$.

13. $(x \cos \theta + y \sin \theta)(x \sin \theta + y \cos \theta) - (x \cos \theta - y \sin \theta)(x \sin \theta - y \cos \theta) = 2xy$.

14. $(a \sin\theta \cos\phi + r\cos\theta\cos\phi)(b\sin\theta\sin\phi + r\sin\theta\cos\phi)$
$- (b\sin\theta\cos\phi - r\sin\theta\sin\phi)(a\sin\theta\sin\phi + r\cos\theta\sin\phi)$
$= r\sin\theta(r\cos\theta + a\sin\theta)$.

15. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, show that $x^2 + y^2 + z^2 = r^2$.

16. If $a = b\cos C + c\cos B$, $b = a\cos C + c\cos A$, $c = a\cos B + b\cos A$, show that $a^2 + b^2 - c^2 = 2ab\cos C$.

17. Given $\sin^2 A + 3\sin A = \frac{7}{4}$, find $\sin A$.

18. Given $\cos^2 A - \sin A = \frac{9}{25}$, find $\cos A$.

19. Solve $\sin A - \cos A = \frac{1}{\sqrt{2}}$, for $\sin A$.

20. Find $\tan A$, when $\tan A + 1 = \sqrt{\frac{3}{2}}.\sec A$.

21. Given $a\cos A = b\sin A + a$, find $\cot A$.

22. Given $\tan^3 A - 7\tan A + 6 = 0$, find $\tan A$.

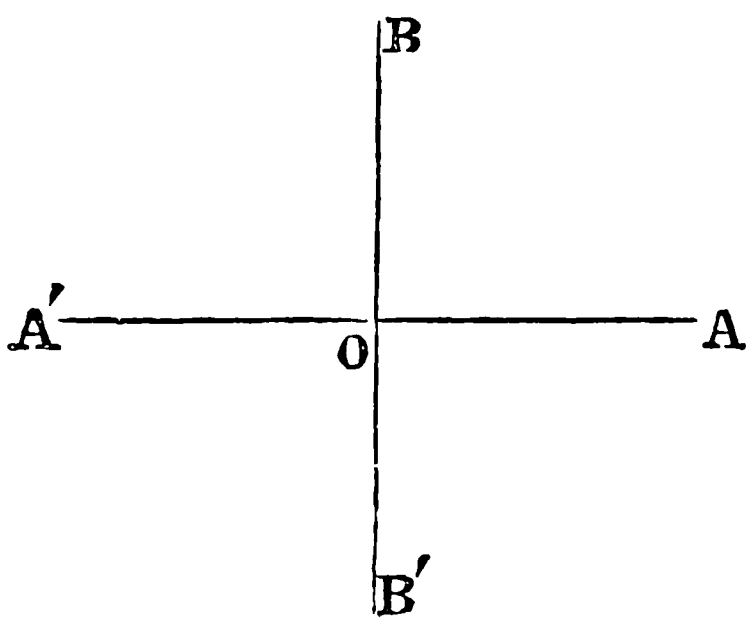
23. Show that $\sqrt{1 + 2\sin A\cos A} + \sqrt{1 - 2\sin A\cos A} = 2\cos A$ or $2\sin A$, according as A is between $0°$ and $45°$, or between $45°$ and $90°$.

24. Given $m\sin^2 A + n\sin^2 B = a\cos^2 A$,
$m\cos^2 A + n\cos^2 B = b\sin^2 A$, find $\sin A$ and $\sin B$.

CHAPTER III.

CONTRARIETY OF SIGNS.—CHANGES OF MAGNITUDE AND SIGN OF THE TRIGONOMETRICAL RATIOS THROUGH THE FOUR QUADRANTS.

9. We have explained at some length the meaning and use of the signs + and − in algebra. They have a similar interpretation in trigonometry.



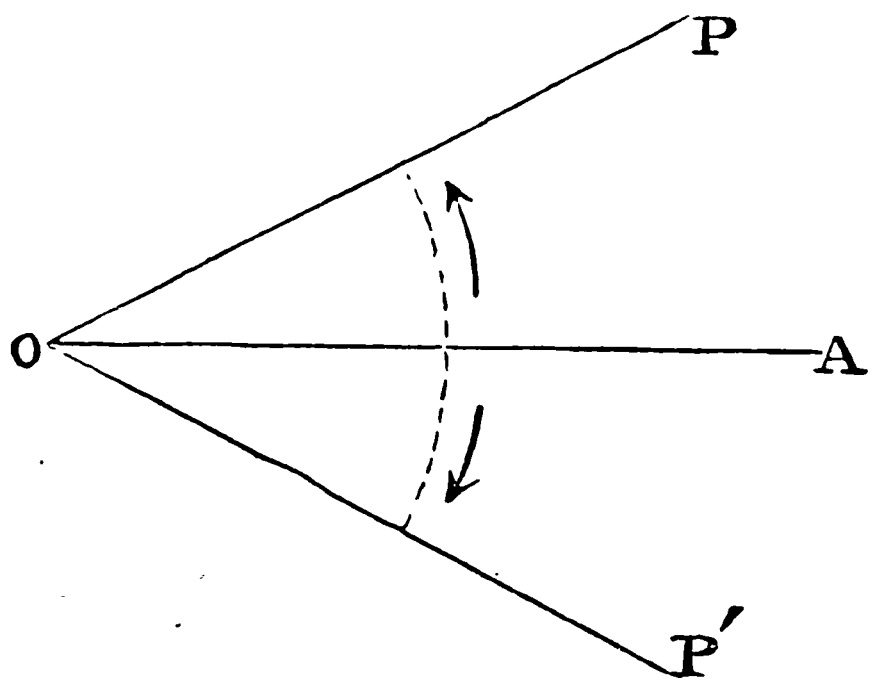
1. *Lines.*—Draw the horizontal line A′A, and draw BB′ at right angles, meeting it in O. Then considering O as origin,

(1.) All lines drawn to the *right* parallel to A′A are called *positive*,

and all lines drawn to the *left* parallel to A′A are called *negative.*

(2.) All lines drawn *upwards* parallel to B′B are called *positive*, and all lines drawn *downwards* parallel to B′B are called *negative.*

(3.) Lines drawn in every other direction are considered positive, as is therefore the revolving line by which angles may be conceived to be generated.



2. *Angles.*—A similar convention is made for angles. Let OA be an initial line, and let a revolving line about the centre O take up the positions OP and OP′. Then—

(1.) That direction of revolution is considered *positive* which is *contrary* to that of the hands of a watch, and the angle generated is a *positive* angle.

(The positive direction is then *upwards.*)

Thus, AOP is a *positive* angle.

(2.) The *negative* direction of revolution is *the same* as that of the hands of a watch, and the angle thus generated is a *negative* angle.

(The negative direction is then *downwards.*)

Thus, AOP′ is a negative angle.

Hence, if the angles AOP and AOP′ be of the same magnitude, and

We have—

$$\angle AOP = A,$$

then $\angle AOP' = -A.$

10. We will now examine the trigonometrical ratios for angles greater than a right angle, and for negative angles.

Let OP_1, OP_2, OP_3, OP_4 represent the position of the revolving line at any period of revolution in the several quadrants respectively,

And let P_1N_1, P_2N_2, P_3N_3, P_4N_4 be the respective perpendiculars from the end of the revolving line upon the initial line.

Then P_1N_1, P_2N_2, P_3N_3, P_4N_4 are respectively the perpendiculars corresponding to the angles generated.

Also, ON_1, ON_2, ON_3, ON_4 are respectively the *bases* of the right-angled triangles with respect to the angles in question.

We have then in the *second* quadrant—

$$\text{Sin } AOP_2 = \frac{P_2N_2}{OP_2}, \cos AOP_2 = \frac{ON_2}{OP_2},$$

$$\tan AOP_2, = \frac{P_2N_2}{ON_2}, \text{ \&c.}$$

It is therefore evident that the relations between the trigonometrical ratios, which were proved to exist in Art. 7, also hold for angles in the second quadrant—that is, angles between 90° and 180°.

And in the same way we may show that they hold for angles in the third, fourth, or any quadrant.

And again, if we suppose the line to revolve in a *negative* direction, and take the position OP', we shall have $P'N'$ the perpendicular corresponding to the negative angle AOP', and ON' the base.

$$\text{Hence, sin } AOP' = \frac{P'N'}{OP'}, \cos AOP' = \frac{ON'}{OP'},$$

$$\tan AOP' = \frac{P'N'}{ON'}, \text{ \&c.}$$

And the relations proved in Art. 7 may be also similarly proved to exist here.

Hence the relations proved in Art. 7 hold for any angles whatever.

Changes of Magnitude and Sign of the Trigonometrical Ratios.

11. Let OP_1, OP_2, OP_3, OP_4 be positions of the revolving line in the several quadrants respectively; P_1N_1, P_2N_2,

P_3N_3, P_4N_4, the respective perpendiculars; and ON_1, ON_2, ON_3, ON_4, the *bases* of the corresponding right-angled triangles.

Then—

(1.) *In the first quadrant*—

$$\text{Sin } AOP_1 = \frac{P_1N_1}{OP_1}, \cos AOP_1 = \frac{ON_1}{OP_1},$$

$$\tan AOP_1 = \frac{P_1N_1}{OP_1}, \text{ \&c.}$$

At the *commencement* of the motion of the revolving line, the angle $AOP_1 = 0°$;

Also, the perpendicular $P_1N_1 = 0$,

And the base $ON = OP_1$.

Hence, we have—

$$^{*}\text{Sin } 0° = \frac{0}{OP_1} = 0, \cos\ 0° = \frac{OP_1}{OP_1} = 1,$$

$$\tan\ 0° = \frac{0}{OP_1}, = 0.$$

As the revolving line moves from OA towards OB, P_1N_1 increases and ON_1 diminishes; and when it arrives at OB, we have $P_1N_1 = OP_1$, and $ON_1 = 0$. But the angle generated is now a right angle. Hence we have—

$$\text{Sin } 90° = \frac{OP_1}{OP_1} = 1, \cos 90° = \frac{0}{OP_1} = 1,$$

$$\tan 90° = \frac{OP_1}{0} = \infty.$$

Hence, as the angle increases from 0° to 90°—

The *sine* changes in magnitude from 0 to 1 and is +.

The *cosine* changes in magnitude from 1 to 0 and is +.

The *tangent* changes in magnitude from 0 to ∞ and is +.

(2.) *In the second quadrant*—

Here the *perpendicular* P_2N_2 is +,

and the *base* ON_2 is −.

* The student ought properly to look upon the values 0, 1, 0 here obtained as the limiting values of the *sine*, *cosine*, and *tangent* respectively, when the angle is indefinitely diminished.

Hence the *sine* during the second quadrant is +, the *cosine* is −, and the *tangent* is −.

Again, as the revolving line moves from OB to OA′, the perpendicular P_2N_2 diminishes until it becomes zero. Also, the base ON_2 increases in magnitude, until it finally coincides with OA′, and ∴ = − OP_2. But the angle now described is 180°.

Hence we have—

$$\text{Sin } 180° = \frac{0}{OP_2} = 0, \cos 180° = -\frac{OP_2}{OP_2} = -1,$$

$$\tan 180° = -\frac{0}{OP_2} = 0, \text{ \&c.}$$

Hence in the second quadrant—
The *sine* changes in magnitude from 1 to 0, and is *positive.*
The *cosine* changes in magnitude from 0 to 1, and is *negative.*
The *tangent* changes in magnitude from ∞ to 0, and is *negative.*

And in the same way may we trace the changes of magnitude and sign in the third and fourth quadrants.

Thus we shall find—

(3.) *In the third quadrant*—
The *sine* changes in magnitude from 0 to 1, and is *negative.*
The *cosine* changes in magnitude from 1 to 0, and is *negative.*
The *tangent* changes in magnitude from 0 to ∞, and is *positive.*

(4.) *In the fourth quadrant*—
The *sine* changes in magnitude from 1 to 0, and is *negative.*
The *cosine* changes in magnitude from 0 to 1, and is *positive.*
The *tangent* changes in magnitude from ∞ to 0, and is *negative.*

Moreover, as the cosecant, secant, and cotangent are respectively the reciprocals of the *sine*, *cosine*, and *tangent*, it follows that their *signs* will follow respectively the latter, and that their magnitudes will be their reciprocals.

CHAPTER IV.

TRIGONOMETRICAL RATIOS CONTINUED. ARITHMETICAL VALUES OF THE TRIGONOMETRICAL RATIOS OF 30°, 45°, 60°, &c.

12. *To prove that* $\sin A = \cos(90° - A)$, and that $\cos A = \sin(90° - A)$.

Using the same figure as in Art. 5, we have—

$$\text{Sin A} = \frac{\text{PM}}{\text{AP}} = \cos \text{APM}.$$

But $\angle$ APM = 90° − A,

$\therefore$ Sin A = cos (90° − A),

and similarly—

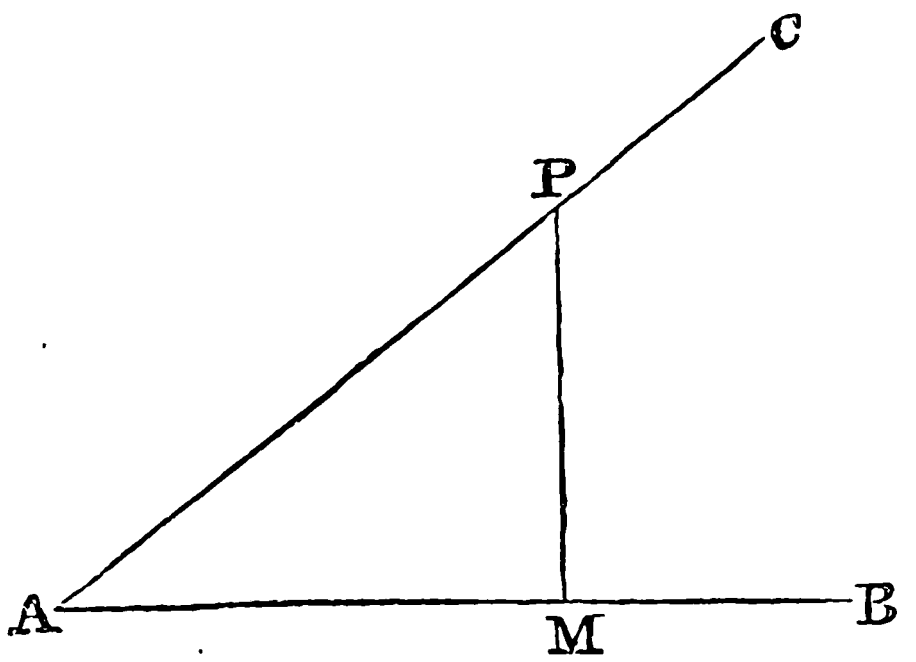
cos A = sin (90° − A),

tan A = cot (90° − A),

cot A = tan (90° − A),

sec A = cosec (90° − A),

cosec A = sec (90° − A).



13. *Ratios of* 45°.

In the last figure, suppose $\angle$ PAM = 45°, then also $\angle$ APM = 90° − 45° = 45°. And hence $\angle$ PAM = $\angle$ APM, and $\therefore$ PM = AM (Euc. I., 6).

Hence, also—

$$\text{AP or } \sqrt{\text{AM}^2 + \text{PM}^2} = \sqrt{2\,\text{AM}^2} \text{ or } \sqrt{2\,\text{PM}^2}.$$

$$\therefore \text{AP} = \text{AM}\sqrt{2} \text{ or } \text{PM}\sqrt{2}.$$

Hence we have—

$$\text{Sin } 45° = \sin \text{PAM} = \frac{\text{PM}}{\text{AP}} = \frac{\text{PM}}{\text{PM}\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45°, \text{ by Art. 8.}$$

$$\text{Tan } 45° = \tan \text{PAM} = \frac{\text{PM}}{\text{AM}} = \frac{\text{PM}}{\text{PM}} = 1 = \cot 45°, \text{ by Art. 8.}$$

$$\text{Sec } 45° = \sec \text{PAM} = \frac{\text{AP}}{\text{AM}} = \frac{\text{AM}\sqrt{2}}{\text{AM}} = \sqrt{2} = \text{cosec } 45°,$$

by Art. 8.

14. *Ratios of* 30° and 60°.

In the same diagram, suppose $\angle$ PAM = 30°, then $\angle$ APM = 90° − 30° = 60°.

Hence, if we conceive another triangle equal in every respect to APM to be described on the other side of AM, the whole would form an equilateral triangle whose side is AP.

Hence, PM = $\frac{1}{2}$ AP.

Now $AM = \sqrt{AP^2 - PM^2}$, $\therefore AM = \sqrt{AP^2 - (\frac{1}{2}AP)^2}$

$= \frac{\sqrt{3}}{2}AP.$

We hence have—

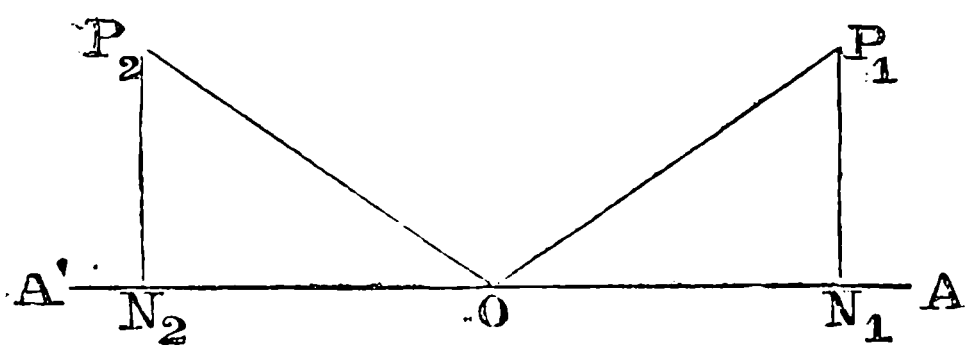
$\text{Sin } 30° = \sin PAM = \frac{PM}{AP} = \frac{\frac{1}{2}AP}{AP} = \frac{1}{2} = \cos 60°$, by Art. 8.

$\text{Cos } 30° = \cos PAM = \frac{AM}{AP} = \frac{\frac{\sqrt{3}}{2}AP}{AP} = \frac{\sqrt{3}}{2} = \sin 60°$, by Art. 8.

$\text{Tan } 30° = \frac{PM}{AM} = \frac{\frac{1}{2}AP}{\frac{\sqrt{3}}{2}AP} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cot 60°$, by Art. 8.

15. *To show that* $\sin (180° - A) = \sin A$,

$\cos (180° - A) = -\cos A$,



Let $\angle AOP_1 = A$,

And let the revolving line describe an angle $AOP_2 = 180° - A$;

Then $\angle A'OP_2 = 180° - (180° - A) = A$;

Hence, $\angle AOP_1 = A'OP_2$.

Hence, also (Euc. I., 26), if P_1N_1, P_2N_2 be drawn perpendicular to AA', $P_1N_1 = P_2N_2$, $ON_2 = -ON_1$,

We have therefore—

$\text{Sin } (180° - A) = \sin AOP_2 = \frac{P_2N_2}{OP_2} = \frac{P_1N_1}{OP_1} = \sin A.$

$\text{Cos } (180° - A) = \cos AOP_2 = \frac{ON_2}{OP_2} = -\frac{ON_1}{OP_1} = -\cos A.$

And similarly—

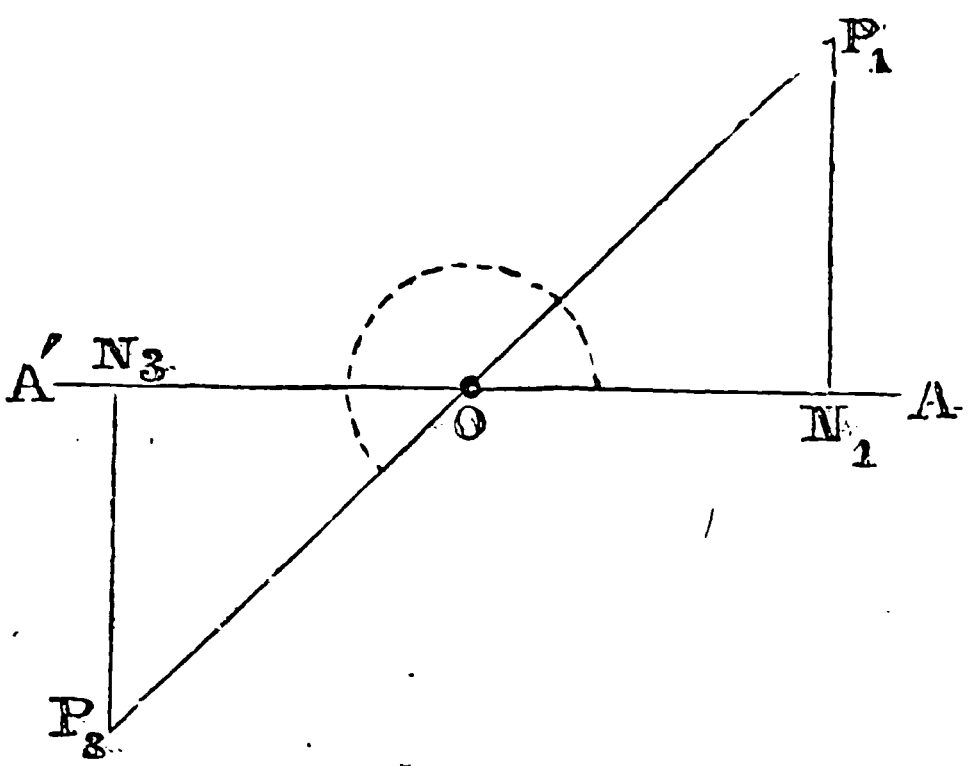
$\text{Tan } (180° - A) = -\tan A$, $\cot (180° - A) = -\cot A.$

$\text{Sec } (180° - A) = -\sec A$, $\text{cosec } (180° - A) = \text{cosec } A.$

16. *To show that* $\sin (180° + A) = - \sin A$,
$\cos (180° + A) = - \cos A$,

Let $AOP_1 = A$,

And let the revolving line take a position such that P_1P_3 is a straight line.



Then, evidently, $\angle AOP_3 = 180° + A$.

Then, as in last Article—

$P_3N_3 = - P_1N_1$,

$ON_3 = - ON_1$.

Hence—

$$\text{Sin } (180° + A) = \sin AOP_3 = \frac{P_3N_3}{OP_3} = -\frac{P_1N_1}{OP_1} = - \sin A.$$

$$\text{Cos } (180° + A) = \cos AOP_3 = \frac{ON_3}{OP_3} = -\frac{ON_1}{OP_1} = - \cos A.$$

And similarly—

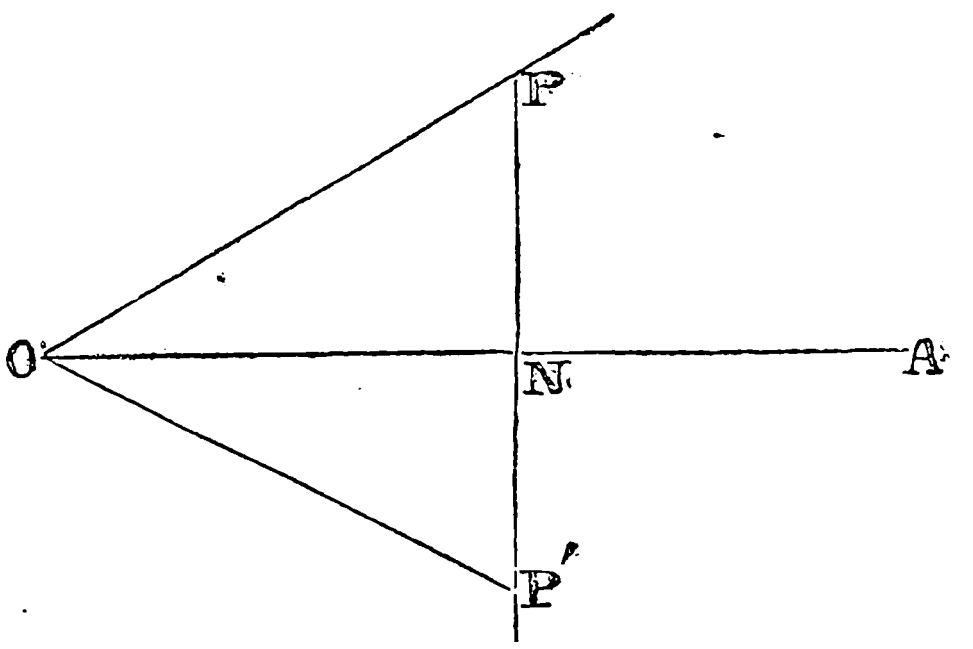
$\text{Tan } (180° + A) = \tan A$, $\cot (180° + A) = \cot A$.

$\text{Sec } (180° + A) = - \sec A$, $\text{cosec } (180° + A) = - \text{cosec } A$.

17. *To show that* $\sin (- A) = - \sin A$, and
$\cos (- A) = \cos A$,

Let $\angle AOP = A$,

And let the revolving line describe an angle $AOP' = - A$.



Then evidently, if PNP' be drawn perpendicular to OA, we have (Euc. I., 26) $P'N = - PN$.

Hence—

$$\text{Sin } (- A) = \sin AOP' = \frac{P'N}{OP'} = -\frac{PN}{OP} = - \sin A,$$

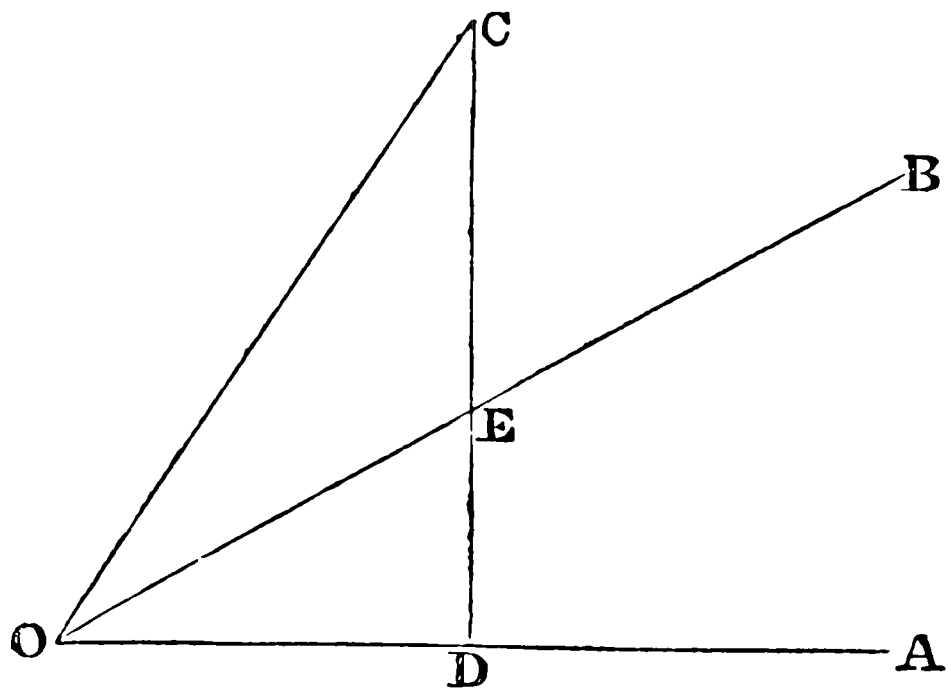
$$\text{Cos } (- A) = \cos AOP' = \frac{ON}{OP'} = \frac{ON}{OP} = \cos A.$$

And similarly—

$\text{Tan } (- A) = - \tan A$, $\cot (- A) = - \cot A$,

$\text{Sec } (- A) = \sec A$, $\text{cosec } (- A) = - \text{cosec } A$.

Although the results of Arts. 12, 15, 16, 17 have been proved from diagrams where A is less than a right angle, the student will have no difficulty, if he has understood the proofs, in deducing the same results for angles of any magnitude whatever.



18. *To show that* $\tan \frac{A}{2} = \frac{1 - \cos A}{\sin A}$,

Let $\angle AOC = A$;

Bisect it by the straight line OB, so that $\angle AOB = \frac{A}{2}$, and draw CD perpendicular to OA, meeting OB in E.

Then $\tan \frac{A}{2} = \tan EOD = \frac{ED}{OD}$.....(1).

Now (Euc. VI., 2), $\frac{OD}{OC} = \frac{ED}{EC}$, and $\therefore \frac{OD}{OC + OD} = \frac{ED}{CD}$, or

$$\frac{ED}{OD} = \frac{CD}{OC + OD} = \frac{CD(OC - OD)}{OC^2 - OD^2} = \frac{CD(OC - OD)}{CD^2}$$

$$= \frac{OC - OD}{CD} = \frac{OC - OC \cos A}{OC \sin A} = \frac{1 - \cos A}{\sin A}. \quad Q.E.D.$$

Cor 1. Hence, squaring—

$$\text{Tan}^2 \frac{A}{2} = \frac{(1 - \cos A)^2}{\sin^2 A} = \frac{(1 - \cos A)^2}{1 - \cos^2 A} = \frac{1 - \cos A}{1 + \cos A}.$$

$\therefore$ Art. 64, page 215, $\frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{\cos A}{1}$.

$$\therefore \text{Cos } A = \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} \text{.....(1).}$$

$$= \frac{1 - \dfrac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}}{1 + \dfrac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}} = \frac{\cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2} + \sin^2 \frac{A}{2}}$$

$$= \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

$$= \cos^2 \frac{A}{2} - \left(1 - \cos^2 \frac{A}{2}\right)$$

$$= 2 \cos^2 \frac{A}{2} - 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

$$= 2\left(1 - \sin^2 \frac{A}{2}\right) - 1 = 1 - 2 \sin^2 \frac{A}{2} \quad (4.)$$

19. *To find the trigonometrical ratios of* 15°, 75°, 120°, 135°, 150°.

(1.) *Ratios of* 120°.

$$\text{Sin } 120° = \sin (180° - 120°) = \sin 60° = \frac{\sqrt{3}}{2},$$

$$\text{Cos } 120° = -\cos (180° - 120°) = -\cos 60° = -\frac{1}{2},$$

$$\text{Tan } 120° = -\tan (180° - 120°) = -\tan 60° = -\sqrt{3}, \text{ \&c.}$$

(2.) *Ratios of* 150°.

$$\text{Sin } 150° = \sin (180° - 150°) = \sin 30° = \frac{1}{2},$$

$$\text{Cos } 150° = -\cos (180° - 150°) = -\cos 30° = -\frac{\sqrt{3}}{2},$$

$$\text{Tan } 150° = -\tan (180° - 150°) = -\tan 30 = -\frac{1}{\sqrt{3}},$$

&c.

(3.) *Ratios of* 15°.

By last Art., $\tan\frac{A}{2} = \frac{1 - \cos A}{\sin A}$; put $A = 30°$, or $\frac{A}{2} = 15°$,

$$\text{then } \tan 15° = \frac{1 - \cos 30°}{\sin 30°} = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

From this result we easily get, Art. 8,

$$\text{Sin } 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}}, \cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}}, \text{ \&c.}$$

(4.) *Ratios of* 75°.

We have, $\sin 75° = \sin(90° - 15°) = \cos 15° = \frac{\sqrt{3} + 1}{2\sqrt{2}}$,

$$\cos 75° = \sin(90° - 15°) = \sin 15° = \frac{\sqrt{3} - 1}{2\sqrt{2}},$$

$$\tan 75° = \tan(90° - 15°) = \cot 15° = \frac{1}{2 - \sqrt{3}}$$

$$= 2 + \sqrt{3}, \text{ \&c.}$$

(5.) *Ratios of* 135°.

We have, $\sin 135° = \sin(180° - 135°) = \sin 45° = \frac{1}{\sqrt{2}}$,

$$\cos 135° = -\cos(180° - 135°) = -\cos 45°$$

$$= -\frac{1}{\sqrt{2}},$$

$$\tan 135° = -\tan(180° - 135°) = -\tan 45°$$

$$= -1, \text{ \&c.}$$

Ex. III.

1. Define a negative angle, and show that $\tan(-A) = -\tan A$, when A lies between $-90°$ and $-180°$.

2. Trace the changes of sign of $\sin A \cdot \cos A$ through the four quadrants.

3. Trace the changes of sign of $\cos A + \sin A$, and of $\cos A - \sin A$, as A changes from $-45°$ to $315°$.

4. Assuming generally that $\cos 2A = \cos^2 A - \sin^2 A$, trace the changes of sign of $\cos 2A$ as A changes from $-45°$ to $315°$.

5. Write down the sines of $210°$, $165°$, $240°$, $-120°$.

6. Show that $\sin (90° + A) = \cos A$, and $\cos (90° + A) = -\sin A$, for any value of A from $0°$ to $180°$.

7. Assuming generally that $2\cos^2 \frac{A}{2} = 1 + \cos A$, and $2\sin^2 \frac{A}{2} = 1 - \cos A$, show that $\sqrt{2}\cos \frac{A}{2} = -\sqrt{1 + \cos A}$ and $\sqrt{2}\sin \frac{A}{2} = -\sqrt{1 - \cos A}$, when A lies between $360°$ and $540°$.

8. Given $\cos A = 1 - 2\sin^2 \frac{A}{2}$, show that $\sin A = 2\sin \frac{A}{2}\cos \frac{A}{2}$.

9. Hence show that $2\cos \frac{A}{2} = -\sqrt{1 + \sin A}$ $-\sqrt{1 - \sin A}$, when $\frac{A}{2}$ lies between $135°$ and $225°$.

Solve the following equations :—

10. $\text{Cos}^2 A + \frac{3}{8}\cos A = 1\frac{7}{16}$.

11. $\text{Tan}\,\theta + 5\cot\theta = 6$.

12. $\text{Sin}\,A + \sec A = \frac{2}{\sqrt{3}} + \frac{1}{2}$.

13. $2\cos^2 A = 3\sin A$.

14. $\text{Sin}\,(A + B) = \cos (A - B) = \frac{\sqrt{3}}{2}$.

15. $\text{Tan}^2 A = 2\sin^2 A$.

16. $\text{Sin}\,(3A + 75°) = \cos (2A - 15°)$.

17. $\text{Sec}\,\theta + \cos\theta = \frac{5}{2\sqrt{3}}\tan\theta$.

18. $\text{Tan}\,\theta + \cot\theta = 4$.

CHAPTER V.

LOGARITHMS.

20. DEF.—The **logarithm** of a number to a given base is the index of the power to which the base must be raised to obtain the number.

Thus, we may obtain the numbers 1, 10, 100, 1,000, 10,000, &c., by raising the base 10 to the powers 0, 1, 2, 3, 4, &c., respectively; and hence, by the above definition, we have—

$$\operatorname{Log}_{10} 1 = 0,\ \log_{10} 10 = 1,\ \log_{10} 100 = 2,\ \log_{10} 1{,}000 = 3,\ \&c.,$$

the suffix 10 being added to the word *log* to indicate that the base is 10. It is usual, however, in common logarithms to omit this suffix; and hence, when there is no base expressed, the student will understand 10.

Again, the numbers 1, 2, 4, 8, 16, &c., may be obtained by finding the values of 2^0, 2^1, 2^2, 2^3, 2^4, &c., respectively, and hence we have by definition—

$$\operatorname{Log}_{2} 1 = 0,\ \log_{2} 2 = 1,\ \log_{2} 4 = 2,\ \log_{2} 8 = 3,\ \&c.$$

So also we find $\log_{4} 16 = 2$, $\log_{5} 125 = 3$, $\log_{3} 81 = 4$, &c.

Ex. Find $\log_{4} 256$, $\log_{36} 216$, and the logarithm of 9 to base $\sqrt{3}$.

$\operatorname{Log}_{4} 256 = \log_{4} 4^4 = 4$, by definition.

$\operatorname{Log}_{36} 216 = \log_{36} 6^3 = \log_{36} (6^2)^{\frac{3}{2}} = \log_{36} 36^{\frac{3}{2}} = \frac{3}{2}$, by definition.

$\operatorname{Log}_{\sqrt{3}} 9 = \log_{\sqrt{3}} 3^2 = \log_{\sqrt{3}} (\sqrt{3})^4 = 4$, by definition.

Characteristics of Ordinary Logarithms.

21. DEF.—The **characteristic** of a logarithm is the integral part of the logarithm, and the fractional part (generally expressed as a decimal) is called the **mantissa.**

1. *Numbers containing integer digits.*

Every number containing n digits in its integral part must lie between 10^{n-1} and 10^n.

Thus, 6 lies between 10^0 and 10^1, 29 lies between 10^1 and 10^2, 839 lies between 10^2 and 10^3, &c.

Hence the ordinary logarithms of all numbers having n integer digits lies between $(n - 1)$ and n.

The integral portion or characteristic of the logarithm of a number having n integer digits is therefore $(n - 1)$.

Hence we have the following rule :—

RULE 1.—*The characteristic of the logarithm of a number having integer digits is* ONE *less than the number of integer digits.*

Thus, the characteristics of the logarithms of 32, 713·54, 8·7168, 56452, 73607·9 are respectively 1, 2, 0, 4, 4.

2. *Numbers less than unity expressed as decimals.*

All such numbers having n zeros immediately after the decimal point lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, or between 10^{-n} and $10^{-(n+1)}$.

Thus, ·3 lies between 1 and ·1, or between 1 and $\frac{1}{10}$, or 10^0 and 10^{-1};

·027 lies between ·1 and ·01, or between $\frac{1}{10}$ and $\frac{1}{10^2}$, or 10^{-1} and 10^{-2};

·000354 lies between ·001 and ·0001, or between $\frac{1}{10^3}$ and $\frac{1}{10^4}$, or 10^{-3} and 10^{-4}, and so on.

Hence, by Def., Art. 20, the logarithm of any number having n zeros immediately after the decimal point lies between $-n$ and $-(n + 1)$. Hence, the logarithm is *negative*, and the integral part of this negative quantity is n. It is however usual to write all the mantissæ of logarithms as *positive* quantities, and the *negative integral* part of the logarithm will be the next higher negative integer, viz., $-(n + 1)$.

We have therefore the following rule :—

RULE 2.—*The characteristic of the logarithm of a number less than unity, and expressed as a decimal, is the negative*

integer next greater than the number of zeros immediately after the decimal point.

Thus, the characteristics of the logarithms of ·3, ·0076, ·02535, ·7687, are respectively – 1, – 3, – 2, – 1.

22. *The logarithm of the* PRODUCT *of two numbers is the* SUM *of the logarithms of the numbers.*

Let m and n be the numbers, and let a be the base. Since m and n must be each some power of a, integral or fractional, positive or negative, assume—

$$\left.\begin{aligned} m &= a^x, \\ n &= a^y. \end{aligned}\right\}$$ Then, by definition of a logarithm,

$$x = \log_a m, \text{ and } y = \log_a n.$$

Now we have $mn = a^x \cdot a^y = a^{x+y}$, and hence, by definition,

$$\log_a mn = x + y;$$ we therefore have—

$$\log_a (mn) = \log_a m + \log_a n. \quad Q.E.D.$$

COR. This proposition may be extended to any number of factors.

Thus, $\log_a (mnpq) = \log_a m + \log_a n + \log_a p + \log_a q.$

23. *The logarithm of the* QUOTIENT *of two numbers is found by* SUBTRACTING *the logarithm of the denominator from the logarithm of the numerator.*

Assuming, as in the last Art., we have—

$$x = \log_a m, \; y = \log_a n.$$

Also, $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$, and hence, by definition,

$$\log_a \frac{m}{n} = x - y;$$ we therefore have—

$$\log_a \frac{m}{n} = \log_a m - \log_a n. \quad Q.E.D.$$

24. *The logarithm of the* POWER *of a number is found by* MULTIPLYING *the logarithm of the number by the* INDEX *of the power.*

Let it be required to find $\log_a N^p$.

Assume $N = a^x$, and therefore $x = \log_a N$.

We have $N^p = (a^x)^p = a^{px}$, and hence, by definition,

$$\log_a N^p = px = p \log_a N. \quad Q.E.D.$$

Ex. 1.—Given log 2 = ·3010300, and log 3 = ·4771213, find the logarithms of 18, 15, ·125, 6·75.

Log 18 = log (2×3^2) = log 2 + 2 log 3 = ·3010300 + 2(·4771213) = 1·2552726.

Log 15 = log $(3 \times \frac{10}{2})$ = log 3 + log 10 − log 2 = ·4771213 + 1 − ·3010300 = 1·1760913.

Log ·125 = $\log\left(\frac{1}{2^3}\right)$ = log 1 − 3 log 2 = 0 − 3 × ·3010300 = − ·9030900 = − 1 + (1 − ·9030900) = − 1 + ·0969100, or, as usually written, = $\bar{1}$·0969100.

Log 6·75 = $\log \frac{27}{4} = \log \frac{3^3}{2^2}$ = 3 log 3 − 2 log 2 = 3(·4771213) − 2(·3010300) = ·8293039.

Ex. 2. Find the logarithm of $\frac{(2{\cdot}4)^{\frac{1}{2}} \times (\cdot 375)^4}{(2{\cdot}43)^5 \times (\cdot 032)^{-\frac{1}{3}}}$, having given log 2 and log 3.

We have—

$$\text{Log N} = \tfrac{1}{2} \log 2{\cdot}4 + 4 \log \cdot 375 - 5 \log 2{\cdot}43 - (-\tfrac{1}{3}) \log \cdot 032.$$

$$= \frac{1}{2} \log \frac{2^3 \times 3}{10} + 4 \log \frac{3}{2^3} - 5 \log \frac{3^5}{10^2} + \frac{1}{3} \log \frac{2^5}{10^3}$$

$$= \tfrac{1}{2}(3 \log 2 + \log 3 - \log 10) + 4(\log 3 - 3 \log 2) - 5(5 \log 3 - 2 \log 10) + \tfrac{1}{3}(5 \log 2 - 3 \log 10)$$

$$= (\tfrac{3}{2} - 12 + \tfrac{5}{3}) \log 2 + (\tfrac{1}{2} + 4 - 25) \log 3 + (-\tfrac{1}{2} + 10 - 1) \log 10$$

$$= -\tfrac{53}{6} \times \cdot 3010300 - \tfrac{41}{2} \times \cdot 4771213 + \tfrac{17}{2} \times 1$$

$$= - 2{\cdot}6590983 - 9{\cdot}7809867 + 8{\cdot}5 = - 3{\cdot}9400850$$

$$= \bar{4} + (4 - 3{\cdot}9400850) = \bar{4}{\cdot}0599150.$$

Ex. IV.

1. Find the logarithm to base 4 of the following numbers: 16, 64, 2, ·25, ·0625, 8.

2. Find the value of $\log_{8} 32$, $\log_{\sqrt[3]{5}} 25$, $\log_{\cdot 81} \cdot 729$.

3. Given log 2 = ·3010300, and log 3 = ·4771213, find the logarithms of 12, 36, 45, 75, ·04, 3·75, $\cdot\dot{6}$, $\cdot\dot{0}7\dot{4}$.

4. Given log 20763 = 4·3172901, what is the logarithm of 2·0763, 2076·3, ·020763, ·0020763?

5. Write down the characteristics of the common logarithms of 29·6, ·25402, ·0034, 6176·003.

6. Given log 20·912 = 1·3203956, what numbers correspond to the following logarithms:—$\bar{2}$·3203956, 6·3203956, $\bar{1}$·3203956, 4·3203956?

7. Given log 20·713 = 1·3162430, and log 20714 = 3·3162640, find log ·2071457.

8. Given log 3·4937 = ·5432856, and log 3·4938 = ·5432980, find the number whose logarithm is 3·5432930.

9. Given log 1·05 = ·0211893, log 2·7 = 1·4313638, log 135 = 2·1303338, find the value of $\log \dfrac{(2\cdot7)^{\frac{1}{3}} \times \sqrt[5]{13\cdot5}}{(1\cdot05)^{6}}$.

10. Given log 18 = 1·2552725, and log 2·4 = ·3802112, find the value of log ·00135.

11. What are the characteristics of $\log_{3} 1167$, and $\log_{4} 1965$?

12. Having log 2 = ·3010300, and log 3 = ·4771213, find x when $18^{x} = 125$.

CHAPTER VI.

THE USE OF TABLES.

25. Tables have been formed of the logarithms of all numbers from 1 to 100,000, and we shall now show how they are

practically used. We shall not enter here upon the method of forming the tables themselves.

The following is a specimen of the way in which the logarithms of numbers are usually tabulated:—

No.	0	1	2	3	4	5	6	7	8	9	D.
7990	9025468	5522	5577	5631	5685	5740	5794	5848	5903	5957	
91	6011	6066	6120	6174	6229	6283	6337	6392	6446	6500	
92	6555	6609	6663	6718	6772	6826	6881	6935	6989	7044	
93	7098	7152	7207	7261	7315	7370	7424	7478	7533	7587	
94	7641	7696′	7750	7804	7859	7913	7967	8022	8076	8130	54
95	8185	8239	8293	8348	8402	8456	8511	8565	8619	8674	
96	8728	8782	8836	8891	8945	8999	9054	9108	9162	9217	
97	9271	9325	9380	9434	9488	9542	9597	9651	9705	9760	
98	9814	9868	9923	9977	$\bar{0}$031	0085	0140	0194	0248	0303	
99	9030357	0411	0466	0520	0574	0628	0683	0737	0791	0846	
8000	0900	0954	1008	1063	1117	1171	1226	1280	1334	1388	
D. {54	P. {	5	11	16	22	27	32	38	43	49	

Thus, if the number consist of four figures only, we have simply to copy out the figures in the column headed 0, prefix a decimal point, and the proper characteristic.

Ex. Log 7991 = 3·9026011, log 7·995 = ·9028185.

When we speak of a number consisting of four figures only, we include such numbers as ·003654, ·07682, &c., the number of zeros immediately following the decimal points not being counted.

Thus, log ·07997 = $\bar{2}$·9029271

log ·007992 = 3·9026555.

When the number contains five figures, as, for instance, 79936, we look along the line containing the first four figures —viz., 7993—of the number until the eye rests upon the column headed 6, the fifth figure. We then take the first three figures of the column headed 0, and affix the four figures of the column headed 6 in the horizontal line of the first four figures of the number.

Thus, log 79936 = 4·9027424

log ·079927 = $\bar{2}$·9026935.

It will be seen from the portion of the logarithmic table above extracted, that when the first three figures of the logarithm—viz., 902—have been once printed, they are not

repeated, but must be understood to belong to every four figures in each column, until they are superseded by higher figures, as 903. When, however, this change is intended to be made at any place not at the commencement of a horizontal row, the first of the four figures corresponding to the change is usually printed either in different type, or, as above, with a bar over it. Thus we have above $\bar{0}031$, indicating that from this point we must prefix 903 instead of 902.

Thus, log 79·986 = 1·9030140,
log ·0079987 = $\bar{3}$·9030194.

26. *To find the logarithm of a number not contained in the tables.*

Ex. Find the logarithm of 799·1635.

Since* the *mantissa* of the number 79916·35 is the same as the *mantissa* of the given number, and that the first five figures are contained in the tables, we may proceed as follows—

(1.) Take out from the tables the mantissa corresponding to the number 79916. This is ·9026337.

(2.) Take out the mantissa of the *next higher* number in the tables—viz., 79917. This is ·9026392.

(3.) Find the difference between these mantissæ. This is called the tabular difference, being the difference of the mantissæ for a difference of *unity* in the numbers. We find tab. diff. = ·0000055, which we call D.

(4.) Then *assuming* that small differences in numbers are proportional to the differences of the corresponding logarithms, we find the difference for ·35 = ·35 × ·0000055 = ·0000019, retaining only 7 figures. This is often called *d*.

(5.) Now adding this value of *d* to the mantissa for the number 79916, we get the mantissa corresponding to the number 79916·35.

(6.) Lastly, prefix to this mantissa the proper characteristic.

The whole operation may stand thus—

M. of log 79916 = ·9026337 ..... (1).
M. of log 79917 = ·9026392
Tabular difference or D = ·0000055

* Thus log 79916·35 = log (100 × 799·1635) = 2 + log 799·1635.

Hence, difference for ·35 or d
$= ·35 \times ·0000055 = ·0000019$ .....(2).

Hence, adding (1.) and (2.)—
M. of log 79916·35 = ·9026356.
∴ log 799·1635 = 2·9026356.

or better thus, omitting the useless ciphers—

M. of log 79916	=	·9026337
M. of log 79917	=	·9026392
∴ D	=	55
Hence, d = ·35 × 55	=	19

∴ M. of log 79916·35 = ·9026356, as before.

In the next article we shall show how the required difference may be obtained by inspection from the tables.

27. *Proportional parts.*

We saw in the example just worked that the tab. diff. (omitting the useless ciphers) is 55, and if we examine the table in Art. 25, we shall find the difference between the mantissæ of any two consecutive numbers there to be 54 or 55—generally 54. The number 54 is therefore placed in a separate column at the right of the table, and headed D.

The student will understand that the tab. diff. changes from time to time, and is not *always* 54 or 55.

Now assuming as in (4.) of the last article, we have—

Diff. for ·1	= 54 × ·1 = 5	Diff. for ·6	= 54 × ·6 = 32
,, ·2	= 54 × ·2 = 11	,, ·7	= 54 × ·7 = 38
,, ·3	= 54 × ·3 = 16	,, ·8	= 54 × ·8 = 43
,, ·4	= 54 × ·4 = 22	,, ·9	= 54 × ·9 = 49
,, ·5	= 54 × ·5 = 27		

We find therefore the numbers 5, 11, 16, 22, 27, 32, 38, 43, 49 placed in a horizontal row at the bottom marked P, in the columns respectively headed 1, 2, 3, 4, &c.

Hence, if we require the difference for (say) ·7, we take out the number 38 from the horizontal row marked P, instead of being at the trouble to find it by actual computation.

The following example will illustrate how we proceed when we require the difference for a decimal containing more than one decimal figure. No explanation is needed.

Ex. Find log 7994·3726—

M. of log 79943	=	·9027804	
Diff. for 7	=	38	
,, 2	=	1	1
,, 6	=		32
∴ M. of log 79943726	=	·9027843	

Hence, log 7994·3726 = 3·9027843.

28. *Having given the logarithm of a number to find the number.*

After the explanations of Art. 26, the method of working the following examples will be easily understood :—

Ex. 1. Find the number whose logarithm is $\bar{1}$·9030173. Taking from the tables the mantissæ next above and below, we have—

·9030194	=	M. of log 79987
·9030140	=	M. of log 79986 .....(1).
∴ 54	=	D.
Again, ·9030173	=	M. of log N.....(2).

Hence, subtracting (1) from (2)—

33 = d, the difference between the logarithms of the required number and the next lower.

Now $\frac{33}{54}$ = ·61, the difference between the next lower number and the required number.

Hence ·9030173 = M. of log 79986·61 ;

∴ $\bar{1}$·9030173 = log ·7998661 ;

∴ ·7998661 is the number required.

Ex. 2.* Find the value of $\frac{(1{\cdot}023)^3 \times (\cdot00123)^{\frac{1}{4}}}{(1{\cdot}32756)^4}$.

We have—

$\log N = 3 \log 1{\cdot}023 + \frac{1}{4} \log \cdot00123 - 4 \log 1{\cdot}32756.$

Now, 3 log 1·023	= 3 × ·0098756		= ·0276268
$\frac{1}{4}$ log ·00123	= $\frac{1}{4}$ ($\bar{3}$·0899051)		
	= $\frac{1}{4}$ ($\bar{4}$ + 1·0899051)		= $\bar{1}$·2724763

* The logarithms used in this example are taken from the tables.

$$\begin{aligned}
\therefore \text{ adding, } 3 \log. 1{\cdot}023 + \tfrac{1}{4} \log {\cdot}00123 &= \overline{1}{\cdot}3001031 \\
\text{Again, M. of log } 13275 &= {\cdot}1230345 \\
\text{and diff. for } {\cdot}6 &= \underline{\quad 196} \\
\therefore \text{ M. of log } 13275{\cdot}6 &= {\cdot}1230541 \\
\therefore 4 \log 1{\cdot}32756 = 4 \times {\cdot}1230541 &= {\cdot}4922164 \\
\text{Then, subtracting, log N} &= \overline{2}{\cdot}5078867
\end{aligned}$$

$$\begin{aligned}
\text{Hence we have, } {\cdot}5078867 &= \text{M. of log N,} \\
\text{and } {\cdot}5078828 &= \text{M. of log } 32202\text{;} \\
\therefore \quad 39 &= d, \\
\text{also } \quad 135 &= \text{D,} \\
\text{and } \frac{39}{135} &= {\cdot}29. \\
\therefore {\cdot}5078867 &= \text{M. of log } 32202{\cdot}29\text{;} \\
\therefore \overline{2}{\cdot}5078867 &= \log {\cdot}03220229.
\end{aligned}$$

Hence ·03220229 is the number required.

Trigonometrical Tables.

29. We use trigonometrical tables much in the same way as we do tables of ordinary logarithms of numbers.

Tables have been formed of natural sines, cosines, &c., and also of logarithmic sines, cosines, &c. It is with the latter only we shall now deal, though many of our remarks apply equally to the former.

As the values of the natural sines and cosines of all angles between 0° and 90° are (Art. 11) less than *unity*, it follows (Art. 21) that their logarithms are negative. To avoid, however, printing them in a negative form, and for other reasons, it is usual to add 10 to their real value, and hence in using them we must allow for this. The same thing is also done in the case of logarithmic tangents, cotangents, secants, and cosecants.

We generally express the *true* logarithmic sine by log sin, and the *tabular* logarithmic sine by L sin.

$$\begin{aligned}
\text{Hence, we have, } \log \sin A &= \text{L} \sin A - 10, \\
\log \cos A &= \text{L} \cos A - 10, \text{ &c.}
\end{aligned}$$

It must be remembered in using the tables that, although (Art. 11) the sine, secant, and tangent of an angle *increase* as

the angle *increases* from 0° to 90°, yet the cosine, cosecant, and cotangent *diminish* as the angle *increases.*

Hence, when any angle is not exactly contained in the tables, we must *add* the difference in the case of a *sine*, *secant*, or *tangent;* but *subtract* it in the case of a *cosine*, *cosecant*, or *cotangent.*

And, conversely, when the given logarithm is not contained exactly in the tables, we must in the case of the *sine*, *secant*, or *tangent* take out the next *lower* tabular logarithm as corresponding to the angle next lower; but in the case of a *cosine*, *cosecant*, or *cotangent*, we must take out the next *higher* tabular logarithm as corresponding to the angle next lower in the tables.

We shall *assume* that small differences in the angles are proportional to the corresponding differences of the logarithmic trigonometrical ratios

Ex. 1. Find L sin 56° 28′ 24″.

Referring to tables, we have—

L sin 56° 28′ = 9·9209393

Tab. diff. for 60″ or D = 836

$\therefore$ diff. for 24″ or $d = \frac{24}{60} \times 836$ = 334

$\therefore$ L sin 56° 28′ 24″ = 9·9209727

Ex. 2. Find L cos 29° 31′ 28″.

Now L cos 29° 31′ = 9·9396253

Tab. diff. for 60″ or D = − 716

$\therefore$ diff. for 28″ $= -\frac{28}{60} \times 716$ = − 334

$\therefore$ L cos 29° 31′ 28″ = 9·9395919

Ex. 3. Find the angle A, when L tan A = 9·8658585

We have 9·8658585 = L tan A.

Next *lower*, 9·8657702 = L tan 36° 17′,

883 = difference or d,

Also, 2648 = tab. diff. for 60″ = D,

And $\frac{d}{D} \times 60'' = \frac{883}{2648} \times 60'' = 20''$.

Hence, 9·8658585 = L tan 36° 17′ 20″.

Ex. 4. Find the angle A, when L cot A = 10·0397936.

We have, 10·0397936 = L cot A,
Next *higher*, 10·0399770 = L cot 42° 22′,
1834 = difference or d,
Also, 2537 = tab. diff. for 60″ or D,

$$\text{And } \frac{d}{D} \times 60'' = \frac{1834}{2537} \times 60'' = 43''.$$

Hence, 10·0397936 = L cot 42° 22′ 43″.

Ex. V.

1. Given log 47582 = 4·6774427, and log 47583 = 4·6774518, find log 47·58275.

2. Given log 5·2404 = ·7193644, and log 524·05 = 2·7193727, find log ·5240463.

3. Given log ·56145 = $\bar{1}$·7493111, and log 56·146 = 1·7493188, find log $\sqrt[7]{\cdot 05614581}$.

4. Given log 61683 = 4·7901655, and log 616·84 = 2·7901725, find the number whose logarithm is $\bar{2}$·7901693.

5. Find the value of $(1{\cdot}05)^{15}$, having given log 1·05 = ·0211893, log 20789 = 4·3178336, and log 20790 = 4·3178545.

6. Find the compound interest of £120 for 10 years at 4 per cent. per annum, having given log 1·04 = ·0170333, log 14802 = 4·1703204, and log 14803 = 4·1703497.

7. A corporation borrows £8,630 at $4\frac{1}{2}$ per cent. compound interest, what annual payment will clear off the debt in 20 years?

Log 1·045 = ·0191163, log 4·1464 = ·6176712, and log 4·1465 = ·6176817.

8. Find the value of $\dfrac{(1{\cdot}032)^{10} \times \sqrt[3]{37{\cdot}62}}{(\cdot 347215)^6}$, having given

Log 1·032 = ·0136797, log 34722 = 4·5406047.
Log 3762 = 3·5754188, log 26202 = ·4183344.
Log 34721 = 4·5405922, log 26203 = ·4183510.

9. Find L sin 32° 28′ 31″, having given L sin 32° 28′ = 9·7298197, and L sin 32° 29′ = 9·7300182.

10. Find L cosec 43° 48′ 16″, having given L sin 43° 48′ = 9·8401959, and L sin 43° 49′ = 9·8403276.

11. Required the angle whose logarithmic cotangent is 10·1322449, having given L cot 36° 25′ = 10·1321127, L cot 36° 26′ = 10·1318483.

12. Construct a table of proportional parts, having given 163 as the tabular difference.

13. In what time will a sum of money double itself at 5 per cent. per annum, compound interest?

14. Find x when $1{\cdot}03^x = 1{\cdot}2143$, having given that log 1·03 = ·0128372, and log 12143 = 4·0843260.

15. Solve the equation $2^{2x-1} - 40 = 9{\cdot}2^x$, having given log 2 = ·3010300.

16. Given L cos 32° 45′ = 9·9341986, D = 752, find L cos 32° 45′ 12″, and L sec 32° 45′ 20″.

17. Given L tan 28° 38′ = 10·2628291, D = 3003, find L tan 28° 37′ 15″, and L cot 28° 38′ 42″.

18. Find the angle whose logarithmic cosine is 9·9590635, having given—

L cos 24° 29′ = 9·9590805,
L cos 24° 31′ = 9·9589653.

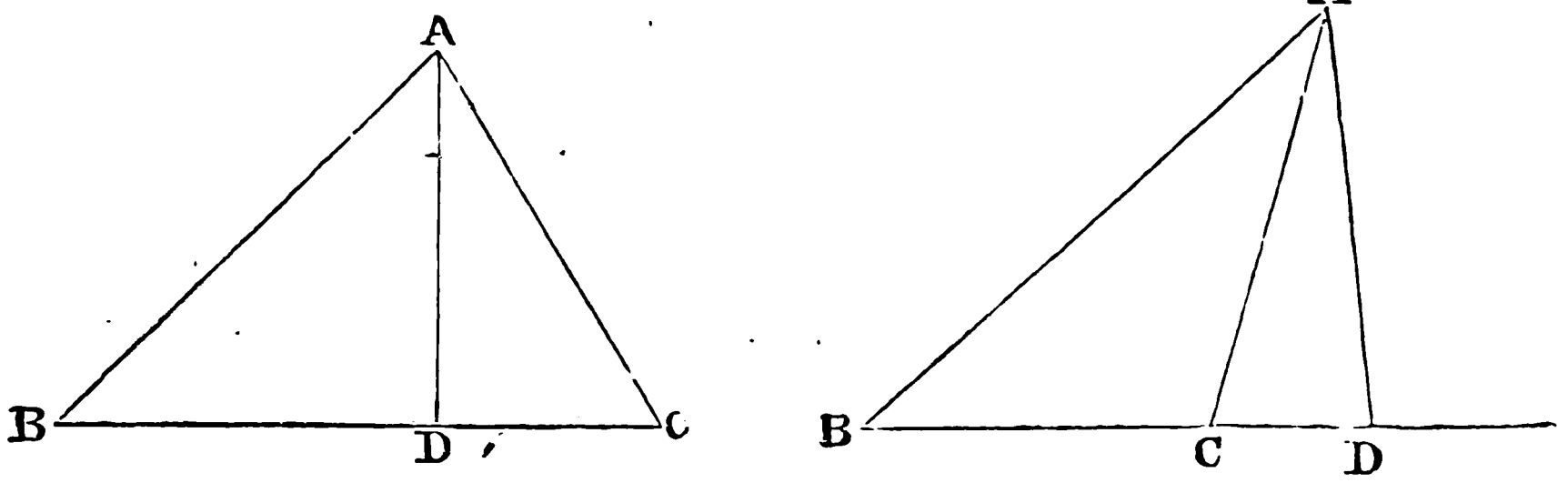
CHAPTER VII.

PROPERTIES OF TRIANGLES.

30. *The sines of the angles of a triangle are proportional to the opposite sides.*

We shall designate the sides opposite to the angles A, B, C, by the small letters *a*, *b*, *c*, respectively.

Draw AD perpendicular to BC, or to BC produced.



Then $\sin B = \frac{AD}{AB} = \frac{AD}{c}$ .....(1).

And $\sin C = \frac{AD}{AC} = \frac{AD}{b}$ .....(2).

Hence, (1) $\div$ (2), $\frac{\sin B}{\sin C} = \frac{AD}{c} \div \frac{AD}{b} = \frac{b}{c}$,

$$\text{or } \frac{\sin B}{b} = \frac{\sin C}{c}.$$

It follows, therefore, from the symmetrical nature of this equation, that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. *Q.E.D.*

31. *In any triangle,* $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$.

Taking the figures of the last article, we have—

(1.) *When* C *is an acute angle*—

By Euc. II., 13, $AB^2 = BC^2 + AC^2 - 2\,BC.CD$.

$$\text{Now } \frac{CD}{AC} = \cos C, \text{ or } CD = AC \cos C.$$

Hence we have, $AB^2 = BC^2 + AC^2 - 2\,BC.AC \cos C$, or $c^2 = a^2 + b^2 - 2ab \cos C$.

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

(2.) *When* C *is an obtuse angle*, as in the second figure—

By Euc. II., 12, $AB^2 = BC^2 + AC^2 + 2\,BC.CD$;

and $CD = AC\cos ACD = AC\cos(180° - C) = -AC\cos C$.

Hence, $AB^2 = BC^2 + AC^2 + 2\,BC(-AC\cos C)$;

or, $c^2 = a^2 + b^2 - 2\,ab\cos C$.

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}, \text{ as before.}$$

From the *form* of this result we have also—

$$\text{Cos A} = \frac{b^2 + c^2 - a^2}{2\,bc},$$

$$\text{Cos B} = \frac{a^2 + c^2 - b^2}{2\,ac}.$$

32. *To express the sine of any angle of a triangle in terms of the sides.*

We have—

$$\text{Sin}^2 A = 1 - \cos^2 A = 1 - \left(\frac{b^2 + c^2 - a^2}{2\,bc}\right)^2$$

$$= \frac{(2\,bc)^2 - (b^2 + c^2 - a^2)^2}{(2\,bc)^2}$$

$$= \frac{\{(b + c)^2 - a^2\}\{a^2 - (b - c)^2\}}{(2\,bc)^2}$$

$$= \frac{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}{(2\,bc)^2}.$$

Hence, taking the square root, and taking the positive sign, because (Art. 11) the sin A is always positive when A is the angle of a triangle, we have—

$$\text{Sin A} = \frac{1}{2\,bc}\sqrt{(a + b + c)(b + c - a)(a + c - b)(a + b - c)}.$$

Let $a + b + c = 2\,s$, then $b + c - a = 2\,(s - a)$,

$a + c - b = 2\,(s - b)$, $a + b - c = 2\,(s - c)$.

$$\text{Hence, sin A} = \frac{1}{2\,bc}\sqrt{2\,s \cdot 2\,(s - a) \cdot 2\,(s - b) \cdot 2\,(s - c)}$$

$$= \frac{2}{bc}\sqrt{s\,(s - a)\,(s - b)\,(s - c)}.$$

From the *form* of this result we have also—

$$\text{Sin B} = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)},$$

$$\sin \text{C} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$$

33. *To find the area of a triangle.*

Using the figures of Art. 30, we have—

$$\Delta \text{ABC} = \tfrac{1}{2}\,\text{BC}.\,\text{AD}, \text{ or, since AD} = \text{AC sin C},$$
$$= \tfrac{1}{2}\,\text{BC}.\,\text{AC sin C}$$
$$= \tfrac{1}{2}\,ab \sin \text{C} \quad \ldots\ldots\ldots\

The *form* of this result gives us also—

$$\text{Sin}\,\frac{B}{2} = \sqrt{\frac{(s-a)\,(s-c)}{ac}},\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}.$$

Again, Art. 18, we have—

$$2\cos^2\frac{A}{2} - 1 = \cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\text{or, } 2\cos^2\frac{A}{2} = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(a+b+c)\,(b+c-a)}{2bc} = \frac{2s \cdot 2\,(s-a)}{2bc},$$

$$\text{or, } \cos^2\frac{A}{2} = \frac{s\,(s-a)}{bc}.$$

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The *form* of this result gives us also—

$$\cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ac}},\ \cos\frac{C}{2} = \sqrt{\frac{s\,(s-c)}{ab}}.$$

Now, (1) ÷ (2), we have—

$$\frac{\text{Sin}\,\frac{A}{2}}{\text{Cos}\,\frac{A}{2}} = \sqrt{\frac{(s-b)\,(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}},\ \text{or}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

The *form* of this result gives us also—

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)\,(s-c)}{s(s-b)}},\ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

35. *In any triangle* ABC, $a = b\cos C + c\cos B$.

Using the figures of Art. 30, we have—

(1.) When C is acute—

$$BD = AB\cos B, \text{ and } DC = AC\cos C.$$

$$\therefore BD + DC = AC\cos C + AB\cos B, \text{ or}$$

$$a = b\cos C + c\cos B.$$

(2.) When C is obtuse—

$$BD = AB \cos B,$$

and $DC = AC \cos ACD = AC \cos (180° - C) = - AC \cos C.$

$$\therefore BD - DC = AC \cos C + AB \cos B,$$

or $a = b \cos C + c \cos B$, as before.

The *form* of this result gives us also—

$$b = a \cos C + c \cos A,\ c = a \cos B + b \cos A.$$

Cor. 1. Hence $\sin (B + C) = \sin B \cos C + \cos B \sin C.$

For we have, $1 = \frac{b}{a} \cos C + \frac{c}{a} \cos B$, or, by Art. 30,

$$= \frac{\sin B}{\sin A} \cos C + \frac{\sin C}{\sin A} \cos B,$$

$$\text{or, } \sin A = \sin B \cos C + \sin C \cos B.$$

But $\sin A = \sin (180° - A) = \sin (B + C).$

$\therefore$ Sin $(B + C) = \sin B \cos C + \cos B \sin C.$

Cor 2. Hence also—

$$\sin (B - C) = \sin B \cos C - \cos B \sin C.$$

For, since this result has been proved for any angles of a triangle, it will be also true for a triangle which contains an angle *supplementary* to B; that is, it will be true, if we put $180° - B$ for B. We then have—

$$\text{Sin } (180° - B + C) = \sin (180° - B) \cos C + \cos (180° - B) \sin C.$$

But $\sin (180° - B + C) = \sin (180° - B - C) = \sin (B - C)$,

$\sin (180° - B) = \sin B$, $\cos (180° - B) = - \cos B.$

Hence, $\sin (B - C) = \sin B \cos C - \cos B \cos C.$

Note.—The results of Cor. 1 and Cor. 2 have been proved only for angles less than two right angles. We shall see in Vol. II. they are generally true.

36. *In any triangle* ABC—

$$\text{Tan } \tfrac{1}{2} (B - C) = \frac{b - c}{b + c} \cot \tfrac{1}{2} A.$$

We have, Art. 30,

$$\frac{\text{Sin } B}{\text{Sin } C} = \frac{b}{c}, \text{ or } \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b - c}{b + c} \ldots\ldots\ldots\ldots (1).$$

Now, $\sin B = \sin\left\{\frac{B+C}{2} + \frac{B-C}{2}\right\}$, or, by Cor. 1, Art. 35,

$$= \sin\frac{B+C}{2}\cos\frac{B-C}{2} + \cos\frac{B+C}{2}\sin\frac{B-C}{2} \ldots\ldots(2).$$

and $\sin C = \sin\left\{\frac{B+C}{2} - \frac{B-C}{2}\right\}$, or, by Cor. 2, Art. 35,

$$= \sin\frac{B+C}{2}\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\sin\frac{B-C}{2} \ldots\ldots(3).$$

(2) − (3), then $\sin B - \sin C = 2\cos\frac{B+C}{2}\sin\frac{B-C}{2}$,

and (2) + (3), then $\sin B + \sin C = 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$.

$$\therefore \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$= \cot\frac{B+C}{2}\tan\frac{B-C}{2} = \cot\left(90° - \frac{A}{2}\right)\tan\frac{B-C}{2}$$

$$= \tan\frac{A}{2}\tan\frac{B-C}{2} \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

(4) = (1), then $\tan\frac{A}{2}\tan\frac{B-C}{2} = \frac{b-c}{b+c}$.

$$\therefore \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\tfrac{1}{2}A. \quad Q.E.D.$$

Ex. VI.

1. If $a = 5$, $C = 30°$, $\sin A = \frac{1}{5}$, find c.
2. Find a, having given $b = 12$, $c = 15$, $A = 60°$.
3. Find $\tan\frac{A}{2}$, when $a = 6$, $b = 7$, $c = 8$.
4. What is the area of a triangle whose sides are 48, 52, 20?
5. Given two sides of a triangle to be 18 and 24, and the included angle 45°, find the area.

6. Two of the sides of a triangle are as 2 : 1, and the included angle is 60°, find the other angles.

7. In any triangle show that—

$$b \cos C - c \cos B = \sqrt{a^2 - 4bc \cos B \cos C}.$$

8. Show that the area of a triangle $= \frac{1}{2} a^2 . \frac{\sin B \sin C}{\sin A}$.

9. An object is observed from two stations 100 yards apart, and the angles subtended by the distance between the object and either station are 45° and 60° respectively. Find the distance of the object from each station.

10. An observation is made from a point known to be distant 120 and 230 yards respectively from two trees, and the angle which the trees subtend is found to be 120°. Find the distance between the trees.

11. If $\left.\begin{aligned} a \sin^2 \theta + b \cos^2 \theta &= m, \\ b \sin^2 \phi + a \cos^2 \phi &= n, \\ a \tan \theta &= b \tan \phi, \end{aligned}\right\}$ then $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$.

12. If a', b', c' be the sides of the triangle formed by joining the feet of the perpendiculars from the angles A, B, C of the triangle ABC upon the opposite sides, then—

$$\frac{a'}{a^2} + \frac{b'}{b^2} + \frac{c'}{c^2} = \frac{a^2 + b^2 + c^2}{2abc}.$$

13. A perpendicular AD is drawn from the angle A of a triangle, meeting the opposite side BC and D; and from D a perpendicular is drawn to AC, meeting it in E. Show that $DE = b \sin C \cos C$.

14. Show that the length of AD in the last example—

$$= \frac{bc \sin A + ac \sin B + ab \sin C}{3a}.$$

15. Show that $\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} ab$, when the triangle is right-angled at C.

16. Show that $(a + b + c)^2 \sin A \sin B = (\sin A + \sin B + \sin C)^2 ab$.

17. Show that in any triangle—

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1.$$

18. The sides of a triangle ABC are in arithmetical progression; show that its area $= \frac{b\sqrt{3}}{2}\sqrt{(2a - b)(3b - 2a)}$.

CHAPTER VIII.

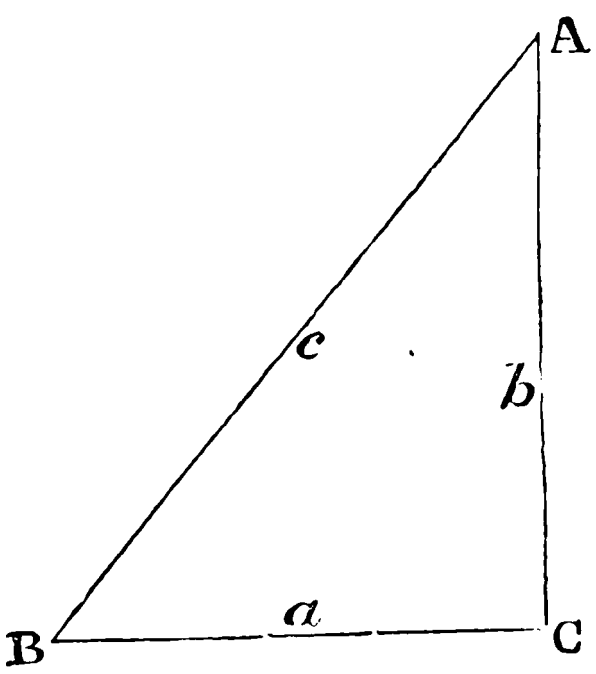
SOLUTION OF RIGHT-ANGLED TRIANGLES.

37. A triangle can always be determined when any three elements, with the exception of the three angles, are given. In the latter case we have only the same data as when two angles are given, for the third can always be found by subtracting the sum of the other two from two right angles (Euc. I., 32).

Hence a right-angled triangle can always be determined when any two elements, other than the two acute angles, are given besides the right angle. And when one of the acute angles is given, the other may be obtained by subtracting it from a right angle.

We have the following cases:—

CASE 1. *When the two sides containing the right angle are given.*



We shall take C as the right angle in every case.

Now $\tan A = \frac{a}{b}$;

or, $\mathrm{L}\tan A - 10 = \log a - \log b$;

or, $\mathrm{L}\tan A = 10 + \log a - \log b$...(1).

This determines A, and we then have

$B = 90° - A$.....(2).

Also, $\frac{a}{c} = \sin A$, or $\log a - \log c = \mathrm{L}\sin A - 10$.

$\therefore \log c = 10 + \log a - \mathrm{L}\sin A$.....(3).

Hence the three elements, A, B, c, are determined.

CASE 2. *When the hypothenuse and a side are given.*

Let a be the given side.

We have $\sin A = \frac{a}{c}$, or $L \sin A - 10 = \log a - \log c$.

$\therefore L \sin A = 10 + \log a - \log c$.....(1),

and $B = 90° - A$.....(2),

also $b^2 = c^2 - a^2 = (c + a)(c - a)$;

$\therefore \log b = \frac{1}{2}\{\log (c + a) + \log (c - a)\}$.....(3).

CASE 3. *When an acute angle and a side are given.*

Let A, a be the given angle and side.

Then $B = 90° - A$.....(1),

also $\frac{b}{a} = \tan B$, or $\log b = L \tan B - 10 + \log a$.....(2),

and $\frac{a}{c} = \sin A$, or $\log a - \log c = L \sin A - 10$,

or $\log c = 10 + \log a - L \sin A$.

CASE 4. *When the hypothenuse and an acute angle are given.*

Let A be the given acute angle.

We have $B = 90° - A$.....(1).

Also $\frac{a}{c} = \sin A$, or $\log a = \log c + L \sin A - 10$...(2).

And $\frac{b}{c} = \cos A$, or $\log b = \log c + L \cos A - 10$....(3).

It is evident, from Art. 30, that when the angles only of a triangle are known, we can determine the *ratio* only of the three sides of the triangle to each other.

Ex. 1. Given $A = 23° 41'$, $a = 35$, solve the triangle.

This is an example of Case 3.

We have $B = 90° - 23° 41' = 66° 19'$.

Again, $\log b = L \tan B - 10 - \log a$

$= L \tan 66° 19' - 10 - \log 35$

$= 10{\cdot}3579092 - 10 + 1{\cdot}5440680$

$= 1{\cdot}9019772 = \log 79{\cdot}795$

$\therefore b = 79{\cdot}795$.

$$\begin{aligned}\text{Also } \log c &= 10 + \log a - \text{L}\sin \text{A}\\ &= 10 + \log 35 - \text{L}\sin 23^\circ\ 41'\\ &= 10 + 1{\cdot}5440680 - 9{\cdot}6038817\\ &= 1{\cdot}9401863 = \log 87{\cdot}134\\ \therefore c &= 87{\cdot}134.\end{aligned}$$

Ex. 2. Given $a = 214$, $b = 317$, solve the triangle.

This falls under Case 1.

$$\begin{aligned}\text{We have L}\tan \text{A} &= 10 + \log a - \log b\\ &= 10 + \log 214 - \log 317\\ &= 10 + 2{\cdot}3304138 - \log 2{\cdot}5010593\\ &= 9{\cdot}8293545,\end{aligned}$$

Next lower in tables is $9{\cdot}8292599 = \text{L}\tan 34^\circ\ 1'$;

$$\therefore d = 946.$$

Also, by tables, $\text{D} = 2724$,

$$\text{And } \frac{d}{\text{D}} \times 60'' = \frac{946 \times 60''}{2724} = 21'' \text{ nearly.}$$

$$\therefore \text{L}\tan \text{A} = \text{L}\tan 34^\circ\ 1'\ 21'',$$

$$\text{or A} = 34^\circ\ 1'\ 21''.$$

$$\text{Hence B} = 90 - 34^\circ\ 1'\ 21'' = 45^\circ\ 58'\ 39''.$$

And similarly may c be determined.

Ex. VII.

1. Given $a = 32$, $\text{A} = 63^\circ\ 45'$, find b.
   Log 32 = 1·5051500, L cot 63° 45′ = 9·6929750,
   Log 15780 = 4·1981070, log 15781 = 4·1901345.

2. Given $c = 151$, $\text{A} = 37^\circ\ 42'$, find a.
   Log 151 = 2·1789769, L sin 37° 42′ = 9·7864157,
   Log 92340 = 4·9653899, tab. diff. = 47.

3. Given $a = 60$, $c = 65$, find b, A.
   Log 2 = ·3010300, log 3 = ·4771213,
   Log 65 = 1·8129134, L sin 67° 22′ = 9·9651953,
   L sin 67° 23′ = 9·9652480.

4. Given $a = 73$, $b = 84$, find A, c.
   Log 73 = 1·8633229, L tan 40° 59′ = 9·9389079,
   Log 84 = 1·9242793, L tan 41° = 9·9391631,
   L sin 40° 59′ = 9·8167975, L sin 41° = 9·8169429,
   Log 111·288 = 2·0464479.

5. Given B = 71° 41′ 10″, c = 24, find b.
Log 24 = 1·3802112, L cos 18° 18′ = 9·9774609,
L cos 18° 19′ = 9·9774191, log 2278·4 = 3·3576300,
Log 2278·5 = 3·3576490.

6. Given a = 293, c = 751, find b.
Log 1044 = 3·0187005, log 458 = 2·6608655,
Log 691·49 = 2·8397830.

7. Given a = 12, A = 30°, find b, c.

8. Given c = 10, B = 75°, find a, b.

9. Given a = 17, c = 34, find A, b.

10. Given $a = 5$, $b = 5\sqrt{3}$, find A.

11. Given a = 28, B = 15°, find the length of the perpendicular from C on AB.

12. CD is the perpendicular from C on AB, and DE the perpendicular from D on BC. Given B = 60°, a = 20, find DE.

CHAPTER IX.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

38. *Given the three sides of a triangle, to find the remaining parts.*

We have, Art. 31, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, from which A may be determined; and from similar formulæ we may find B and C. These formulæ are not however adapted to logarithmic computation. We shall therefore find it generally advisable to use the formulæ of Art. 34.

We have, $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$,

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

From either of these formulæ we can determine A, and from similar formulæ determine the other angles.

39. *Given one side and two angles, to find the remaining parts.*

Of course the third angle is at once known. Let a be the given side.

We have, Art. 30, $b = \dfrac{a \sin \mathrm{B}}{\sin \mathrm{A}}$,

$$c = \frac{a \sin \mathrm{C}}{\sin \mathrm{A}},$$

both of which formulæ are adapted to logarithmic computation.

40. *Given two sides and the included angle, to find the remaining parts.*

Let b, c be the given sides, and A the included angle.

We have, Art. 36, $\tan \frac{1}{2}(\mathrm{B} - \mathrm{C}) = \dfrac{b - c}{b + c} \cot \frac{1}{2}\mathrm{A}$.

This formula is adapted to logarithmic computation, and determines $\frac{1}{2}(\mathrm{B} - \mathrm{C})$.

We know also $\frac{1}{2}(\mathrm{B} + \mathrm{C})$, for it is the complement of $\frac{1}{2}$ A.

Hence, B and C are easily determined.

Then, Art. 30, $a = \dfrac{b \sin \mathrm{A}}{\sin \mathrm{B}}$, which determines a.

NOTE.—When the two given sides are equal, the solution may be effected more easily by drawing a perpendicular from the given angle upon the opposite side, and so bisecting it. By drawing a figure, it is easily seen that, in this case, $\mathrm{B} = \mathrm{C} = 90° - \frac{1}{2}\mathrm{A}$, and $a = 2b \cos \mathrm{B}$.

41. *Given two sides and an angle opposite to one of them, to find the remaining parts.*

Let a, b, B be the given elements.

Then we have, Art. 30, $\sin \mathrm{A} = \dfrac{a}{b} \sin \mathrm{B}$.

(1.) Let the value of $\dfrac{a}{b} \sin \mathrm{B}$ be *unity*.

We then have $\sin \mathrm{A} = 1 = \sin 90°$, and $\therefore$ A $= 90°$.

Hence the triangle is right-angled at A.

(2.) Let the value of $\frac{a}{b}$ sin B be > 1.

We then have sin A > 1, which is impossible.

Hence, in this case, it is impossible to form a triangle with the given elements.

(3.) Let the value of $\frac{a}{b}$ sin B be < 1.

Then, since, Art. 15, the sine of an angle is the same as the sine of its supplement, there are two values of A which satisfy the equality, sin A $= \frac{a}{b}$ sin B, and these values are supplementary.

Let A, A′ be the two values, then the relation between them is A + A′ = 180°.

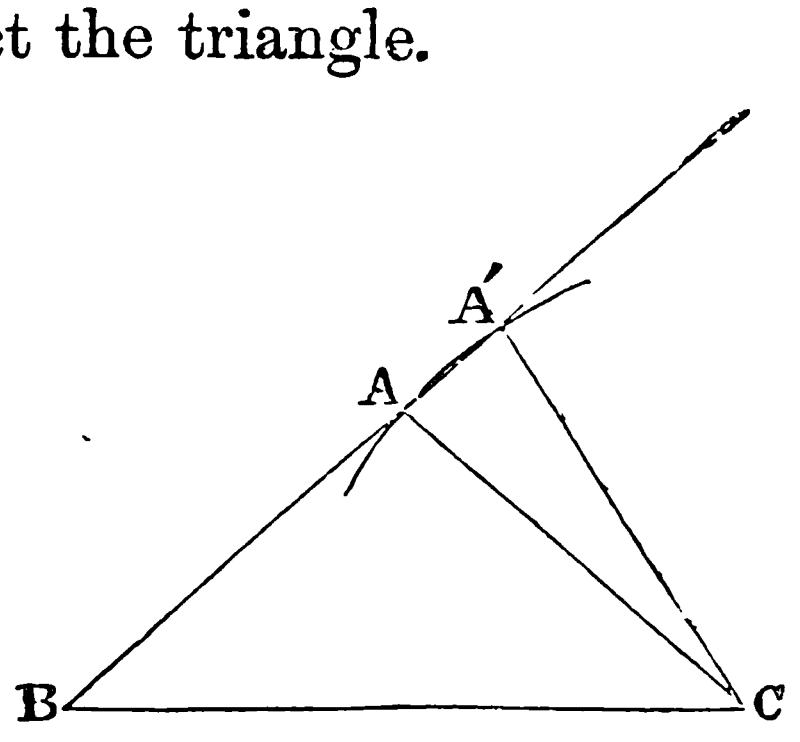
If a is not greater than b, then A is not greater than B, and there is no doubt as to which value of A is to be taken. If, however, a is greater than b—that is, if the given angle is opposite to the less of the given sides, we must have A greater than B, and both values of A may satisfy this condition. This particular case, when *the given angle is opposite to the less of the given sides*, is called *the ambiguous case.* We will illustrate this geometrically.

42. *The ambiguous case.*

Let a, b, B be given to construct the triangle.

Draw the line BC equal to the given side a, and draw BA making an angle B with BC.

Then with centre C and radius CA equal to b describe an arc meeting BA or BA produced in A and A′, and join CA and CA′.



Each of the triangles ABC, A′BC satisfies the given conditions.

For, in the triangle ABC, we have BC = a, AC = b, and ∠ ABC = B; and in the triangle A′BC, we have BC = a, A′C = b, and ∠ A′BC = B.

Again, the sides BA and BA′ correspond to the two values

of c which are obtained from the two values of A in the equality $\sin A = \frac{a}{b} \sin B$ (see last Art.).

And the angles ACB and A′CB correspond to the two values of C which would also be found.

Cor. If a perpendicular CD be drawn from C upon AA′, and if c' and c be the lengths of BA′ and BA respectively, it may be easily shown that $c' + c = 2a \cos B$, and $c' \sim c = 2b \cos A$.

Ex. VIII.

Solve the following triangles, having given—

1. $b = 12, c = 6, A = 60°$.

2. $a = 18, b = 18\sqrt{2}, A = 30°$.

3. $a = 5\sqrt{3}, b = 5\sqrt{2}, c = \frac{5}{2}(\sqrt{6} + \sqrt{2})$.

4. $a = 12, B = 60°, C = 15°$.

5. $a = 3\sqrt{2} + \sqrt{6}, b = 6, C = 45°$.

6. $a = 10\sqrt{3}, b = 15\sqrt{2}, A = 45°$.

Given—

7. $b = 251, c = 372, A = 40° 32'$, find B and C.

   Log 121 = 2·0827854, L cot 20° 16′ = 10·4326795,
   Log 623 = 2·7944880, L tan 27° 44′ = 9·7207827,
   L tan 27° 45′ = 9·7210893.

8. $a = 237, b = 341, B = 28° 24'$, find A.
   Log 237 = 2·3747483, L sin 28° 24′ = 9·6772640,
   Log 341 = 2·5327544, L sin 19° 18′ = 9·5191904,
   L sin 19° 19′ = 9·5195510.

9. $C = 26° 32'$, and $a : b :: 3 : 5$, find A, B.
   Log 2 = ·3010300, L cot 13° 16′ = 10·6275008,
   L tan 46° 40′ = 10·0252805, L tan 46° 41′ = 10·0255336.

10. $a = 14, b = 16, c = 18$, find A, B.
    log 2 = ·3010300, log 3 = ·4771213,
    L tan 24° 5′ = 9·6502809, L tan 24° 6′ = 9·6506199,
    L tan 29° 12′ = 9·7473194, L tan 29° 13′ = 9·7476160.

11. $a = 3$, $b = 2$ A $= 60°$, find B, C, and c.
Log 2 = ·3010300, log 3 = ·4771213,
L sin 35° 15′ = 9·7612851, L sin 35° 16′ = 9·7614638,
Log 1·3797 = ·1397847, log 1·3798 = ·1398161.

12. $a = 5$, $b = 6$, $c = 7$, find A.
Log 2 = ·3010300, L tan 22° 12′ = 9·6107586,
Log 3 = ·4771213, L tan 22° 13′ = 9·6111196.

13. If c, c' be the two values of the third side in the ambiguous case when a, b, A are given, show that—

$$(c - c')^2 + (c + c')^2 \tan^2 \mathrm{A} = 4a^2.$$

14. If a, b, A are given, show from the equation—

$$b^2 + c^2 - 2bc \cos \mathrm{A} = a^2;$$

that if c and c' be the two values of the third side—

$$cc' = b^2 - a^2, \text{ and } c + c' = 2b \cos \mathrm{A}.$$

15. Show also from the same equation that there is no ambiguous case when $a = b \sin \mathrm{A}$, and that c is impossible when $a < b \sin \mathrm{A}$.

16. Having $a - b$, A, B, solve the triangle.

17. Given the ratios of the sides, and the angle A, solve the triangle.

18. If in a triangle $\tan \frac{1}{2}(\mathrm{B} - \mathrm{C}) = \tan^2 \frac{\phi}{2} \cot \frac{\mathrm{A}}{2}$, show that $b \cos \phi = c$.

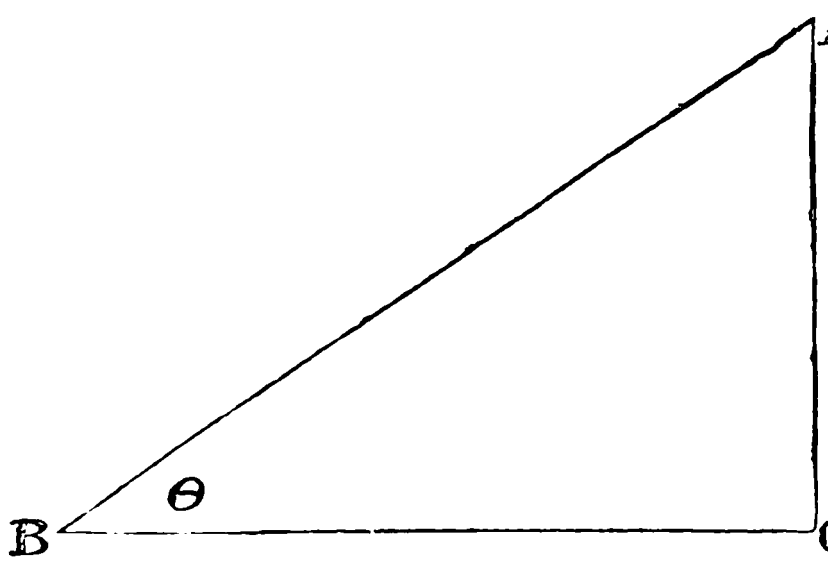
CHAPTER X.

HEIGHTS AND DISTANCES.

43. We shall now show how the principles of the previous chapters are practically applied in determining heights and distances.

We have not space here to describe the instruments by which angles are practically measured, but we shall assume that they can be measured to almost any degree of accuracy.

44. *To find the height of an accessible object.*



Let AC be the object, and let any distance BC from its foot be measured.

At B let the *angle of elevation* ABC be observed.

Suppose BC = a, $\angle$ ABC = θ.

Then, we have—

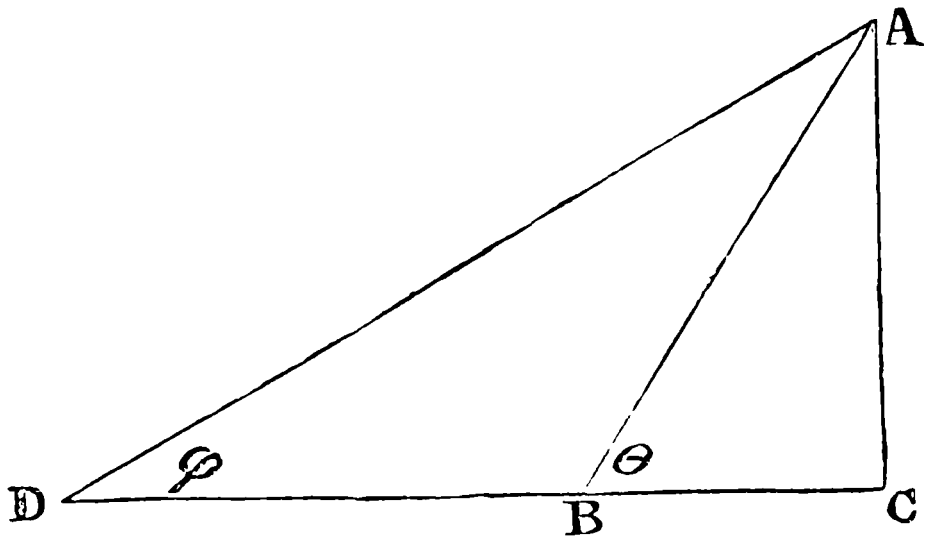
$$\frac{AC}{BC} = \tan ABC \text{ or } \frac{AC}{a} = \tan\theta.$$

$\therefore$ AC = $a \tan\theta$, the height required.

Ex. Let $a = 200$, and $\theta = 30°$.

Then $AC = 200 \tan 30° = 200 . \frac{1}{\sqrt{3}} = \frac{200}{3}\sqrt{3}$.

45. *To find the height of an inaccessible object.*



At any point B in the horizontal plane of the base let the *angle of elevation* ABC be observed.

Measure a convenient distance BD in the straight line CB produced, and observe the *angle of elevation* ADC.

Let BD = a, $\angle$ ABC = θ, $\angle$ ADB = ϕ.

Then, Euc. I. 32, $\angle$ BAD = $\theta - \phi$.

Now, $\frac{AB}{BD} = \frac{\sin ADB}{\sin BAD}$ or $\frac{AB}{a} = \frac{\sin\phi}{\sin(\theta-\phi)}$,

$$\therefore AB = a . \frac{\sin\phi}{\sin(\theta-\phi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Again, $\frac{AC}{AB} = \sin ABC = \sin\theta$, $\therefore AC = AB \sin\theta$,

$$\text{or, from (1) } AC = a\frac{\sin\theta\sin\phi}{\sin(\theta-\phi)}.$$

Ex. Let BD = 120, $\theta = 60°$, $\phi = 45°$.

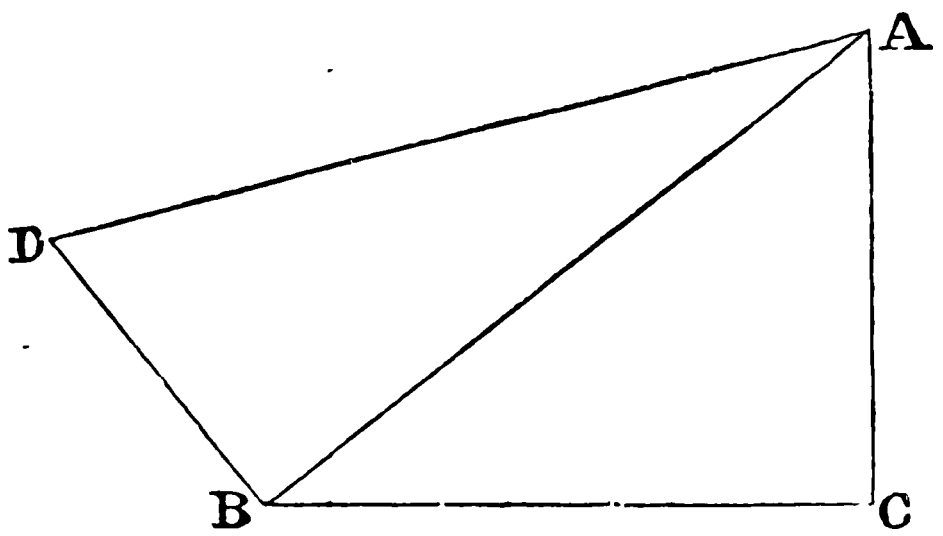
$$\text{Then, } AC = 120 . \frac{\sin 60° \sin 45°}{\sin(60° - 45°)}.$$

$$= 120\,\frac{\sin 60^\circ \sin 45^\circ}{\sin 15^\circ} = 120 \cdot \frac{\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= 120 \cdot \frac{\sqrt{3}}{\sqrt{3}-1} = 60\,(3 + \sqrt{3}).$$

46. *To find the height of an inaccessible object when it is not convenient to measure any distance in a line with the base of the object.*

Let a distance BD be measured in any direction in the same horizontal plane as BC, and let the angles ABC, ABD, ADB be observed.



Let BD = a, $\angle$ ABC = α
$\angle$ ABD = β, $\angle$ ADB = γ.

Then, Euc. I., 32, $\angle$ BAD $= 108^\circ - (\beta + \gamma)$.

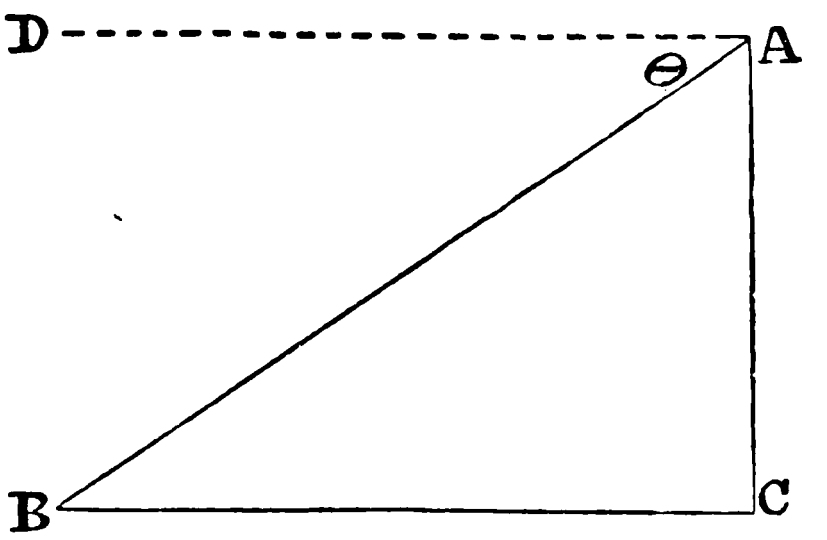
Now, $\frac{AB}{BD} = \frac{\sin ADB}{\sin BAD} = \frac{\sin \gamma}{\sin\{180^\circ - (\beta + \gamma)\}}$, or

$$\frac{AB}{a} = \frac{\sin \gamma}{\sin (\beta + \gamma)}, \quad \therefore \quad AB = a \cdot \frac{\sin \gamma}{\sin (\beta + \gamma)} \ldots\ldots\ldots (1).$$

Again, $\frac{AC}{AB} = \sin ABC = \sin \alpha$, $\therefore$ $AC = AB \cdot \sin \alpha$,

or from (1), $AC = a \cdot \frac{\sin \alpha \cdot \sin \gamma}{\sin (\beta + \gamma)}$.

47. *To find the distance of an object by observation from the top of a tower whose height is known.*



Let B be the object in the same horizontal plane with c the foot of the tower, and let *the angle of depression* DAB be observed.

Let AC = h, $\angle$ DAB = θ.

Then we have, $\frac{BC}{AC} = \cot ABC = \cot DAB$, or $\frac{BC}{h} = \cot \theta$,

$\therefore BC = h \cot \theta$, the distance required.

Cor. Suppose B to be not in the same level with C. Let m be the height of B above C, then B is on the same level with a point which is $h - m$ from A. $\therefore BC = (h - m) \cot \theta$.

Ex. IX.

1. Find the height of a tower 200 yards distant when it subtends an angle of 15°.

2. From the top of a tower, the angle of depression of a point in the horizontal plane at the foot of the tower was 30°. Given the height to be 60 ft., find the distance of the point.

3. The angle of depression of two consecutive milestones in a direct line with the summit of a hill were observed to be 60° and 30°. Find the height of the hill.

4. An object is observed from a ship to be due E. After sailing due S. for six miles it is observed to be N.E. Find the distance from the last position of the ship.

5. There is an object A, and two stations B and C are taken in the same plane. Given that $BC = 50$, $\angle ABC = 60°$, $\angle ACB = 30°$, find AB, AC.

6. The elevation of a tower is found to be 45°, and on approaching 60 feet nearer the elevation is 75°. Find the height of the tower.

7. Wishing to know the breadth of a river I observed an object on the opposite bank, and, having walked along the side of the river a distance of 100 yards, found the angle subtended by the object and my first station to be 30°. Find the breadth of the river.

8. A person, standing exactly opposite to the centre of an oblong which measures 16 ft. by 12 ft., and such that the line drawn from the centre to his eye is at right angles to the oblong, observes that the diagonal subtends an angle of 60°. Find his distance.

9. The angles of elevation of the summit of one tower, whose height is h, are observed from the base and summit of

another, and found to be θ and ϕ respectively. Show that the height of the second tower—

$$= h \cdot \frac{\tan\theta}{\tan\theta - \tan\phi}.$$

10. From the top of a tower the angles of depression of two objects in a direct line, and whose distance from each other is a, are α, β respectively. Show that the height of the tower—

$$= \frac{a}{\cot\beta - \cot\alpha}.$$

11. A person, having walked a distance a from one corner along a side of an oblong, observes that the side immediately behind him subtends an angle α, and the side in front an angle β. Show that the dimensions of the oblong are—

$$a\tan\alpha,\ a(1 + \tan\alpha\cot\beta).$$

12. Three points A, B, C form a triangle whose sides are a, b, c respectively, and a person standing at a point S, such that SA is at right angles to BC, observes that the side AC subtends an angle θ. Show that the distance of S from B—

$$= \frac{1}{2a}\sqrt{(a^2 + c^2 - b^2)^2 + (a^2 + b^2 - c^2)^2\cot^2\theta}.$$

13. A person walks a yards from A to E along AB the side of a triangle ABC, and observes that the angle AEC $= \alpha$; he also walks b yards from B to F along BA, and observes $\angle$ CFB $= \beta$. Having given AB $= c$, find BC and AC.

14. A tower is observed from three stations A, B, C, in a straight line not meeting the tower, to subtend angles α, β, γ respectively. Show that if AB $= a$, BC $= b$, the height of the tower—

$$= \sqrt{\frac{ab(a + b)}{a\cot^2\gamma - (a + b)\cot^2\beta + b\cot^2\alpha}}.$$

When are the conditions impossible?

15. From two stations whose distance apart is a, and which are due W. and due S. respectively of one end of a wall, the angles subtended by the wall are each α. Show that the length of the wall is $a\sin\alpha$.

16. The angles of elevation of the top of a tower, whose height is h and standing on a hill, are α, β, when observed from two stations a miles distant, and in a direct line up the hill. Show that if θ be the slope of the hill—

$$\cos\theta = \frac{a}{h} \cdot \frac{\sin\alpha \sin\beta}{\sin(\beta - \alpha)}.$$

17. The elevation of a tower was observed to be α, but on walking in the horizontal plane a distance a at right angles to the line joining the first position and the foot of the tower, the elevation was β. Show that the height of the tower was—

$$\frac{a}{\sqrt{\cot^2\beta - \cot^2\alpha}}.$$

18. The angles of depression of two objects in the same horizontal plane, as seen from the top of a tower, are θ and ϕ respectively, and the angle they subtend is α. Show that if h be the height of the tower, the distance between the objects—

$$= h\sqrt{\operatorname{cosec}^2\theta + \operatorname{cosec}^2\phi - 2\operatorname{cosec}\theta \operatorname{cosec}\phi \cos\alpha}.$$

ANSWERS.

I.—Page 15.

1. 45·23, 290, ·2367, ·7, &c.

2. ·0005, 11·11, ·040020, ·45, &c.

3. Three thousand four hundred and sixty-seven thousandths; thirty-four, and sixty-seven hundredths; three thousand four hundred and sixty-seven millionths; three, and four hundred and sixty-seven thousandths.

4. 35·90846, 29130·19391, 60·0239.

5. ·7237, 3·32091.

6. 69·5289, 5·06679, ·41481.

7. ·026, 7708·71.

8. ·09, 24·356706, ·003627, ·289, ·0096, ·00016384.

9. 200, ·00125, 4000, ·2295, ·006, ·5002, &c.

10. 170000. 11. 4·97 &c., 1. 12. ·2.

II.—Page 23.

1. $\frac{22}{7}$, $\frac{43}{9}$, $\frac{1007}{10}$, $\frac{607}{17}$, $\frac{1010}{999}$, $\frac{729}{64}$.

2. $\frac{114}{6}$, $\frac{190}{10}$, $\frac{247}{13}$, $\frac{342}{18}$, $\frac{361}{19}$, $\frac{380}{20}$.

3. 6, 7, 12, 11, 9, 14; $\frac{24}{4}$, $\frac{42}{6}$, $\frac{96}{8}$, $\frac{110}{10}$, $\frac{108}{12}$, $\frac{196}{14}$.

4. $7\frac{1}{7}$, $1\frac{7}{8}$, $7\frac{3}{5}$, $16\frac{2}{3}$, $71\frac{1}{9}$, $31\frac{1}{4}$, &c. 5. 52.

6. $\frac{37}{42}$, $\frac{7}{51}$, $\frac{11}{36}$, $\frac{23}{39}$, $17\frac{4}{21}$, $1\frac{10}{11}$; $\frac{37}{49}$, &c.

7. $\frac{2}{5}$, $\frac{7}{52}$, $\frac{20}{27}$; $\frac{12}{35}$, $\frac{3}{26}$, $\frac{40}{63}$; $\frac{3}{10}$, $\frac{21}{208}$, $\frac{5}{9}$.

8. $\frac{5}{24}$, 6, 1, 1, 6. 9. $\frac{1}{2}$, $\frac{32}{45}$, $9\frac{1}{3}$, $\frac{9}{112}$, $\frac{1}{5}$, $\frac{2}{5}$.

10. $11\frac{3}{7}$. 11. 3. 12. 28.

III.—Page 28.

1. 4×11, 2^6, $2 \times 3 \times 5^2$, $2^2 \times 3^2 \times 7$, $3^3 \times 47$, $2 \times 3 \times 7 \times 11$.

2. $2 \times 7 \times 3^2 \times 11$, $2^3 \times 5 \times 43$, $2^9 \times 3^2$, $5^2 \times 7 \times 11^2$, $2^2 \times 3 \times 11^3$, $2^4 \times 3^4 \times 11$.

5. $\frac{2}{5}$, $\frac{3}{4}$, $\frac{7}{8}$, $\frac{11}{90}$, $\frac{4}{25}$, $\frac{4}{9}$; $\frac{19}{32}$, $\frac{25}{36}$, $\frac{15}{116}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{17}{19}$; $\frac{7}{9}$, $\frac{3}{5}$, $\frac{6}{11}$, $\frac{17}{36}$, $\frac{16}{35}$, $\frac{11}{4919}$.

6. 19, 7, 13, 6, 41, 729, 14, 11, 39.

7. $\frac{3}{5}$, $\frac{7}{9}$, $\frac{3}{11}$, $\frac{4}{13}$, $\frac{12}{17}$, $\frac{5}{9}$; $\frac{2}{3}$, $\frac{6}{7}$, $\frac{13}{15}$, $\frac{1}{9}$, $\frac{31}{32}$, $\frac{41}{54}$.

9. 2 days out of every 7.

10. 2 : 7. 12. 15, 10, 6.

IV.—Page 31.

1. 24, 1260, 2520, 1260, 5460, 33300.

2. $\frac{45}{60}$, $\frac{84}{60}$, $\frac{8}{60}$, $\frac{33}{60}$; $\frac{24}{84}$, $\frac{35}{84}$, $\frac{39}{84}$, $\frac{44}{84}$; $\frac{336}{252}$, $\frac{56}{252}$, $\frac{231}{252}$, $\frac{234}{252}$; $\frac{35}{105}$, $\frac{21}{105}$, $\frac{15}{105}$; $\frac{972}{1296}$, $\frac{567}{1296}$, $\frac{304}{1296}$, $\frac{630}{1296}$; $\frac{187}{306}$, $\frac{38}{306}$, $\frac{54}{306}$.

3. 12. 4. $\frac{17}{21}$, $\frac{7}{13}$, $\frac{11}{28}$, $\frac{15}{39}$. 5. $1065\frac{3}{7}$.

6. $\frac{4}{129}$. 7. $1\frac{1}{9}$. 8. $\frac{4}{19}$. 10. $\frac{16}{41}$, $\frac{1}{3}$, $\frac{7}{45}$.

V.—Page 34.

1. $\frac{7}{8}$, $2\frac{1}{30}$, $1\frac{1}{4}$. 2. $29\frac{13}{24}$. 3. $14\frac{211}{360}$.

4. $\frac{1}{8}$, $\frac{1}{30}$, $\frac{1}{56}$. 5. $2\frac{1}{4}$, $\frac{13}{45}$, $3\frac{19}{30}$. 6. $2\frac{25}{84}$.

7. $\frac{9}{56}$. 8. $5\frac{1}{12}$. 9. $\frac{3}{130}$.

10. $2\frac{22}{351}$. 11. $12\frac{14}{117}$. 12. $1\frac{1}{60}$.

VI.—Page 36.

1. $3\frac{7}{8}$, $1\frac{3}{8}$, $3\frac{23}{32}$. 2. $5\frac{67}{100}$. 3. 5.

4. $\frac{5}{162}$, $\frac{5}{12168}$. 5. $3\frac{7}{24}$. 6. 4.

7. $\frac{33}{1036}$. 8. $2\frac{662}{11125}$, $15\frac{11}{1912}$.

9. 77. 10. £8. $12\frac{2}{19}$s.

11. £352. 6s. 8d. 12. $20\frac{3}{4}$.

VII.—Page 41.

1. 2·203125. 2. $\frac{7}{20}$, $\frac{13}{500}$, $\frac{16}{99}$, $\frac{1}{7}$, $\frac{1}{6}$, $\frac{3}{7}$.
3. 3·703059239. 4. 5·098809263. 5. $15\frac{4}{11}$, $\frac{9}{11}$.
6. 1. 7. 2·718281. 8. ·321750.
9. ·367879. 10. 1·015873.
11. 3·141592. 12. ·857142.

VIII.—Page 43.

1. 13s. 4d., $7\frac{1}{5}$d., 2s. 8d., £1. 5s. $8\frac{4}{7}$d.
2. £4. 7s. 6d., 6s., 1s. $1\frac{7}{11}$d., 3s. 9d.
3. £5. 17s. 4d., £8. 15s., £25. 3s. $11\frac{4}{11}$d.
4. 17 cwt. 16 lbs., $16\frac{4}{5}$ lbs., $6\frac{2}{9}$ lbs., $2\frac{5}{11}$ lbs.
5. 3 m. 2 f. $62\frac{6}{7}$ yds., $293\frac{1}{3}$ yds., 4 po. $1\frac{4}{7}$ yd., $82\frac{3}{3}\frac{4}{5}$ yds.
6. 144 days, 32 days, 4 hrs. 54′ 47″.
7. 38 lbs. 7 oz. 2 dwt. $15\frac{5}{7}$ grs., 12 dwt. $6\frac{6}{49}$ grs.
8. – 245 ac. 1 r. 27 po. $12\frac{3}{2}\frac{1}{8}\frac{1}{8}$ yds.
9. 40° 3′ $28\frac{1}{8}$″, 35°.
10. 6153 grains.
11. $7\frac{7}{13}$; 4 cwt. 1 qr. $14\frac{2}{9}$ lbs.
12. 1 day 3 hrs. 8′ $24\frac{1}{1}\frac{5}{5}\frac{4}{7}\frac{9}{9}\frac{7}{5}$″.

IX.—Page 45.

1. $\frac{1}{12}$, $\frac{1}{16}$. 2. $\frac{7}{32}$, $\frac{13}{27}$. 3. $\frac{9}{2}\frac{9}{4}\frac{9}{0}$, $\frac{2}{2}\frac{2}{6}\frac{9}{8}\frac{9}{8}$.
4. 3 lbs. 7 oz. 15 dwts., $8\frac{7}{1}\frac{6}{7}\frac{6}{5}$ lbs.
5. $\frac{4}{5}\frac{1}{1}\frac{1}{6}$, $\frac{2}{5}\frac{2}{4}\frac{1}{0}$. 6. $\frac{1}{3}\frac{1}{3}\frac{2}{7}\frac{7}{5}$, $\frac{7}{2}\frac{7}{2}$.
7. $\frac{3}{14}$, $\frac{1}{51}$. 8. $1185\frac{5}{27}$, $\frac{1}{4}\frac{1}{8}\frac{4}{4}\frac{7}{0}\frac{7}{0}$.
9. $\frac{1}{17}\frac{1}{8}\frac{2}{8}\frac{1}{8}\frac{7}{6}\frac{5}{4}$. 10. $\frac{1}{8}\frac{6}{9}\frac{6}{25}$.
11. $\frac{5}{6}\frac{9}{0}$. 12. $\frac{1}{1}\frac{1}{4}\frac{1}{4}\frac{1}{0}$ day of 24 hrs.

X.—Page 47.

1. 7s. 6d., 19s. $7\frac{1}{2}$d., 16s. $3\frac{3}{4}$d.
2. £1. 5s., 2s. 9d., £1. 6s. 3d.

3. 12 cwt. 2 qrs., 2 cwt. 3 qrs. 7 lbs., 6 cwt. 1 qr. 25 lbs.

4. 1 ac. 2 r. $26\frac{3}{4}$ po., $13\frac{9}{10}$ po., 1 r. 22 po.

5. 4 quires 4 sheets, 11 sheets, 23 quires $4\frac{3}{5}$ sheets.

6. 38 galls. 3 qts. $0\frac{1}{4}\frac{1}{0}$ pts., 1 hhd. 60 galls. 2 qts. $1\frac{1}{5}$ pt. nearly, 1 pk. 0 gall. 3 qts. $1\frac{3}{4}$ pt. nearly.

7. 4000 grains, 8 oz. 6 dwts. 16 grs. 8. ·698 lbs.

9. 15s. 2d. 10. 24 tons 9 cwt. 2 qrs. 8 lbs.

11. 4 cwt. 1 qr. 10 lbs. 12. 3 oz. 14 dwts.

XI.—Page 48.

1. ·625, ·53125, ·55625, ·928125.

2. ·$\dot{8}46153\dot{}$, ·$\dot{6}92307\dot{}$, ·$\dot{6}15384\dot{}$. 3. ·$36803\dot{}$, ·$11805\dot{}$.

4. ·015625, ·$065\dot{4}76190\dot{}$. 5. ·003125, ·$00223\dot{2}142857\dot{}$.

6. ·$003472\dot{}$, ·$\dot{0}40293\dot{}$. 7. ·3125, ·150625.

8. ·$000223\dot{2}142857\dot{}$, ·$00083\dot{}$. 9. 1·$13085\dot{3}17460\dot{}$.

10. ·$82285714\dot{}$. 11.

12. ·0544575

XII.—Page 54.

1. 150000, 20000, 100, 270, 2·5, 1, 3·45, 5·294.

2. 46000000, 3000000, 2950000, 1500, 395, 29·5.

3. 2 myriag. 0 kilog. 2 hectog. 9 dekag., 1 myriag. 8 kilog. 0 hectog. 0 dekag. 8 grm. ·5 decig., 1 myriag. 2 kilog. 3 hectog. 0 dekag. 0 grm. 1 decig. 3 centig., 1 hectog. 2 dekag. 0 grm. 2 decig. 9 centig. 6 millig., 1 grm. 5 decig. 3 centig. 3 millig., 3 grm. 4 decig. 2 centig. 7 millig.

4. 160001·2, 25100, 396·45, 203550.

5. 1000, 2·96, 2900·03, 300·$12\dot{}$, 3765·43.

6. 10000000000, 10000000, 50000, 349800, 4600.

7. 150, 39·4, 90·2, 1860·3, 3·764, ·4.

8. 10, 1·234567, ·372456126, 1, ·000639, ·293.

9. 3203, ·4, 2000·003, 76·384, 29·34, 8300, 3457·6.

10. 18300·453, 1830·0453, 1830045·3.

11. 1, 73·6, 246·45, 2·55, 16·95.

12. 1300, 130; 713, 71·3; 1235, 123·5; 2·9, 320, 32, 1804, 180·4.

XIII.—Page 56.

1. 16287·599 m., 10738767 m., 1322·371 sq. m., 69·548396 cub. m., 39129·99 grm., 65632·02 ares, 368·93 st., 78603·982 lit., 288·06 fr.

2. 1600·688 m., 11696·359 m., 96·18 sq. m., 5·967600029 cub. m., 5972·935 grm., 2450·94 ares, 409·6 st., 694003·024 lit., 2·35 fr., 98·56 fr.

3. (1) 70·245 m., 110·385 m., 130·455 m.; (2) 486082·89 m., &c.; (3) 49 sq. m., &c.; (4) 76·190000095 cub. m., &c.; (5) 11046013·965 gr., &c.; (6) 36000·9 ar., &c.; (7) 41629 st., &c.; (8) 7347660·72 lit., &c.; (9) 2932·50 fr., &c.

4. 864 fr. 91·5 c. **5.** 3408 fr. $1\frac{7}{8}$ c.

6. (1) 1·56 m., 1·43 m., 1·32 m.; (2) 27788 m., 2613·75 m., 2460 m.; (3) 2·0910 sq. m., &c.; (4) 41·82 cub. m., &c.; (5) 188·263 grm., &c.; (6) 13·2 ar., &c.; (7) 10·01 st., &c.; (8) 40446·3 lit., &c.; (9) 293·58 fr., &c.

7. 25 fr., ·15010 fr., 24120 fr., 328 fr., 1·80 fr., 2857 fr. 64 c. nearly, 16·5 fr., 12 c., ·01 fr.

8. 25, 80, 2400, 1440, 480000, 14400, 4, 96, 125.

9. 1150 fr. **10.** 45. **11.** 36. **12.** 3877 nearly.

XIV.—Page 60.

1. 1609·314 m. **2.** 57319·8975 ft. **3.** 239·613 sq. ft.
4. 862784. **5.** 271·7. **6.** 1128 fr.
7. 67 fr. 62 c. nearly. **8.** 1053268765 galls.
9. £4. 3s. 4d. **10.** 2204 fr. 61 c.
11. 447·39. **12.** 1·0392.

XV.—Page 64.

1. £480. **2.** 8s. $11\frac{1}{2}$d. **3.** 344 fr. $48\frac{1}{4}$ c.
4. $39' 22\frac{1}{2}''$. **5.** ·70. **6.** 7 fr. $97\frac{1}{2}$ c.

7. £4. 9s. 7d. 8. 31 days. 9. £305.
10. 3 h. $34\frac{1}{1}\frac{0}{4}\frac{3}{4}\frac{8}{3}'$ P.M. 11. $32' 43\frac{7}{1\,1}''$ past 2.
12. 4 months.

XVI.—Page 67.

1. 15. 2. $30\frac{3}{4}\frac{3}{3}$ ft.
3. 1125 miles (take 8 kilom. = 5 miles).
4. £15,000. 5. $11\frac{1}{2}$ days. 6. $92\frac{2}{1\,9}$ days.
7. 2s. 8d. 8. 52·5 m. 9. £154.
10. 88 horses. 11. $7\frac{1}{2}$ miles. 12. 12 days.

XVII.—Page 69.

1. £70. 2. £41. 6s. $2\frac{1}{2}$d. 3. £39. 8s. $4\frac{1}{2}$d.
4. £4. 6s. $9\frac{3}{7}\frac{0}{3}$d. 5. £4. 4s. 6. £21. 18s. 5d.
7. $1\frac{1}{1}\frac{4}{5}\frac{1}{5}\frac{0}{3}$. 8. £236. 11s. 8d. 9. $3\frac{8}{9}\frac{3}{3}\frac{9}{5}$.
10. £385. 11. $5\frac{7}{9}$ months. 12. 15s. $5\frac{1}{2}$d.

XVIII.—Page 72.

1. £23. 3s. 5·856d. 2. £49. 5s. 6·84d.
3. £20. 3s. 10·149d. 4. £102. 17s. 4·941d.
5. 10s. 9·6d. 6. £441. 2 fl. 1 c. 1·40 m.
7. £270. 12s. 1·929d. 8. £200 ÷ $(1{\cdot}045)^3$.
9. £231525. 10. £71. 1s.
11. $£50\left\{(1{\cdot}05)^3 + (1{\cdot}05)^2 + (1{\cdot}05)\right\} = £165.\ 10s.\ 1\frac{1}{2}d.$
12. £450 ÷ $(1{\cdot}04)^3$.

XIX.—Page 75.

1. £12. 7s. 2·4d. 2. $3\frac{1}{2}\frac{9}{7}$d.
3. £4. 13s. 3·38d. 4. £4. 17s. 11·66d.
5. £3. 0s. $2\frac{7}{1\,1}\frac{4}{1}\frac{6}{0}\frac{}{7}$d. 6. £10. 11s. 8·7d.
7. £5. 12s. 6d. 8. £42. 10s.
9. £39. 7s. 6d. 10. £35.
11. £2212. 10s. 12. £326. 7s. $3\frac{2}{3}\frac{3}{6}$d.

XX.—Page 77.

1. £690. 18s. 9d.
2. £304. 14s. $8\frac{7}{10}$d.
3. £5528. 10s. nearly.
4. £382. 10s. 10d.
5. £6911. 3s. $6\frac{81}{100}$d.
6. £126. 14s. 10d. nearly.
7. £842. 12s. 6d.
8. £7768. 5s. $3\frac{3}{4}$d.
9. £863. 7s. 6d. nearly.
10. £134.
11. £281. 5s.
12. 50,000 francs.

XXI.—Page 80.

1. £374. 11s. $10\frac{2}{25}$d.
2. £1069. 10s. $11\frac{1}{2}$d. nearly.
3. £1773. 14s. $6\frac{1}{2}$d.
4. £393. 7s. $8\frac{1}{2}$d. nearly.
5. £308·8693.
6. 20.
7. 12s. 10d.
8. £522·0411.
9. £217·7937.
10. £1000 ÷ $(1{\cdot}05)^2$.
11. £23. 11s. nearly.
12. £369·8728.

XXII.—Page 82.

1. 5s. 5d.
2. 10s. 10d.
3. £1. 10s.
4. 6 to 5.
5. £48.
6. $4\frac{1}{2}$ gallons.
7. He loses $12\frac{1}{2}$ per cent.
8. £72,123. 12s. 6d.
9. $£\frac{2}{1}\frac{5}{8}\frac{5}{3}$.
10. $2\frac{1}{2}$.
11. £4. 5s.
12. $33\frac{1}{3}$.

Miscellaneous Examples.—Page 85.

1. 10.
2. £3. 5s. 10d.
3. 721 lbs., 321 kilog.
4. $\sqrt{\cdot 5} + \sqrt{\cdot 3}$, $\frac{1}{2}$.
5. ·0078125, $\frac{3}{1}\frac{3}{4}\frac{3}{8}$.
6. 2933·73.
7. 33·1, 3·236.
8. £2. 16s. $0\frac{1}{4}$d., he lost $12\frac{1}{3}$ per cent.
9. 12.
10. $\frac{4}{5}$, 6s., ·04895.
11. 0, 4, 2500, $1\frac{1}{1}\frac{1}{6}$.
12. £29. 6s. $7\frac{1}{2}$d., £0. 9s. $8\frac{1}{2}$d.
13. £8000 stock, £7530.
14. £·002, 34·3168.
15. 83·8967, 1·9387.
16. $£5\frac{1}{1}\frac{0}{9}$, $£11\frac{1}{1}\frac{6}{9}$, $£12\frac{1}{1}\frac{2}{9}$.
17. ·2036.
18. 240.
19. 4s. 7·4d. per ounce.
20. £1763. 1s.

21. £723, £3. 6s. 4d.
22. £60, £40, £100.
23. 5s., £1. 17s. 6d.
24. £1. 1s. $5\frac{1}{2}$d, $11\frac{20}{91}$.
25. £336, $\frac{291}{311}$.
26. $\frac{6664}{3330}\frac{1}{000}$, $9{\cdot}0\dot{9}$.
27. £3375. 28. $6\frac{4}{81}$. 29. $\dot{3}$. 30. 3000 days.

ALGEBRA—STAGE I.

I.—Page 149.

1. 8, 27, 6, − 16. 2. 6, 2, 4, 0. 3. 5, 11, − 7.
4. 11, 16. 5. − 2, − 15. 6. 8. 7. − 5, − 7.
8. − 11, − 3. 9. 7, − 69. 10. 72, − 68.
11. − 2, − 8. 12. − 10, 32.

II.—Page 154.

1. 51, 6, 3. 2. 125, 35. 3. − 513, − 65.
4. − 1224, 30. 5. 0, 2. 6. 1, − 27. 7. 2, − 2.
8. 1591, 0. 9. 144. 10. 1521.
11. 145. 12. $-\frac{37}{\sqrt[3]{1029}}$.

III.—Page 156.

1. $10a + 3b$. 2. $11a^2$. 3. $12b + 8c$. 4. 0.
5. $8a^2 + ab$. 6. $3x^4 - x^2 - 15x - 2$. 7. $4ab - 4$.
8. $x^3 + y^3 + z^3 - 3xyz$. 9. $x^4 + x^2y^2 + y^4$.
10. $a^3 + b^3 + c^3 + 3a^2b + 3ab^2 - a^2c - ac^2 - b^2c - bc^2$ $2abc$.
11. $x^4 + y^4 + z^4 - 4x^3y + 4x^3z - 4xy^3 - 4y^3z + 4xz^3$ $4yz^3 + 6x^2y^2 + 6x^2z^2 + 6y^2z^2 - 12x^2yz + 12xy^2z$ $12xyz^2$. 12. 0.

IV.—Page 157.

1. $4a + 2b + 5c$. 2. $-4x + 2y - 6z$.
3. $a^2 - 3ab - b^2 - 5a - 7b - 8$.

4. $2a^4 - 2a^2x^2 + x^4$. 5. $4a^4 + 8a^2b^2 + 4b^4$.

6. $-3x^3 - 3z^3 - 3x^2z - 3xz^2$.

7. $x^4 - ax^3 - 9a^2x^2 - 3a^3x - 2a^4$. 8. 0.

9. $-a - b - c - d + e + f + g + h$.

10. $2x^4 - 8x^3y + 12x^2y^2 - 8xy^3 + 2y^4$.

11. $x^4 - 2x^2y^2 + y^4$.

12. $a^2 + b^2 - 2c^2 + 2ab - 2ac - 2bc$.

V.—Page 159.

1. $-x + 8y + z$. 2. $a^3 - b^3$. 3. $13 - 6x$.

4. $5x^2 - 3x - 7$.

5. $(a + b) + (c - d)$, $a - (b - c + d)$,
$\{a - (b - c)\} - d$.

6. $-(6a - 7b) - (3c - 5d)$, $-6a + (7b - 3c + 5d)$,
$-\{6a - (7b - 3c)\} + 5d$.

7. $-(4x^3 - 12x^2y) - (12xy^2 - 4y^3)$,
$-4x^3 + (12x^2y - 12xy^2 + 4y^3)$,
$-\{4x^3 - (12x^2y - 12xy^2)\} + 4y^3$.

8. $(a^3 - b^3) - (c^3 - 3abc)$, $a^3 - (b^3 + c^3 - 3abc)$,
$\{a^3 - (b^3 + c^3)\} + 3abc$.

9. $-(a - b + d)x^2 - (a - b - 3c)xy$
$-(2b - c - e - f)y^2$.

10. $(a - b + c - d)x - (c - d + e - f)y$
$-(e - f + g - h)z$.

11. $(a - b - d)x - (a - 7b - c)y - (b - c + d)z$.

12. $(z - x)a^2 + (x - y)ab + (y - z)b^2$.

VI.—Page 161.

1. $12a^2 - ab - 6b^2$, $18x^2 - 9xy - 35y^2$.

2. $x^3 - 6xy^2 + 4y^3$, $30x^3 + 49x^2y + 9xy^2 - 4y^3$.

3. $a^4 - 2a^2b^2 + b^4$, $a^4 - b^4$. 4. $x^4 - y^4$, $x^8 - y^8$.

5. $a^3 + b^3 + c^3 - 3abc$. 6. $x^8 - y^8$.

7. $a^3 + 3a^2b + 3ab^2 + b^3$.

8. $5a^5 + 5a^4 - 405a - 405$.

9. $-7x^3+17x^2-5x-2$, a^3-x^3.
10. $a^8+2a^6b^2+3a^4b^4+2a^2b^6+b^8$.
11. $af+(ac+bf)x+(b+f)cx^2+(c^2+df)x^3+cdx^4$.
12. $x^3+(p-a)x^2+(q-ap)x-aq$.
13. $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$.

VII.—Page 165.

1. $x^2-2xy+y^2$, $9a^2-30ab+25b^2$, $16c^4+8c^2d^2+d^4$, $9x^4-12x^2y^2+4y^4$.
2. a^8-b^8. 3. $m^2x^2-n^2y^2$, $50a^4-18b^4$.
4. $(a+c)^2-(b+d)^2$, $(a+b)^2-(c+d)^2$.
6. x^2+4x-5, $x^2+7x+10$, $x^2+2x-15$, x^2-25.
7. $x^4-37x^2-24x+180$.
8. $\frac{1}{16}\{4a^2b^2-(a^2-b^2-c^2)^2\}$.
9. $4(a^2+b^2+c^2+d^2)$.

VIII.—Page 171.

1. $4a^2-ab+2b^2$, $xy-4y^2$.
2. $2a^4-5a^3b+\frac{20}{3}a^2b^2$, $2x^2+3xy+2y^2$.
3. $ax^{m-n}+bx^{-n}y^n+cx^{-(m+n)}y^{2n}$,
$ax^{m-n}+by^n+cx^{n-m}y^{2n}$.
4. $6x+4y$, $5x-3y$. 5. $1+x+x^2$.
6. $-3x-4$, $-7x+3$.
7. $x^3-3x^2y+5xy^2+27y^3+\dfrac{72y^4}{x-3y}$,
$x^3-x^2y-3xy^2+15y^3-\dfrac{24y^4}{x+y}$.
8. $x^4+x^3y+x^2y^2+xy^3+y^4$, $x^4-x^3y+x^2y^2-xy^3+y^4$.
9. ax^m+by^n. 10. $a^3+a^2b+ab^2+b^3$.
11. $b+c$, $a+b$. 12. $a+bx+cx^2$.
13. $a^4-(p-1)a^3-(p-q-1)a^2-(p-1)a+1$.
14. The given expression is—
$(a+b+c-d)(a+b-c+d)(a-b+c+d)(a-b-c-d)$.

16. $-x^4y^2 - x^2y^4$, $x + x^{-1}$.

17. $(x+y)^3 + (x+y)^2z - (x+y)z^2 - z^3$.

20. $(a-c)^2 - 2(a-c)(b-d) + (b-d)^2$.

IX.—Page 177.

1. $(x+3a)(x-3a)$, $(4y^2+5z^2)(4y^2-5z^2)$,
$6(2a+3b)(2a-3b)$, $(2x-3y)(4x^2+6xy+9y^2)$.

2. $x(x-y)(x^2+xy+y^2)$,
$(a-b)(a+b)(a^2+b^2)(a^4+b^4)$,
$xy(x+y)(x^2-xy+y^2)$,
$2xy^2z(x+2z)(x-2z)$.

3. $(a^2-2b^2)(a^2+2b^2)$, $(x^2+xy+y^2)(x^2-xy+y^2)$,
$(a+b)^2(a-b)^3(a+b+c)(a+b-c)$.

4. $(a+b+c+d)(a+b-c-d)$,
$(a+b-c+d)(a-b+c+d)$,
$(a+b-c)(a-b+c)$.

5. $5(2x+9)$, $3(2x+7)$, $(3a+b-c)(a+b-c)$.

6. $(x^2+xy+y^2)\{(x-y)(x^3-y^3)+4x^2y^2(x^2-xy-y^2)\}$,
$(x^2+xy+y^2)(x^2-xy+y^2)(x+y)^2(x-y)^2$.

7. $(x-10)(x+7)$, $(x+1)(x+10)$,
$(a-7b)(a-8b)$, $(x-16)(x+12)$.

8. $(ax+7by)(ax-6by)$, $3a(x-10)(x+2)$,
$ac(c-8)(c+3)$.

9. $(3x+5)(2x-7)$, $(2x+9)(4x-15)$,
$3(2x+3)(3x-8)$, $(4x-7)(5x+6)$.

10. $xy(3x+y)(x+3y)$, $x(5x+3a)(4x+5b)$,
$(mx+p)(nx+q)$.

11. x^3+2x^2+4x+8, $3x^4-6x^3+12x^2-24x+48$,
x^4+3x^2+9.

12. $(a+b)^2-(a+b)(c+d)+(c+d)^2$, $a-b+c+d$.

13. $a^4+pa^3+qa^2+ra+s$, $2a^4$. 14. -102, 17.

X.—Page 179.

1. a^4b^6, $-27a^3b^9$, $16a^8b^2c^2$, $-x^9y^2z^6$.

2. $a^4 + 12\,a^3b + 54\,a^2b^2 + 108\,ab^3 + 81\,b^4$,
$16\,a^4 + 32\,a^3b + 24\,a^2b^2 + 8\,ab^3 + b^4$,
$a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5$,
$27\,a^3 - 108\,a^2b + 144\,ab^2 - 64\,b^3$.

3. $16\,m^4 + 32\,m^3 + 24\,m^2 + 8\,m + 1$,
$125\,x^3 + 150\,x^2 + 60\,x + 8$,
$81\,a^4 - 432\,a^3c + 864\,a^2c^2 - 768\,ac^3 + 256\,c^4$,
$-a^3 - 3\,a^2b - 3\,ab^2 - b^3$.

4. $x^4 + 2\,x^3 + 3\,x^2 + 2\,x + 1$,
$9\,a^2 + b^2 + 16\,c^2 + d^2 - 6\,ab + 24\,ac - 6\,ad - 8\,bc + 2\,bd - 8\,cd$,
$a^2 + 4\,b^2 + c^2 + 4\,ab - 2\,ac - 4\,bc$,
$9\,(a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc)$.

5. $1 + 3\,x - 5\,x^3 + 3\,x^5 - x^6$, $(ax + by)^3 + 3\,(ax + by)\,cz$ + &c.

6. $1 + 7\,x + 21\,x^2 + 35\,x^3 + 35\,x^4 + 21\,x^5 + 7\,x^6 + x^7$,
$(1 + x)^4 + 4\,(1 + x)^3x^2 + 6\,(1 + x)^2x^4 + 4\,(1 + x)x^6 + x^8$,
$(a + bx)^4 + 4\,(a + bx)^3\,cx^2$ + &c.

7. $a^8x^8 - 8\,a^7bx^7y + 28\,a^6b^2x^6y^2 - 56\,a^5b^3x^5y^3 + 70\,a^4b^4x^4y^4 - 56\,a^3b^5x^3y^5 + 28\,a^2b^2x^2y^6 - 8\,ab^2xy^2 + b^8y^8$,
$729\,x^6 - 243\,x^4y^2 + 27\,x^2y^4 - y^6$,
$x^9 - 3\,x^6y^3 + 3\,x^3y^6 - y^9$.

8. $x^{12} - 2\,x^6y^6 + y^{12}$,
$a^{12} - 6\,a^{10}b^2 + 15\,a^8b^4 - 20\,a^6b^6 + 15\,a^4b^8 - 6\,a^2b^{10} + b^{12}$.

9. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$. 10. $(a - c)^3$.

11 $2\,(1 + 3\,x^2)^4$. 12. $81\,x^4y^4\,(x + y)$.

XI.—Page 193.

1. $2\,xy^2z^3$, $4\,a^2y^2$, $x^2 + a^2$. 2. $2\,x^3 - 3\,x^2y + 5\,xy^2$.
3. $5\,a^2 - 3\,ab + b^2$. 4. $1 - 2\,x + 3\,x^2 - 4\,x^3$.
5. $a + bx + cx^2 + dx^3$. 6. $a^2x^n - 3\,ax^{n-1} + 4\,x^{n-2}$.
7. $x + x^{-1}$, $ax^{-1} - a^{-1}x$ 8. $3\,x^{\frac{m}{2}} - \frac{a^2}{5}\,a$.
9. 36, 79, 207, 289. 10. $\dot{1}03{\cdot}2$, $\cdot024$, $\cdot\dot{3}$, $\frac{4}{9}$.
11. $4{\cdot}123$, $1{\cdot}224$, $\cdot618$, $3{\cdot}732$.
12. $\cdot0203$, $4{\cdot}\dot{6}$, $3{\cdot}5036$. 13. $2\,ab^2y^4$, $5\,x^4y$, $a + 2\,b$.

14. $x^4 + 3x^2 - 7$.
15. $x + y - c$.
16. $x + x^{-1}$, $xy^{-1} + 1$.
17. 18018, 11·11.
18. 2·73, $\cdot\dot{3}$, $\frac{3}{7}$.
19. ·5, ·2599.
20. $\sqrt[3]{5} = 1{\cdot}709$.
21. 7·6457, 50·2487.
22. 4·4142, 3·7320.
23. ·25.
24. 0.

XII.—Page 200.

1. $x - 3$.
2. $x^2 + 4x + 3$.
3. $x + 3$.
4. $x + a$.
5. $a^2 + ab + b^2$.
6. $x - 1$.
7. $x - 3$.
8. $3x - 2$.
9. $12x^2 + 5x$.
10. $3a + 2b$.
11. $x - 1$.
12. $2x^3 - 4x^2 + x - 1$.
13. $2a + 3b + c$.
14. $x^2 + 1$.
15. $x - 1$.
16. $2x^2 - 3x + 9$.
17. 1.
18. $a + b + c$.

XIII.—Page 203.

1. $12a^3x^3y^3$.
2. $60a^2b^2c^2$.
3. $(a - b)(b - c)(c - a)$.
4. $a^2x(x^2 - a^2)$.
5. $(x + 1)(x + 2)(x + 3)$.
6. $(x - 6)(x + 5)(x - 5)$.
7. $(2x + 7)(3x + 8)(4x + 5)$.
8. $210(x^2 + 1)(x^3 + 1)$.
9. $a^2(x^4 + a^2x^2 + a^4)$.
10. $(x + a)(x + b)(x + c)$.
11. $1 - x^{16}$.
12. $(x + 1)(x + 2)(x + 3)(x - 5)(x - 6)(x + 5)$.
13. $a(a - b)^3(a^2 - ab + b^2)$.
14. $(x - 1)(x + 1)(x^2 + 1)(6x^3 + 5x^2 + 2x - 1)$.
15. $(a^2 - b^2)^4$.
16. $2x(x^2 - 1)(3x^2 + 3x - 1)(2x^2 + 2x - 5)(2x^2 - 2x + 5)$.
17. $3(x - 2)^2(x + 4)(5x^3 + 10x^2 + 20x + 18)$.
18. $a^2b^2(a + b)(a - b)$.
19. $15(3x - 10)(x^4 - 16)$.
20. $(x^2 - y^2)^2(x^2 + y^2)(x^4 + y^4)$.
21. $x^6 - a^6$.
22. $(a + b + c + d)(a + b - c - d)(a - b - c + d)$.
23. $(a + b + c)(a^3 + b^3 + c^3 - 3abc)$.
24. $(a + b + c + d)(b + c + d - a)(a + c + d - b)$
$(a + b + d - c)(a + b + c - d)(a + b - c - d)$.

XIV.—Page 206.

1. $\frac{x-1}{x+6}, \quad \frac{x^2+x-2}{x^2+5x+5}$.

2. $\frac{3x+7}{7x+2}, \quad \frac{2x^2+2x+9}{5x^2+2x-2}$.

3. $\frac{(a-b)(a+b)^2}{a^2-3ab+b^2}, \quad \frac{4a^2+b^2}{a+3b}$.

4. $\frac{x(y-z)-y^2}{x(y+z)-y^2}, \quad \frac{1}{x^2-y^2}$.

5. $\frac{2a}{a^2-b^2}, \quad \frac{2b}{a^2-b^2}$.

6. $\frac{a-b}{a+b}$.

7. $\frac{x^3-x^2+3x-1}{2(x-1)(x^2+1)^2}$.

8. $\frac{10}{(x+1)(x+2)^2(x^2+1)}$.

9. $\frac{8x-20}{(x+1)^2(x+3)}$.

10. $\frac{8x^2+8}{(x-1)^4(x^3+1)}$.

11. 0. 12. 0. 13. 1. 14. $a+b+c$. 15. 1. 16. 0.

17. $-\frac{16a^5x}{(a^2-x^2)^2}$.

18. $\frac{a^2-ab+b^2}{a^2+ab-b^2}$.

19. 1. 20. 1.

21. $\frac{8}{(x-y)(y-z)(x-z)}$.

22. $\frac{1}{x^3}$.

23. $\frac{1}{(x-a)(x+b)(x+c)}$.

24. $\frac{3x^2}{a}-4x+5a$.

25. 1.

26. $\frac{4a^2b^2}{(a^2-b^2)^2}$.

27. $\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{a-b}\right)^2$.

28. $\frac{a^2(x-y)}{(a+x)(a+y)^2}$

29. 1.

30. $\frac{1}{x^2}$.

XV.—Page 212.

1. 3. 2. 4. 3. $a+b$. 4. 5.

5. 6. 6. 3. 7. 5. 8. 2.

9. c. **10.** $a + b$. **11.** $a + b + c$. **12.** $\dfrac{a^2 + ab + b^2}{b}$.

13. abc. **14.** $-(a + b)$. **15.** abc. **16.** $\dfrac{2a(2b^2 - 5)}{4a - 3b}$

17. 2. **18.** 8. **19.** $\frac{1}{4}$. **20.** $\dfrac{cd - ab}{a + b - c - d}$.

21. $\dfrac{a^2 + ab + b^2}{a + b}$. **22.** $\dfrac{1}{abc}$. **23.** $a + b + c$.

24. $\dfrac{1}{a + b + c}$.

XVI.—Page 217.

1. 6. **2.** $\dfrac{ab - cd}{a + b - c - d}$. **3.** 40.

4. $\frac{3}{3}\frac{9}{9}\frac{5}{7}\frac{6}{1}$. **5.** $-\frac{3}{1}\frac{1}{5}\frac{1}{9}$. **6.** $-\frac{7}{8}$.

7. $1\frac{1}{2}$. **8.** 3. **9.** $-1\frac{1}{2}$.

10. 8. **11.** $1\frac{4}{9}\frac{1}{3}$. **12.** $\dfrac{3 - m - n}{2a}$.

13. $-\dfrac{a^2 + b^2}{a + b}$. **14.** $2\left(\dfrac{1 - b}{1 + b}\right)^2 - \left(\dfrac{1 - b}{1 + b}\right)^4$.

15. 3. **16.** -8. **17.** 1. **18.** $\dfrac{b^2 + 1}{2ab}$.

19. $(\sqrt{a} + \sqrt{b})^2$. **20.** $-a$. **21.** $a^2 + 2a$.

22. $\frac{5}{7}$. **23.** $-2\frac{1}{2}$. **24.** $\dfrac{af(bc - ad) + (cf - ae)cd}{a^2(bc - ad) - (cf - ae)c^2}$.

XVII.—Page 221.

1. 10. **2.** 45, 25. **3.** £360, £240, £120.

4. 20, 15 miles per hour. **5.** 24.

6. 36s., 48s. **7.** 12. **8.** 36s. **9.** 30s.

10. 13. **11.** 10. **12.** 24 lbs., 40 lbs.

13. 13. 14. 500. 15. 15s. 16. $46\frac{6}{19}$.
17. 18 miles from A. 18. $\frac{2}{6}\frac{4}{4}\frac{4}{7}$ hr.
19. 15 per cent. 20. 28 miles. 21. 10.
22. $4\frac{1}{3}$, 4 miles per hour. 23. 12 miles.
24. 20 miles per hour in same direction.
25. 120. 26. 6s., 4s. 27. 2, 1, $\frac{4}{3}$, 6 hours.
28. 98 per cent. 29. £1,000. 30. $38\frac{2}{21}$ miles per hour.
31. 2 inches. 32. 16 vols. of hydrogen, 8 of oxygen.

33. $\frac{1 - 100\,c}{b - c}$, $\frac{100\,b - 1}{b - c}$. 34. $\frac{ab'c - a'bc'}{cc'(a'b - ab')}$.

35. $-40°$. 36. $\frac{a^2b}{c - ab}$.

XVIII.—Page 228.

1. 3, 4. 2. 5, 2. 3. 3, 7. 4. 4, 5.
5. 12, 8. 6. 2, 6. 7. 1, 2, 3. 8. 1, 2, 3.

9. 4, 1, 2. 10. $\frac{b_1c - bc_1}{ab_1 - a_1b}$, $\frac{ac_1 - a_1c}{ab_1 - a_1b}$.

11. $x = \frac{a + c - b}{2}$, &c. 12. $x = \frac{a + f - e}{2\,a}$.

13. 4, 5. 14. 3, 8. 15. 4, 8. 16. 3, 4.

17. 3, 5. 18. 8, 1. 19. $x = \frac{2}{a + c - b}$, &c.

20. 2, 3, 4, 1. 21. 3, 4, 5. 22. $x = \frac{a + c + d - 2b}{3}$, &c.

23. $x = \frac{m\,(a + c - b)}{2}$, &c. 30. $x = y = z = 1$.

XIX.—Page 231.

1. 12, 8. 2. 37. 3. 6 lbs., 5 stones.
4. $\frac{3}{7}$. 5. 17 horses, 24 cows.

6. 6s., 3s. 7. 220 and £2. 14s.
8. 200, 300, 400. 9. 48, 23, 18. 10. 5, 3, $4\frac{1}{2}$.
11. 4, 6, 8 hours. 12. 6, 10, 18. 13. $7\frac{1}{2}$, 12, 3.
14. 4, 3 miles per hour. 15. 16, 32, 48 miles per hour.
16. $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$,
$a_3x + b_3y + c_3z = d_3$, from which x, y, z.

ALGEBRA—STAGE II.

I.—Page 300.

1. ± 3. 2. ± 4. 3. ± 5.
4. $\pm \frac{3}{2}$. 5. ± 2. 6. $\pm (a + b)$.
7. ± 1. 8. $\pm \sqrt{a^2 - b^2}$. 9. $\pm \sqrt{2ab - b^2}$.
10. $\pm \frac{2}{3a}\sqrt{3}$. 11. $\pm a\sqrt{\frac{3b^2 + 1}{b^3 + 3b}}$.
12. $\pm \frac{a + 1}{2\sqrt{a}}$. 13. 2, 3. 14. 9, − 8.
15. 1, $-\frac{2}{3}$. 16. 4, $-\frac{7}{4}$. 17. $-\frac{7}{3}$, $-\frac{5}{2}$.
18. − 5, $-\frac{8}{5}$. 19. $\frac{1}{2}(b \quad \sqrt{b^2 - 4ac})$.
20. 5, $3\frac{1}{2}$. 21. 7, − 6. 22. 6, $-\frac{3}{2}$.
23. 1, $\frac{ad - (a + b)c}{bc}$. 24. b, a.
25. $a + b$, $\frac{ab}{a + b}$. 26. a, b. 27. $\frac{c}{a - b}$, $\frac{d}{a - b}$.
28. $\frac{a^2 + ab + b^2}{a - b}$, $\frac{a - b}{a^2 - ab + b^2}$. 29. 4, $\frac{3}{10}$.
30. 5, $-\frac{7}{5}\frac{3}{6}$. 31. 3, − 5. 32. 2, $-3\frac{1}{2}$.
33. 2, $-\frac{1}{2}\frac{3}{2}\frac{3}{9}$. 34. 5, $\frac{1}{4}$. 35. 8, $-\frac{8}{7}$.
36. 3, $-\frac{2}{2}\frac{6}{3}$. 37. 2, $2\frac{1}{2}$. 38. 4, $-\frac{5}{3}$.
39. 15, − 1. 40. 2, 0. 41. 0, $2ab$.

42. a, $3\frac{2}{5}\,a$. 43. 0, $\dfrac{a^2 + b^2 + 2\,b\sqrt{ab}}{(b - a)b}$.

44. 1, $\dfrac{a^2b^2(a^2 - b^2)^2}{\left\{(a^2 + b^2)^2 + 2\,ab(a + b)^2\right\}^2 - 4\,a^4b^4}$.

45. $a + b + c$, $-\dfrac{a^2 + b^2}{a - b}$.

II.—Page 306.

1. ± 3. 2. 2, $-\sqrt[3]{\frac{2}{5}^3}$. 3. ± 5, $\pm \dfrac{1}{\sqrt{5}}$.

4. 2, -5. 5. $\pm \frac{1}{5}$. 6. $\sqrt[n]{2}$, $\sqrt[n]{3}$. 7. 4, 69.

8. ± 4, $\pm \sqrt{7}$. 9. 3, -4, $3 \pm \sqrt{21}$. 10. $\frac{5}{3}$, 4.

11. -2, $-2 \pm \sqrt{3}$. 12. $\dfrac{a}{2}$.

13. 4, $\frac{1}{9}$, $-\frac{1}{6}(-13 \pm \sqrt{145})$. 14. 0, $b - a$.

15. $\pm \frac{1}{2}\,a\,\sqrt{3}$. 16. $\dfrac{a}{2} \pm \sqrt{b - 2\,a\,\sqrt{b} + \dfrac{a^2}{2}}$.

17. 0, $\left(\dfrac{a \pm b}{a \mp b}\right)^{\frac{2pq}{p - q}}$. 18. $\pm \dfrac{a}{\sqrt{2}}\sqrt{\sqrt{5} - 1}$.

19. $\left(\dfrac{a^{2m} \pm 1}{a^{2m} \mp 1}\right)^2$. 20. ± 5, $\pm \sqrt{\frac{3}{2}\frac{5}{3}\frac{1}{3}}$.

21. 2, $-\frac{1}{3}$, $\frac{1}{6}(-11 \pm \sqrt{97})$. 22. 4, $\frac{5}{6}$.

23. 5, $3\frac{1}{2}$, $\frac{1}{4}(-7 \pm \sqrt{53})$. 24. $3\,a$.

25. 3, -2. 26. 4, $-14\frac{1}{2}$, $\frac{1}{2}(-19 \pm \sqrt{741})$.

27. 3, $-2\frac{1}{2}$, $\frac{1}{1}\frac{5}{3}$. 28. $\frac{1}{2}(1 \pm \sqrt{4\,a - 3})$.

29. ± 1, and $x^4 + x^2 + 1 = 0$. 30. $\pm \sqrt{\dfrac{a + b + c}{3}}$.

31. a, $\frac{1}{2}\left\{\sqrt{4\,c^2 - 3\,a^2 + 2\,ab + b^2} - (a - b)\right\}$.

32. $x = m$ is one solution.

33. $x = a + b + c$ is one solution.

34. $x = \frac{a}{m}$, and $a^2 + ax + x^2 = 0$.

35. $x = \frac{b + d}{a + c}$ is one solution.

36. $x^2 - x + 1 = 0$ gives two solutions.

III.—Page 310.

1. $x = 2, 3, y = 3, 2.$ 2. $x = 5, -3, y = 3, -5.$

3. $x = 3, 4, y = 4, 3.$ 4. $x = 2, 4, y = 4, 3.$

5. $x = 5, -2, y = 2, -5.$ 6. $\pm 7, \pm 6.$

7. $\pm 6, \pm 3.$ 8. $4, 2.$ 9. $\pm 2, \pm 7.$

10. $\pm 4, \pm 3.$ 11. $x = \pm 3, \frac{1}{\sqrt{3}}, y = \pm 2, -\frac{8}{\sqrt{3}}.$

12. $x = 9, 4, y = 4, 9.$ 13. $x = 2, 8, y = 8, 2.$

14. $x = 3, 2, y = 2, 3.$

15. $x = \frac{2}{a \pm \sqrt{2b^2 - a^2}}, \; y = \frac{2}{a \mp \sqrt{2b^2 - a^2}}.$

16. $\pm \frac{(a + 1)b}{\sqrt{2(a^2 + 1)}}, \; \pm \frac{(a - 1)b}{2\sqrt{a^2 + 1}}.$

17. $\pm \sqrt{\frac{a + b^2}{2b}}, \; \pm \sqrt{\frac{a - b^2}{2b}}.$

18. $x = 2, 3, y = 3, 2.$ 19. $x = 2, 3, y = 3, 2.$

20. $x = 2, 3, y = 3, 2.$ 21. $x = 5, -3, y = 3, -5.$

22. $x = \frac{a + \sqrt{2b - a^2}}{\sqrt[3]{4a^2 - 4b}}.$ 23. $x = 2, 3, y = 3, 2.$

24. $x = \pm 3, \mp 1\frac{5}{8}\sqrt{3\frac{2}{5}}, \; y = \pm 2, \pm \sqrt{3\frac{2}{5}}$

25. $x = 4, 9, y = 9, 4.$ 26. $x = 4, 1, y = 8, 0.$

27. $x = 4, 3, y = 3, 4.$ 28. $\pm 3, \pm 1.$

29. $x = 0, 3, -1\frac{3}{1}\frac{5}{4}\frac{2}{5}, \; y = 0, 2, 1\frac{7}{1}\frac{6}{4}\frac{8}{5}.$

30. 0, − 1. 31. $x = 4, \frac{1}{5}\frac{7}{5}\frac{0}{9}\frac{8}{}$, $y = 3, -\frac{6}{5}\frac{2}{5}\frac{4}{9}\frac{3}{}$.

32. $x = 5, 6, y = 4$. 33. $x = 3, 1, y = 1, 0$.

34. $x = 3, -\frac{1}{4}\frac{3}{5}\frac{8}{}$, $y = 1, -\frac{4}{4}\frac{6}{5}$.

35. $x = 2, 7, y = 4, -14$.

36. $x + y = c$, and $xy = bx + ay$.

37. $x = 0, a - b, y = 0, a - b$.

38. $\frac{5}{2}\frac{1}{} \pm 9\sqrt{\frac{3}{1}\frac{5}{2}}, \frac{1}{2} \pm \sqrt{\frac{3}{1}\frac{5}{2}}$.

39. $x = \sqrt{\dfrac{(a + c - b)(a + b - c)}{2(b + c - a)}}$.

40. 1, 2, 3. 41. ± 1, 0, 0.

42. $x = y = z = 0$, also $\dfrac{1}{xy} = \frac{1}{2}\left(\dfrac{1}{b^2} + \dfrac{1}{c^2} - \dfrac{1}{a^2}\right)$,

$\dfrac{1}{yz} = \frac{1}{2}\left(\dfrac{1}{a^2} + \dfrac{1}{c^2} - \dfrac{1}{b^2}\right)$, &c.

43. $\dfrac{a}{\sqrt[4]{a^2 + b^2 + c^2}}, \dfrac{b}{\sqrt[4]{a^2 + b^2 + c^2}}, \dfrac{c}{\sqrt[4]{a^2 + b^2 + c^2}}$.

44. $x = y = z = u = a + b$.

45. $\dfrac{a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}, \dfrac{a^{\frac{1}{2}}b^{\frac{1}{2}}c^{-\frac{1}{2}}}{(a + b + c)^{\frac{1}{4}}}$, &c.

46. $x = 6, 4, y = 4, 6, z = 5$.

IV.—Page 315.

1. 9, − 6. 2. Numerator $2\frac{8}{9}\frac{9}{5}$, denominator 3.

3. 35 or 23. 4. 10, − 16. 5. 78, − 68 ft.

6. 2s., 9s. 7. 20 hours. 8. 48 shillings.

9. 3 ft., 2 ft. 10. 1, $\frac{1}{2}$. 11. 3 ft., 4 ft., 5 ft.

12. $\dfrac{\sqrt[n-1]{a_1a_2a_3\ldots a_n}}{a_1}, \dfrac{\sqrt[n-1]{a_1a_2a_3\ldots a_n}}{a_2}$, &c.

13. 2, 3, 6 days.

14. Solve the equations $P(R^3 - 1) = a$, $PR^2(R - 1) = b$, where P is the principal, and R the amount of £1 for one year.

15. Let x be the equated time, then x is the root of the equation—$240\left\{a(m-x) - b(x-n)\right\} = b(x-n)(m=x)$.

16. $x = y = z = \dfrac{1}{\sqrt{2}}$.

17. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

V.—Page 318.

1. a^2, a^2, $a^{\frac{1}{2}}$, a.
2. $a + x$, $(a - x)^{\frac{1}{2}}$.
3. $a + a^{\frac{1}{2}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$.
4. $x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}$.
5. $xy^{-1} + x^{-1}y + 1$.
6. $a^2 + ab^{\frac{1}{3}} + b^{\frac{2}{3}}$, $a^{\frac{3}{8}} + b^{\frac{3}{8}}$.
7. $x - x^{\frac{1}{2}}y$.
8. $a^{\frac{2}{3}} + b^{\frac{2}{3}} + c^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{3}}c^{\frac{1}{3}} - b^{\frac{1}{3}}c^{\frac{1}{3}}$.
9. $a^2b^{-\frac{1}{3}} - b^{\frac{2}{3}}$.
13. $a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}}$.
14. $2\,x^{\frac{1}{2}}y^{\frac{2}{3}} - 3\,xy^{\frac{1}{3}} + 2\,x^{\frac{3}{2}}$.
15. $ab^{-\frac{1}{2}} - 2 + 2\,a^{-1}b^{\frac{1}{2}}$.
16. $x^{\frac{4}{3}} + 3\,x^{\frac{2}{3}} - 7$.
17. $x^2y^{-\frac{1}{3}} + 1$.
18. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^2$.

VI.—Page 325.

1. $x^{\frac{5}{2}}$, $a^{\frac{5}{3}}b^{\frac{2}{3}}$, $x^{\frac{1}{2}}y^{\frac{3}{4}}$, $a^{\frac{1}{4}}b^{\frac{1}{2}}$.
2. $xya^{-\frac{1}{2}}b^{-\frac{1}{2}}$, $c^{\frac{3}{2}}d^{\frac{1}{2}}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$, $a^{-\frac{n}{m}}x^{\frac{m}{n}}$.
3. $\sqrt{27}$, $\sqrt{24}$, $\sqrt{2\frac{7}{5}}$, $\sqrt[3]{9}$.
4. $\sqrt{32}$, $\sqrt{27}$, $\sqrt[4]{32}$, $\sqrt{2\frac{7}{8}}$.
5. $\sqrt{9\,ab}$, $\sqrt{abc}$, $\sqrt[3]{a^2 - x^2}$.
6. $\sqrt[6]{4}$, $\sqrt[6]{27}$.
7. $\sqrt[12]{8}$, $\sqrt[12]{81}$.
8. $\sqrt{8}$, $\sqrt[3]{135}$.
9. $\sqrt[mn]{a^n}$, $\sqrt[mn]{b^m}$.
10. $\sqrt[6]{(a + x)^3}$, $\sqrt[6]{(a - x)^2}$.
11. $a^{\frac{p+q}{pq}}$, $b^{\frac{q-p}{pq}}$.
12. $2\sqrt{3}$, $2\sqrt[3]{6}$, $6\sqrt{7}$, $3\sqrt[3]{3}$.
13. $2\,a\sqrt{a + b}$, $b\sqrt[3]{a^3 + b^3}$, $\dfrac{1}{4\,a}\sqrt[3]{a^3 + a^2b}$.
14. $\dfrac{1}{x} - a$, $\dfrac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}} - a^{\frac{2}{3}}x^{\frac{1}{3}}$.

15. $\frac{x+1}{3x}\sqrt{3x}$, $\frac{1}{x-a}\sqrt[3]{(x^2-a^2)^2}$.

16. $4\sqrt{3}$, $5\sqrt[3]{7}$. 17. $\frac{ay-ab+c}{b}\sqrt{ab^2x^2}$.

18. $(3a^2b - 2a^{m+3} - 12)\sqrt[3]{a^m}$.

19. $a^{\frac{3}{2}} - b^{\frac{3}{2}}$, $a - b$. 20. $(x+y)^{\frac{5}{6}}$, $a^3 - b^3d^{\frac{3}{2}}$.

21. $\frac{ac^3}{d} - c\sqrt{\frac{ab^3}{a+b}} + \frac{c^2}{d}\sqrt{\frac{a}{b}(a+b)} - b$.

22. $a - b^{\frac{2}{3}} + c^{\frac{1}{2}} - d^{\frac{2}{5}} + 2a^{\frac{1}{2}}c^{\frac{1}{4}} - 2b^{\frac{1}{3}}c^{\frac{1}{5}}$.

23. $x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

24. $x^{\frac{5}{2}} - x^2y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - xy + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{3}}$.

25. $\frac{1}{3}\sqrt{15}$, $\frac{4}{7}\sqrt{7}$, $\frac{1}{5}\sqrt[3]{175}$.

26. $3(2 - \sqrt{3})$, $\frac{2}{3}(3\sqrt{2} - 2\sqrt{3})$, $\frac{1}{2}(\sqrt{5} + \sqrt{3})$.

27. $\frac{3}{4}(2 + \sqrt{2} - \sqrt{6})$, $\frac{1}{6}(2\sqrt{3} + 3\sqrt{2} - \sqrt{30})$, $5 + 2\sqrt{6}$.

28. $a(x^{\frac{5}{2}} - x^2y^{\frac{1}{3}} + x^{\frac{3}{2}}y^{\frac{2}{3}} - xy + x^{\frac{1}{2}}y^{\frac{4}{3}} - y^{\frac{5}{3}})$,

$\frac{1}{2^{\frac{1}{3}}}(3^{\frac{5}{2}} - 3^2 . 2^{\frac{1}{3}} + 3^{\frac{3}{2}} . 2^{\frac{2}{3}} - 3 . 2 + 3^{\frac{1}{2}} . 2^{\frac{4}{3}} - 2^{\frac{5}{3}})$,

$\frac{b}{x^2+xy+y^2}(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$.

29. $2 + \sqrt{7}$, $\sqrt{5} + \sqrt{3}$, $5 - \sqrt{5}$.

30. $\sqrt{6} + \sqrt{2}$, $\sqrt{6} - \sqrt{3}$, $\sqrt{15} - \sqrt{5}$.

VII.—Page 330.

1. $a^2 + b^2 : a^2 - b^2$ is the greater, if $a > b$.

2. The former. 3. $b - \frac{ad}{c}$.

4. $(1 - y)(1 + x) : 1 + x^2$. 5. 30, 35.

6. Less than 5 when x lies between 2 and 3 ; greater than 5 for all values beyond these limits.

8. x satisfies the equation—
$4x^2 - (a + b - 6c)x + 2(c^2 - ab) = 0.$

9. $\frac{b}{2}$. 10. 1. 14. $2b - a$, $\frac{b^3}{a^2}$, $\frac{ab}{2a - b}$.

PLANE TRIGONOMETRY.

I.—Page 335.

1. $43{\cdot}750^g$, $11° 50' 3{\cdot}03''$. 2. $88^g\ 88'\ 88{\cdot}\dot{8}''$.

3. $\frac{10}{9}n$ the number of grades.

4. $180 - (a + \frac{9}{10}b)$ degrees, $200 - (\frac{10}{9}a + b)$ grades.

5. $\frac{1}{3}$. 7. $\frac{9}{8}$. 8. 25^g, 75^g, $90°$. 9. $576°$, 648^g.

10. 2nd, 1st, 4th, 3rd, 4th. 11. $\frac{9}{10}$, $\frac{b}{a}$. 12. $\frac{90°.p}{qa}$.

II.—Page 341.

1. Cos A $= \frac{5}{13}$, tan A $= \frac{12}{5}$ &c.

2. Sin A $= \frac{24}{\sqrt{1201}}$, cos A $= \frac{25}{\sqrt{1201}}$, &c.

17. $\frac{1}{2}$. 18. $\frac{1}{10}(2\sqrt{89} - 5)$. 19. $\frac{\sqrt{3} + 1}{2\sqrt{2}}$.

20. $2 + \sqrt{3}$. 21. $\infty, -\frac{a^2 - b^2}{2ab}$. 22. 1.

24. Sin A $= \sqrt{\frac{a - (m + n)}{a + b}}$.

III.—Page 352.

2. From 0° to 90°, + ; from 90° to 180°, − ; from 180° to 270°, + ; from 270° to 360°, − .

3. Cos A + sin A is +, +, −, −, as A lies between − 45° and 45°, 45° and 135°, 135° and 225°, 225° and 315° respectively. The corresponding signs of cos A − sin A are +, −, −, +.

4. Cos 2 A is +, −, +, − respectively, as in Question 3.

6. $-\frac{1}{2}, \frac{\sqrt{3}-1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$.

10. A = 60°. 11. A = 45°, and tan A = − 6.

12. A = 30°. 13. A = 30°.

14. A = 45°, B = 15°. 15. A = 0°, 45°, 135°.

16. A = 6°. 17. A = 60°. 18. $\theta = 2 \pm \sqrt{3}$.

IV.—Page 358.

1. 2, 3, $\frac{1}{2}$, − 1, − 2, 1·5. 2. $\frac{5}{3}$, 6, 1·5.

3. 1·0791812, 1·5563025, 1·6532125, 1·8750613, $\bar{2}$·6020600, ·5740313, $\bar{1}$·8239087, $\bar{2}$·8696761.

4. 1·3172901, 3·3172901, $\bar{2}$·3172901, $\bar{3}$·3172901.

5. 1, $\bar{1}$, $\bar{3}$, $\bar{3}$.

6. ·020912, 2091200, ·20912, 20912. 7. $\bar{1}$·3162760.

8. 3493·768. 9. ·2427189. 10. $\bar{3}$·1303338.

11. 6, 5. 12. 1·67.

V.—Page 365.

1. 1·6774495. 2. 6·7193696. 3. $\bar{1}$·8213310.

4. ·0616835. 5. 2·07892. 6. £177·63 nearly.

7. £663·449. 8. 2620·248. 9. 9·7299222.

10. 9·8401608. 11. 36° 26′ 30″.

VI.—Page 372.

1. $c = 12{\cdot}5$. 2. $\sqrt{189}$. 3. $\operatorname{Tan}\frac{A}{2} = \frac{5}{27}$.

4. 480. 5. $108\sqrt{2}$. 6. $B = 90°$, $C = 30°$.

9. $C = 50\sqrt{6}(\sqrt{3}-1)$, $b = 100(\sqrt{3}-1)$.

10. $a = 10\sqrt{949}$.

VII.—Page 376.

1. $b = 15{\cdot}78065$. 2. $a = 92{\cdot}34057$.

3. $b = 25$, $A = 67° 22' 49''$.

4. $A = 40° 59' 32''$, $c = 111{\cdot}288$.

5. $b = 22{\cdot}78438$. 6. $b = 691{\cdot}49$.

7. $b = 12\sqrt{3}$, $c = 24$.

8. $a = \frac{5}{2}(\sqrt{6}-\sqrt{2})$, $b = \frac{5}{2}(\sqrt{6}+\sqrt{2})$.

9. $A = 30°$, $b = 17\sqrt{3}$. 10. $A = 30°$.

11. $CD = 7(\sqrt{6}-\sqrt{2})$. 12. $DE = 5\sqrt{3}$.

VIII.—Page 380.

1. $a = 6\sqrt{3}$, $B = 90°$, $C = 30°$.

2. $B = 45°$ or $135°$, $C = 105°$ or $15°$, $c = 9(\sqrt{6}\pm\sqrt{2})$.

3. $A = 60°$, $B = 45°$, $C = 75°$.

4. $A = 105°$, $b = 6(3\sqrt{2}-\sqrt{6})$, $c = 12(2-\sqrt{3})$.

5. $A = 75°$, $B = 60°$, $c = 2\sqrt{6}$.

6. $B = 60°$ or $120°$, $C = 75°$ or $15°$, $c = 5(3\pm\sqrt{3})$.

7. $B = 41° 59' 23''$, $C = 97° 28' 37''$.

8. $A = 19° 18' 11''$.

9. $A = 30° 3' 26''$, $B = 123° 24' 34''$.

10. $A = 48° 11' 22''$, $B = 58° 24' 42''$.

11. $B = 35° 15' 52''$, $C = 84° 44' 8''$, $c = 137{\cdot}9796$.

12. $A = 44° 24' 56''$.

IX.—Page 384.

1. $200\,(2 - \sqrt{3})$.
2. $60\sqrt{3}$.
3. $\frac{1}{2}\sqrt{3}$.
4. $6\sqrt{2}$.
5. $6\,(\sqrt{3} - 1), 3\sqrt{6}\,(\sqrt{3} - 1)$.
6. $30\,(\sqrt{3} + 1)$.
7. $\frac{100}{3}\sqrt{3}$.
8. $10\sqrt{3}$.

13. $\sqrt{\left\{\left(\frac{x + y - c}{2}\sec\theta\right)^2 + (c - y)\,x\right\}}$,
$\sqrt{\left\{\left(\frac{x + y - c}{2}\sec\theta\right)^2 + (c - x)\,y\right\}}$.

WILLIAM COLLINS AND COMPANY, PRINTERS, GLASGOW.

Collins' Advanced Science Series.

PURE MATHEMATICS,

INCLUDING THE HIGHER PARTS OF

ALGEBRA AND PLANE TRIGONOMETRY,

TOGETHER WITH

ELEMENTARY SPHERICAL TRIGONOMETRY.

BY

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LONDON AND GLASGOW:
WILLIAM COLLINS, SONS, & COMPANY.
1875.

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TO THE REVEREND

DAVID JAMES VAUGHAN, M.A.,

HONORARY CANON OF PETERBOROUGH CATHEDRAL;

VICAR OF ST. MARTIN'S, LEICESTER;

AND FORMERLY FELLOW OF TRINITY COLLEGE, CAMBRIDGE,

This Volume is Dedicated

IN APPRECIATION OF

HIS ZEAL IN THE CAUSE OF EDUCATION,

AND IN GRATEFUL ACKNOWLEDGMENT OF

HIS PERSONAL SYMPATHY

DURING MANY YEARS.

PREFACE.

THIS Volume is intended to form, with the volume previously written, a complete treatise on Algebra and Plane Trigonometry, or at least to include as much of these subjects as is usually read.

I have, therefore, not at all confined myself to the Government Pass Syllabus for Pure Mathematics, Stage III., but have treated the subjects with sufficient fulness to enable a student to take up the Honours Papers.

The book, which is complete in itself, may be considered as an Advanced Text-Book in those subjects; and will, I presume to hope, be found useful to all who wish to pass beyond the mere rudiments of Mathematical Science.

It is unnecessary for me to say anything here as to the place Mathematics should hold as a branch of education. I may, however, remark that the recent alterations in this respect, made in the Training College Curriculum of the Education Department, show that the Government are thoroughly alive to its importance.

I have included in the volume the elementary portions of Spherical Trigonometry, for the use of those students who may prefer to take it up as an alternative subject in the Pass Examination of the Science and Art Department.

E. A.

LEICESTER, *April, 1875.*

CONTENTS.

SECTION I.—ALGEBRA.

CHAP. I.—THE THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS, ... 9

,, II.—VARIATION, ... 14

,, III.—PERMUTATIONS AND COMBINATIONS, . . . 18

,, IV.—ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION, ... 28

,, V.—THE BINOMIAL THEOREM, ... 42

,, VI.—INDETERMINATE COEFFICIENTS AND PARTIAL FRACTIONS, ... 66

,, VII.—THE EXPONENTIAL THEOREM—LOGARITHMIC SERIES, ... 73

,, VIII.—THE MULTINOMIAL THEOREM, ... 85

,, IX.—INTEREST AND ANNUITIES, ... 92

,, X.—CONTINUED FRACTIONS, ... 103

,, XI.—SERIES, ... 123

,, XII.—SCALES OF NOTATION, ... 149

,, XIII.—PROPERTIES OF NUMBERS, ... 158

,, XIV.—THEORY OF INDICES—IMAGINARY QUANTITIES, 174

SECTION II.—PLANE TRIGONOMETRY.

CHAP. I.—FUNCTIONS OF THE SUM AND DIFFERENCE OF ANGLES, ... 185

,, II.—CIRCULAR MEASURE. GENERAL VALUES OF ANGLES, ... 210

CHAP. III.—INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES OF A TRIANGLE. QUADRILATERALS, . 219

„ IV.—SUBSIDIARY ANGLES, ... 228

„ V.—HEIGHTS AND DISTANCES, ... 236

„ VI.—DEMOIVRE'S THEOREM. CALCULATION OF THE VALUE OF π, ... 246

„ VII.—SUMMATION OF TRIGONOMETRICAL SERIES, . 279

„ VIII.—CONSTRUCTION OF TRIGONOMETRICAL TABLES, . 290

SECTION III.—SPHERICAL TRIGONOMETRY.

CHAP. I.—DEFINITIONS AND INTRODUCTORY PROPOSITIONS, 303

„ II.—SPHERICAL TRIANGLES, ... 306

„ III.—RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SIDES AND ANGLES OF SPHERICAL TRIANGLES, ... 312

„ IV.—SOLUTION OF SPHERICAL TRIANGLES, . . 326

„ V.—INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES, ... 350

„ VI.—AREA OF A SPHERICAL TRIANGLE. SPHERICAL EXCESS, ... 355

„ VII.—POLYHEDRONS, ... 359

ANSWERS, ... 368

MATHEMATICS.

THIRD STAGE.

SECTION I.

ALGEBRA.

CHAPTER I.

THE THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS.

1. *A quadratic equation cannot have more than two roots.*

It has been shown (Vol. I., 176) that $f(x)$ is divisible by $x - a$, when it vanishes on putting a for x; that is, when a is a root of the equation $f(x) = 0$.

Hence, if α, β are roots of the quadratic equation

$$x^2 + px + q = 0,$$

$x - \alpha$ and $x - \beta$ are each of them factors of the expression $x^2 + px + q$, whatever be the value of x.

Moreover, $x - \alpha$ and $x - \beta$ are the only linear factors of $x^2 + px + q$, since their product gives an expression of two dimensions whose first term is x^2, the same as that of the expression $x^2 + px + q$. We therefore have identically

$$x^2 + px + q = (x - \alpha)(x - \beta).$$

Hence no value of x can make $x^2 + px + q$ vanish unless

it make one of the factors $x - \alpha$, $x - \beta$ vanish; that is, $x^2 + px + q$ cannot vanish for any other values of x than α and β.

It follows, therefore, that α, β are the only roots of the equation

$$x^2 + px + q = 0.$$

2. *If* α, β *are the roots of the equation* $x^2 + px + q = 0$, *then* $\alpha + \beta = -p$, *and* $\alpha\beta = q$.

For (Art. 1) we have *identically*,

$$x^2 + px + q = (x - \alpha)(x - \beta), \text{ or,}$$
$$= x^2 - (\alpha + \beta)x + \alpha\beta.$$

Here we have the coefficient p on the first side represented by $-(\alpha + \beta)$ on the second side, and the constant term q represented by $\alpha\beta$.

We then have

$$-(\alpha + \beta) = p, \text{ or } \alpha + \beta = -p,$$
$$\text{and } \alpha\beta = q. \quad Q.E.D.$$

Cor. 1. It easily follows that

(1.) $\alpha^2 + \beta^2 = p^2 - 2q.$

(2.) $\alpha - \beta = \sqrt{p^2 - 4q}.$

(3.) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = -\dfrac{p}{q}.$

Cor. 2. If the given equation be of the form $ax^2 + bx + c = 0$, we may write it thus:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

and we have

(1.) $\alpha + \beta = -\dfrac{b}{a}.$

(2.) $\alpha\beta = \dfrac{c}{a}.$

(3.) $\alpha^2 + \beta^2 = \dfrac{b^2 - 2ac}{c^2}$

$$(4.) \qquad \alpha - \beta = \frac{1}{a}\sqrt{b^2 - 4ac.}$$

$$(5.) \qquad \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}.$$

3. ***The quadratic expression*** $ax^2 + bx + c$ ***is a perfect square when*** $b^2 = 4ac.$

When $ax^2 + bx + c$ is a perfect square, its *linear* factors must be identical; and hence the condition is that the equation

$$ax^2 + bx + c = 0$$

must have *equal roots.*

Solving this equation we get

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}).$$

Now these two values of x can be equal only when

$$b^2 - 4ac = 0, \text{ or when } b^2 = 4ac.$$

Hence the condition that $ax^2 + bx + c$ shall be a perfect square is that $b^2 = 4ac.$

Cor. The roots of the equation $ax^2 + bx + c = 0$ are *rational* when $b^2 - 4ac$ *is a perfect square,* and are *impossible* when $b^2 < 4ac.$

4. ***To investigate the relations between the coefficients*** A, B, C, a, b, c, ***in order that the expression***

$$Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy$$

shall be a perfect square.

We must have *identically*

$$\begin{aligned} &Ax^2 + By^2 + Cz^2 + ayz + bxz + cxy \\ &= (\sqrt{A}\,.\,x + \sqrt{B}\,.\,y + \sqrt{C}\,.\,z)^2 \\ &= Ax^2 + By^2 + Cz^2 + 2\sqrt{BC}\,.\,yz + 2\sqrt{AC}\,.\,xz \\ &\quad + 2\sqrt{AB}\,.\,xy. \end{aligned}$$

Equating the coefficients of identical terms, we have

$$a = 2\sqrt{BC}, \quad b = 2\sqrt{AC}, \quad c = 2\sqrt{AB},$$

the relations required.

Cor. Solving for A, B, C, we may put these relations in the following form :

$$A = \frac{bc}{2a}, \quad B = \frac{ac}{2b}, \quad C = \frac{ab}{2c}.$$

5. *The values of* x *and* y *derived from the equations*

$$ax + by = \frac{a^3}{x} + \frac{b^3}{y} = c^2$$

are rational, when $a^2 + b^2 = c^2$.

For, solving for x, we get—

$$ac^2x^2 + (b^4 - c^4 - a^4)\,x + a^3c^2 = 0.$$

Hence (Art. 3, Cor.) the values of x are rational when $(b^4 - c^4 - a^4)^2 - 4\,ac^2 \,.\, a^3c^2$ is a perfect square.

The values of x are therefore rational when

$(b^4 - c^4 - a^4)^2 = 4\,a^4c^4$, or

$b^4 - c^4 - a^4 = \pm 2\,a^2c^2$, or

$(a^2 \pm c^2)^2 = b^4$, or when $a^2 \pm c^2 = \pm b^2$.

And we may similarly show that the values of y are rational when

$$b^2 \pm c^2 = \pm a^2.$$

Now, on examination it will be found that the only condition common to these two sets is that

$$a^2 + b^2 = c^2.$$

Hence the values of x and y are both rational when

$$a^2 + b^2 = c^2. \quad Q.E.D.$$

6. *To investigate the condition that the expression*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

is resolvable into elementary factors.

If the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ be solved either for x or y, the condition that either can be expressed in terms of the other in a rational form will (Art. 1) be the condition that the given expression can be broken up into elementary rational factors.

Solving for x, we find x equal to

$$-\frac{By + D \pm \sqrt{(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)}}{2A}.$$

Now, in order that this may be rational, we must have

$$(B^2 - 4AC)y^2 + 2(BD - 2AE)y + (D^2 - 4AF)$$

a perfect square.

Hence (Art. 3) we must have

$$\{2(BD - 2AE)\}^2 = 4(B^2 - 4AC)(D^2 - 4AF)$$

or $$(BD - 2AE)^2 = (B^2 - 4AC)(D^2 - 4AF) \ldots (1).$$

This may be transformed into either of the following forms:

$$AE^2 + CD^2 + FB^2 - BDE - 4ACF = 0 \ldots\ldots\ldots (2).$$
$$(BE - 2CD)^2 = (B^2 - 4AC)(E^2 - 4CF) \ldots\ldots\ldots (3).$$
$$(DE - 2BF)^2 = (D^2 - 4AF)(E^2 - 4CF) \ldots\ldots\ldots (4).$$

Ex. I.

1. Find the difference of the squares of the roots of the equation $3x^2 - 12x - 43 = 0$.

2. Construct an equation whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$.

3. If α, β are the roots of the equation $x^2 - px + q = 0$, construct the equation whose roots are $\alpha - \gamma$, $\beta - \gamma$.

4. Find the value of a when the equation

$$ax^2 - 36x + 81 = 0$$

has equal roots.

5. Find the relation between the quantities a, b, c, a', b', c', so that the equations

$$ax^2 + bx + c = 0, \text{ and } a'x^2 + b'x + c' = 0$$

shall have a common root.

6. The equations $ax^2 + bx + c = 0$, and $2ax + b = 0$, have common roots when the former has two equal roots.

7. When is the equation

$$\sqrt{m^2x + a^2} + \sqrt{n^2x + b^2} + \sqrt{p^2x + c^2} = 0$$

reducible to a simple equation?

8. If α, β are the roots of the equation $x^2 - px + q = 0$, form the equation whose roots are $\alpha + \beta$, and $\alpha - \beta$.

9. If $x^2 + px + q$ and $x^2 + mx + n$ have a common factor, show that this common factor is $x + \dfrac{n - q}{m - p}$.

10. Hence show that the following relation holds, viz.,

$$(n - q)^2 + n(m - p)^2 = m(m - p)\,(n - q).$$

11. Show that the values of x and y derived from the equations $ax^2 + by^2 = a^2x + b^2y = c^3$ are rational when $a^3 + b^3 = c^3$.

12. Show that the expression $6x^2 + 17xy + 12y^2 + 4x + 7y - 10$ can be broken into elementary factors, and determine the factors.

CHAPTER II.

VARIATION.

7. If two quantities are connected together by some constant multiplier, either integral or fractional, one is said to *vary* as the other.

The symbol $\propto$ placed between the quantities is used to express this relation; and if we suppose m, n to be constant quantities, we may distinguish the different kinds of *variation* as follows:

(1.) If $A = mB$, then A varies *directly* as B, or $A \propto B$.

(2.) If $A = \dfrac{m}{B}$, then A varies *inversely* as B, or $A \propto \dfrac{1}{B}$.

(3.) If $A = mBC$, then A varies jointly as B and C, or $A \propto BC$.

(4.) If $A = m\dfrac{B}{C}$, then A varies *directly* as B, and *inversely* as C, or $A \propto \dfrac{B}{C}$.

(5.) If $A = mB^n$, then A varies *directly* as the nth power of B, or $A \propto B^n$.

(6.) If $A = \frac{m}{B^n}$, then A varies *inversely* as the *n*th power of B, or $A \propto \frac{1}{B^n}$.

(7.) If $A = mB + \frac{n}{C}$, then A varies as the sum of two quantities, one of which varies *directly* as B, and the other *inversely* as C.

8. *If* $A \propto B$, *and* $B \propto C$, *then* $A \propto C$.

For, let $A = mB$..... (1),
and $B = nC$;
then, substituting in (1) for this value of B, we have
$A = m \cdot nC = mnC$; or, since mn is constant,
$A \propto C$. *Q.E.D.*

9. *If* $A \propto B$, *then will* $A^2B + AB^2 \propto (A^2 + B^2)(A - B)$.

For, let $A = mB$, then
$A^2B + AB^2 = (mB)^2 \cdot B + mB \cdot B^2 = (m^2 + m) B^3$.... (1).
And so, $(A^2 + B^2)(A - B) = (m^2 + 1)(m - 1) \cdot B^3$ ... (2).
(1) ÷ (2), then $\frac{A^2B + AB^2}{(A^2 + B^2)(A - B)} = \frac{m^2 + m}{(m^2 - 1)(m - 1)}$, a constant quantity.

$\therefore A^2B + AB^2 \propto (A^2 + B^2)(A - B)$. *Q.E.D.*

10. *If* $A \propto B$, *when* C *is constant*,
and $A \propto C$, *when* B *is constant*;
then $A \propto BC$, *when* BOTH *are* VARIABLE.

Let a_1, b be corresponding values of A and B, when C is constant,

Then $A : a_1 :: B : b$, or $a_1 = \frac{Ab}{B}$..... (1).

Now this value a_1 of A will, like A itself, vary as C.
Let then a, c be corresponding values of a_1, C, when B is constant.

Then $a_1 : a :: C : c$, or $a_1 = \frac{Ca}{c}$..... (2).

(1) = (2), then

$$\frac{Ab}{B} = \frac{Ca}{c}, \text{ or } A = \frac{a}{bc} BC.$$

Hence A bears a constant ratio $\frac{a}{bc}$ to the product of the variable quantities B and C.

$\therefore$ A $\propto$ BC, when both B and C are variable. *Q.E.D.*

This proposition is the foundation of the so-called Double Rule of Three in Arithmetic.

We will illustrate this by an example,

Ex. If 15 men earn £180 in 6 weeks, how much will 22 men earn in 13 weeks?

We have

Number of pounds $\propto$ number of men, when time is not considered; that is, when number of weeks is constant.

Hence,

15 : 22 :: 180 : number of pounds required, when *time* is not considered.

Again,

Number of pounds $\propto$ number of weeks, when *men* are not considered; that is, when number of men is constant.

Hence,

6 : 13 :: 180 : number of pounds required, when *men* are not considered.

These two statements correspond to the ordinary statement of a Double Rule of Three sum.

But, by the above proposition:

Number of pounds $\propto$ number of men $\times$ number of weeks, when *both* are considered.

Hence,

15 $\times$ 6 : 22 $\times$ 13 :: 180 : number of pounds required.

This is the ordinary complete statement.

Cor. If A varies as each of the quantities B, C, D, &c.,

when all the rest are constant, then when all are variable, A varies as their product.

Ex. II.

1. If $x \propto y$, and when $x = 3$, $y = 7$; find the value of x when $y = 63$.

2. If A $\propto$ BC, and that C = 3, when A = 60, and B = 1; find the value of B, when A = 80, and C = 4.

3. If $x = my^2 + nz^2$, find m, n, when 17, 2, 1, and 23, 1, 2, are corresponding sets of values of x, y, z.

4. If a cylinder whose length is l, and diameter d, has a weight w, what will be the weight of a cylinder of the same material whose length is l_1, and diameter d_1?

5. If t be the time of a complete vibration of a pendulum of length l, then $t \propto \sqrt{l}$. Hence, find the length of a two-second pendulum, when the length of the second pendulum is 39·4 inches.

6. The time taken by a falling body varies as the square root of the space. Now, a body falls through 64 feet in 2 seconds. Hence, find the relation between time and space.

7. The strength of a beam supported at its ends varies inversely as its length, directly as its breadth, and directly as the square of its depth. Now, a beam whose length, breadth, and thickness are respectively 12 feet, 6 inches, 4 inches, will support 3 tons. Find the breaking-weight of a beam of the same material whose corresponding dimensions are 16 feet, 8 inches, 6 inches.

8. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns; compare the values of equal bulks of gold and silver.

9. The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ the cube of their weight. A diamond of a carats is worth m times a ruby of b carats, and both together are worth c£. Required the value of a diamond and a ruby, each weighing n carats.

10. The heat generated by the discharge of a Leyden jar varies jointly as the *quantity* and the *tension* of the electricity; and the tension varies *directly* as the *quantity* of electricity, and *inversely* as the *surface* of the jar. Show that when the surface is constant, the quantity of electricity varies directly as the square root of the heat.

11. A rod of iron weighs 10 lbs. per foot run, what is the length in metres per kilogram?

12. British units of length (L), mass (M), and time (T) are respectively a foot, a pound, and a second; and the corresponding absolute units are a metre, a kilogram, and a second. Now, the dimensions of the units of force and work are respectively $\frac{LM}{T^2}$ and $\frac{L^2M}{T^2}$. Hence express the absolute units of force and work as British units.

CHAPTER III.

PERMUTATIONS AND COMBINATIONS.

Permutations.

11. The different arrangements that can be made of a number of things are called their **Permutations.**

Thus, *ab*, *ac*, *bc*, *ba*, *ca*, *cb*, are called the permutations of *a*, *b*, *c*, taken *two together;* and *abc*, *acb*, *bca*, *bac*, *cab*, *cba*, are the permutations of the same things taken *three together.*

In some works, the term *permutations* is limited to the arrangements of the things when taken *all together*. When they are *not taken all together*, they are called Variations. We shall include under the term "permutations" all such arrangements.

12. The *different groups* that can be made of a number of *things without* regard to the order of the things are called *Combinations,*

Thus ab, ba form the same combination; so that the combinations of a, b, c taken two together are ab, ac, bc.

13. *The number of permutations of* n *things taken* r *together is*

$$n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-r+1).$$

Let $a, b, c \ldots\ldots k$ be the n things. We will first take them *two* together. Now, there must be $\overline{n-1}$ things before which a stands first, and the same number before which b stands first, and so on for all the letters. Hence the total number of arrangements must be n times $(n-1)$, or $n\,(n-1)$. And hence the *number of permutations of* n *things taken two together* $= n\,(n-1)$.

Again, if we first consider the $(n-1)$ things $b, c, \ldots\ldots k$, we have just shown—as may be seen by putting $(n-1)$ for n in the above result—that

Number of permutations of $(n-1)$ things taken *two* together $= (n-1)\,(\overline{n-1}-1) = (n-1)\,(n-2)$.

Now, a may stand before each of these permutations of $b, c, \ldots\ldots k$.

Hence there are $(n-1)\,(n-2)$ permutations *in which* a *stands first*, when the n letters $a, b, c \ldots\ldots k$ are permuted *three* together. And, of course, there are just as many in which $b, c, \ldots\ldots k$ each stand first.

Hence, the whole number of permutations of the n things taken *three* together $= n$ times $(n-1)\,(n-2) =$

$$n\,(n-1)\,(n-2).$$

And by similar reasoning we may show that the number of permutations of n things taken *four* together $=$

$$n\,(n-1)\,(n-2)\,(n-3).$$

If we look at the *last term* of the *last factor* of each of these results we see that its value (neglecting the sign) is *one less than the number of things taken together.*

It is easy then to perceive *an apparent general law*, viz.: Number of permutations of n things taken r together

$$= n\,(n-1)\,(n-2)\,\ldots\ldots\,(n-\overline{r-1}).$$

By what is called *Mathematical Induction* we may assure *ourselves of the truth of* this law.

We will *assume* the law to be true when p things are taken together, and will then show that it must be true when $(p + 1)$ things are taken together.

Fixing our mind upon $(n - 1)$ things, viz., $b, c, \ldots . k$, we have by our assumption—putting therein $(n - 1)$ for n,

Number of permutations of $b, c, \ldots . k$, taken p together

$$= (n - 1)\ (n - 2) \ldots . (n - 1 - \overline{p - 1})$$
$$= (n - 1)\ (n - 2) \ldots . (n - p).$$

Now, a may stand before each of these, and hence the number of permutations of n things taken $(p + 1)$ together, where a stands first, is $(n - 1)\ (n - 2) \ldots . (n - p)$.

And the same may be said for each of the letters $b, c, \ldots . k$.

Hence the whole number of permutations of n things taken $(p + 1)$ together is

$$n \text{ times } (n - 1)\ (n - 2) \ldots . (n - p),$$

or $$n\ (n - 1)\ (n - 2) \ldots . \left\{ n - (\overline{p + 1 - 1}) \right\}.$$

Hence, *on the assumption that the law is true for any particular case*, we have shown it to be also true when the things are taken *one more together*.

Now, we know it to be true when the things are taken 2, 3, 4 together. It is therefore true when taken 5 together, and therefore also when taken 6, 7, &c., together. Hence, it is generally true.

We have therefore generally

Number of permutations of n things taken r together

$$= n\ (n - 1) \ldots . (n - \overline{r - 1}), \text{ or}$$
$$= n\ (n - 1) \ldots . (n - r + 1).$$

Cor. 1. Let $r = n$, then the whole number of permutations of n things taken *all* together $= n\ (n - 1) \ldots . (n - n + 1)$

$$= n\ (n - 1) \ldots . 1, \text{ or, as often written,}$$
$$= \underline{\lfloor n}, \text{ read } \textit{factorial} \text{ n.}$$

Cor. 2. Every *combination* of r things will produce $\underline{\lfloor r}$ permutations, for the number is that of r things taken all together.

14. *To find the number of permutations of* n *things taken all together when they are not all different things.*

Let there be n letters, in which the letter a occurs p times, b occurs q times, c occurs r times, &c. Then the number of permutations of the n things when taken altogether is

$$\frac{\lfloor n}{\lfloor p \quad \lfloor q \quad \lfloor r \text{ \&c.}}.$$

For (Art. 13, Cor. 1) the number of permutations of n different things taken all together $= \lfloor n$.

Let P be the required number of permutations. Then, since every permutation contains a repeated p times, it is capable of forming $\lfloor p$ permutations on the supposition that the a's are all different letters. The whole number of permutations would then be P times $\lfloor p$, or P . $\lfloor p$. And similarly, on the supposition that all the b's are different, every permutation would form $\lfloor q$ permutations, and so on.

Hence, on the supposition that all the letters are different, the total number of permutations $=$ P . $\lfloor p \quad \lfloor q \quad \lfloor r$ &c.

Hence we have P . $\lfloor p \quad \lfloor q \quad \lfloor r$ &c. $= \lfloor n$,

And $$\therefore \text{P} = \frac{\lfloor n}{\lfloor p \quad \lfloor q \quad \lfloor r \text{ \&c.}}.$$

15. *To find the number of permutations of* n *things when each may be repeated once, twice, &c., times.*

Let $a, b, c, \ldots . k$ be the n quantities.

When taken *one* at a time, their number is n.

If we place a before each of these letters, we get n permutations $aa, ab, ac, \ldots . ak$, in which a stands first; and, we must also have as many where b stands first, where c stands first, &c.

Hence, on the whole, when the letters are taken *two* together, the number of permutations $= n$ times $n = n^2$.

And so we may show that, when taken *three* together, the number of permutations $= n$ times $n^2 = n^3$.

Hence it is easy to show by induction that the number

of permutations of n things when each may be repeated r time $= n^r$.

Ex. 1. Find the number of permutations of 8 things taken 4 together.

Generally (Art. 13) $\mathrm{P} = n\,(n-1)\,\ldots\,(n-r+1)$.

Here $\left.\begin{matrix} n = 8 \\ r = 4 \end{matrix}\right\}$ $\therefore \mathrm{P} = 8\,(8-1)\,\ldots\,(8-4+1) = 8.7.6.5.$

Ex. 2. Find the number of permutations which can be made out of the letters of the word *Wagga-Wagga* when taken all together.

Generally (Art. 14) $\mathrm{P} = \dfrac{\lfloor n}{\lfloor p \;\lfloor q \;\lfloor r}$

Here $n = 10$, and, if p, q, r represent respectively the number of times w, a, g appear,

$$p = 2,\ q = 4,\ r = 4.$$

$$\text{Hence P} = \frac{\lfloor 10}{\lfloor 2 \;\lfloor 4 \;\lfloor 4} = 5.7.9.10.$$

Combinations.

16. *To find the number of combinations of* n *things taken* r *together.*

Let C_r be the required number of combinations.

Now (Art. 13, Cor. 2) every combination of r things is capable of forming $\lfloor r$ permutations.

Hence the whole number of permutations capable of being formed by C_r combinations $= \lfloor r$ times $\mathrm{C}_r = \lfloor r\ \mathrm{C}_r$.

Now this is the number of permutations of n things taken r together.

Hence, by Art. 13,

$$\lfloor r\ \mathrm{C}_r = n\,(n-1)\,\ldots\,(n-r+1).$$

$$\therefore \mathrm{C}_r = \frac{n(n+1)\,\ldots\,(n-r+1)}{\lfloor r}.$$

Cor. Since

$$\frac{n\,(n-1)\,\ldots\,(n-r+1)}{\lfloor r}$$

$$= \frac{n\,(n-1)\,\ldots\,(n-r+1)\,.\,(n-r)\,\ldots\,2\,.\,1}{\lfloor r \;\cdot\; \lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor r \;\; \lfloor n-r}, \quad \text{we have}$$

$$C_r = \frac{\lfloor n}{\lfloor r \;\; \lfloor n-r}.$$

17. *To show that* $C_r = C_{n-r}$ *or that the number of combinations of* n *things taken* r *together is equal to the number taken* (n − r) *together.*

We have

$$C_r = \frac{n\,(n-1)\,\ldots\,(n-r+1)}{\lfloor r},$$

and therefore putting $n - r$ for r,

$$C_{n-r} = \frac{n\,(n-1)\,\ldots\,(n-\overline{n-r}+1)}{\lfloor n-r}$$

$$= \frac{n\,(n-1)\,\ldots\,(r+1)}{\lfloor n-r}$$

$$= \frac{n\,(n-1)\,\ldots\,(r+1)\,.\,r\,(r-1)\,\ldots\,2\,.\,1}{\lfloor n-r \;\cdot\; \lfloor r}$$

$$= \frac{\lfloor n}{\lfloor n-r \,.\, \lfloor r}.$$

Hence (Art. 16, Cor.) $C_r = C_{n-r}$. *Q.E.D.*

18. *To find the number of combinations that can be formed of* n *sets of things, in which there are* p *of one sort,* q *of another,* r *of another, &c., by taking one from each set.*

First, if there be *two* sets, p of one sort and q of another, we may place each of the former sort before each of the latter, and thus obtain p times q of combinations.

The number of combinations is therefore pq.

Again, if there be a third set containing r things, we may

place each one of this set before all the above combinations in turn.

Hence the total number of combinations thus formed is r times pq, or pqr.

And so on for any number of sets.

Hence, number of combinations required $= pqr\ldots.$

19. *To find the number of combinations of* m *things and* n *things when* p *of the former are combined with* q *of the latter.*

The number of combinations of m things taken p together

$$= \frac{m\,(m-1)\ldots.(m-p+1)}{\lfloor p}.$$

And the number of combinations of n things taken q together

$$= \frac{n\,(n-1)\ldots.(n-q+1)}{\lfloor q}.$$

Now each of the former combinations may be combined in turn with all of the latter. Hence the total number of combinations

$$= \frac{m\,(m-1)\ldots.(m-p+1)}{\lfloor p} \cdot \frac{n\,(n-1)\ldots.(n-q+1)}{\lfloor q}.$$

20. *To find when the number of combinations is greatest.*

We have $C_r = \dfrac{n\,(n-1)\ldots.(n-r+1)}{\lfloor r}$,

and $C_{r-1} = \dfrac{n\,(n-1)\ldots.(n-\overline{r-1}+1)}{\lfloor \overline{r-1}}$

$$= \frac{n\,(n-1)\ldots.(n-r+2)}{\lfloor r-1}.$$

Hence, by division, $\dfrac{C_r}{C_{r-1}} = \dfrac{n-r+1}{r} = \dfrac{n+1}{r} - 1.$

Now as r continually increases, $\left(\dfrac{n+1}{r} - 1\right)$ continually diminishes, and therefore must eventually become *less than unity.*

But as long as its value is > 1, the number of combinations found must be greater than the number previously found. Hence the greatest number of combinations will be for that value of r *next before* the one which makes $\left(\frac{n+1}{r} - 1\right)$ less than unity.

When $\frac{n+1}{r} - 1 = 1$, of course $C_r = C_{r-1}$; and since the next higher value of r will make $\left(\frac{n+1}{r} - 1\right) < 1$, the value of r obtained from this equation will give the greatest number of combinations.

Solving the equation, we have $r = \left(\frac{n+1}{2}\right)$;

and, since r is integral, n must be *odd*.

Hence, when n is *odd*,

The greatest number of combinations is when taken $\frac{n+1}{2}$ *together, and also when taken* $\frac{n+1}{2} - 1$, *or* $\frac{n-1}{2}$ *together.*

When n is even, we require the value of r next before the one which makes $\frac{n+1}{r} - 1 < 1$, or $\frac{n+1}{r} < 2$, or $n + 1 < 2r$, or $r > \frac{n+1}{2}$.

The value of r required is $\therefore r = \frac{n}{2}$.

Hence, when n is *even*,

The greatest number of combinations is when they are taken $\frac{n}{2}$ *together.*

Ex. 1. Find the number of combinations of 7 things taken 3 together.

$$\text{Number required} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

Ex. 2. How many words can be formed out of 19 consonants and 7 vowels by taking 4 consonants and 2 vowels for each word?

Number of combinations formed by 19 *consonants* taken 4 together

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4}.$$

And number formed by 7 vowels taken 2 together $= \frac{7 \cdot 6}{1 \cdot 2}$.

Hence (Art. 19),

Number of *combinations* by taking 4 *consonants* and 2 *vowels.*

$$= \frac{19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{7 \cdot 6}{1 \cdot 2} = 81396.$$

Now each combination contains 6 letters, and these may (Art. 13) be permuted in $\lfloor 6$ ways.

Hence total number of words required $= 81396 \times \lfloor 6.$

$= 58605120.$

Ex. III.

1. The number of permutations of some things taken 5 together : the number taken 3 together :: 42 : 1. Find the number of things.

2. How many different permutations can be made of the letters of the word *Proportion* taken all together?

3. How many can be made of each of the words *Kilimaudjaro, Hammada?*

4. Show that the number of triangles which may be formed by joining the angular points of a polygon is $\frac{1}{6} n (n - 1) (n - 2)$, n being the number of sides.

5. How many different numbers can be made by using all the figures of the number 302342044?

6. In how many ways can 10 persons be arranged about a round table?

7. The number of combinations of a number of things taken 5 together is the same as the number taken 7 together. Find the number of things.

8. The number of combinations of n things taken 2 together exceeds by 6 the number of combinations of $(n - 1)$ things taken two together. Find n.

9. If P_r represent the number of permutations of n things taken r together, show that, when $m > 2$,

$$(P_1 - 1)(P_2 - P_1)(P_3 - P_2)(P_4 - P_3) \ldots (P_m - P_{m-1}) = P_2 P_3 P_4 \ldots P_{m-1} \cdot P_{m+1}.$$

10. The number of combinations of $2n$ things taken n together

$$= 2^n \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{1 \cdot 2 \cdot 3 \ldots n}.$$

11. The ratio of the combinations of $4n$ things taken $2n$ together to the combinations of $2n$ things taken n together

$$= \frac{1 \cdot 3 \cdot 5 \ldots (4n - 1)}{(1 \cdot 3 \cdot 5 \ldots 2n - 1)^2}.$$

12. Find the *total number* of signals which can be made by m needles, each of which can assume n distinct positions.

13. Show that $C_r^2 = \left(1 + \frac{1}{r}\right)\left(1 + \frac{1}{n - r}\right) C_{r-1} \cdot C_{r+1}$.

14. If ${}^nP_r$ represent the number of permutations of n things taken r together, show that ${}^{n+1}P_r - {}^nP_r = r\,{}^nP_{r-1}$.

15. Show that $2P_2 + 3P_3 + \&c. + rP_r = (n - 1)(P_1 + P_2 + \&c. + P_r) - P_{r+1}$.

16. At a Parliamentary election for two members there were four candidates. In how many ways can votes be recorded?

17. A firing party, consisting of one sergeant and 12 men, is to be chosen from a company of 100 men and 3 sergeants. In how many ways can this be done?

18. In taking a handful of shot from a bag, show that the chance of getting an even number is greater than the chance of getting an odd number.

(The truth of the Binomial Theorem may be assumed.)

CHAPTER IV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

Arithmetical Progression.

21. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a common difference.

Thus, the numbers 1, 3, 5, 7, &c., are in A. P., and so are the numbers 6, $5\frac{1}{2}$, 5, $4\frac{1}{2}$, &c.

In the first series the common difference is 2, and in the second $-\frac{1}{2}$.

If a be the first term, and d the common difference, the series is evidently $a,\ a + d,\ a + 2d,\ a + 3d$, &c., and the nth term is $a + \overline{n - 1}\ .\ d$.

When the series contains n terms, the nth term is the *last* term. Hence, representing the last term by l, we have

$$l = a + \overline{n - 1}\ .\ d.$$

22. *To find the sum of* n *terms of an* A. P.

Let a be the first term, d the common difference, and S the required sum;

Then $S = a + \overline{a + d} + \overline{a + 2d} + \ldots + l \ldots$ (1).

And, writing the series in reverse order, and remembering that the terms will then diminish by d, we have

$$S = l + \overline{l - d} + \overline{l - 2d} + \ldots + a \ldots (2).$$

Then adding together (1) and (2) we have

$$2S = \overline{a + l} + \overline{a + l} + \overline{a + l} + \ldots + \overline{a + l};$$

and, since there are n of the terms $(a + l)$, we have

$$2S = (a + l)\,n, \text{ or } S = (a + l)\,\frac{n}{2}.$$

Hence we have the following practical rule to find the sum:

Add together the first and last terms, and multiply the result by half the number of terms.

Cor. Since (Art. 21), $l = a + \overline{n-1} \cdot d$, we have

$$S = (a + a + \overline{n-1} \cdot d)\frac{n}{2} = (2a + \overline{n-1} \cdot d)\frac{n}{2};$$

which expresses the sum of an A. P. in terms of the first term, the common difference, and the number of terms.

23. *To insert* m *arithmetic means between* a *and* b.

As there are m means, there must be altogether $m + 2$ terms.

Hence (Art. 21),

$$l = a + \overline{(m+2)-1} \cdot d = a + \overline{m+1} \cdot d.$$

Here $l = b$;

and therefore $a + \overline{m+1} \cdot d = b$, from which $d = \frac{b-a}{m+1}$.

Hence the means required are

$$a + \frac{b-a}{m+1}, \; a + 2 \cdot \frac{b-a}{m+1}, \text{ &c., } a + m \cdot \frac{b-a}{m+1};$$

$$\text{or } \frac{ma+b}{m+1}, \; \frac{\overline{m-1} \cdot a + 2b}{m+1}, \text{ &c., } \frac{a+mb}{m+1}.$$

Ex. 1. Find the sum of 16 terms of the series 2, 4, 6, &c.

Generally, $S = (2a + \overline{n-1} \cdot d)\frac{n}{2}$.

Here $\left.\begin{array}{l} a = 2 \\ d = 2 \\ n = 16 \end{array}\right\}$

$$\therefore S = \left\{2(2) + \overline{16-1} \cdot 2\right\}\frac{16}{2} = (4 + 30)8 = 272.$$

Or thus, $l = a + \overline{n-1} \cdot d = 2 + \overline{16-1} \cdot 2 = 32.$

Hence, by Rule (Art. 22),

$$S = (2 + 32)\frac{16}{2} = 272, \text{ as before.}$$

Ex. 2. Sum the series 12, $11\frac{1}{4}$, $10\frac{1}{2}$, &c., to 8 terms.

We *have* $S = (2a + \overline{n-1} \cdot d)\frac{n}{2}$.

Here $\left.\begin{aligned} a &= 12 \\ d &= -\tfrac{3}{4} \\ n &= 8 \end{aligned}\right\}$

$$\therefore S = \left\{ 2(12) + \overline{8-1}\,(-\tfrac{3}{4}) \right\} \frac{8}{2} = \left(24 - \frac{21}{4}\right) 4 = 75.$$

Ex. 3. Insert 8 arithmetical means between 3 and 21.

We have, Art. 23, $d = \dfrac{b - a}{m + 1}$.

Here $\left.\begin{aligned} a &= 3 \\ b &= 21 \\ m &= 8 \end{aligned}\right\}$

$$\therefore d = \frac{21 - 3}{8 + 1} = \frac{18}{9} = 2, \text{ the common difference.}$$

And since the first term is 3, the 8 means are 5, 7, 9, 11, 13, 15, 17, 19.

Ex. 4. The sum of an A. P. is 140, the first term is 5, and common difference 2; find the number of terms.

We have $S = (2a + \overline{n-1}\,.\,d)\,\frac{n}{2}$.

Here

$\left.\begin{aligned} S &= 140 \\ a &= 5 \\ d &= 2 \end{aligned}\right\}$ Hence $140 = \left\{ 2(5) + \overline{n-1}\,.\,2 \right\} \frac{n}{2} = (n+4)\,n,$

or, transposing, $n^2 + 4n - 140 = 0$, or

$(n - 10)(n + 14) = 0$; from which

$n = 10$ or -14.

The value $n = 10$ tells us that, commencing from the first term of the series and counting 10 terms, the sum of the terms is 140. The value $n = -14$ tells that, commencing from the *opposite end* of the series, and writing down 14 terms, the sum of the terms is also 140. The following would be the two series:

5, 7, 9, 11, 13, 15, 17, 19, 21, 23;

and $-3, -1, 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23.$

When the two values of n are *both* positive and integral,

the series must in both cases be commenced at the first term. It will always be found that the *extra* terms which are given by the larger value will of themselves give *zero* for a sum.

Should one of the values of n be fractional, we may still give it an interpretation. This will be best understood from the following example:

Ex. 5. The sum of an A. P. is 270, the first term 9, and the common difference 4; find the number of terms.

We have $$S = (2a + \overline{n-1}\,.\,d)\,\frac{n}{2}.$$

Here
$$\left.\begin{aligned} S &= 270 \\ a &= 9 \\ d &= 4 \end{aligned}\right\}, \therefore\ 270 = \left\{2\,(9) + \overline{n-1}\,.\,4\right\}\frac{n}{2} = 2n^2 + 7n.$$

Transposing, then $2n^2 + 7n - 270 = 0$, or
$$(n - 10)(2n + 27) = 0;\ \text{from which}$$
$$n = 10 \text{ or } -13\tfrac{1}{2}.$$

The first value $n = 10$ gives the series
9, 13, 17, 21, &c., 45.

The second value, $n = -13\frac{1}{2}$, being negative, tells us to commence with the *opposite end* of the series, viz., the term 45, and write down 13 complete terms, and *another term* which is made up of the 13th term and *half the common difference*. The 13 complete terms beginning with the term 45, and writing from right to left, are

$$-3,\ 1,\ 5,\ 9,\ 13,\ 17,\ 21,\ 25,\ 29,\ 33,\ 37,\ 41,\ 45.$$

The *incomplete term* is $-3 - \frac{1}{2}(4) = -5$.

And the sum of the numbers $-5, -3, 1, 5$ is *zero*, agreeably to the remark made in Ex. 4 above.

Ex. IV.

1. Find the 10th term of the series 2, 6, 10, &c., and the sum to 12 terms.

2. Find the last *term* of the series 3, $3\frac{3}{4}$, $4\frac{1}{2}$, &c., to 20 terms.

Sum the following series :

3. 10, 8, 6, &c., to 9 terms.

4. $5\frac{1}{6}$, $6\frac{1}{2}$, $7\frac{5}{6}$, &c., to 12 terms.

5. 1, 3, 5, 7, &c., to 50 terms.

6. $a + 6\,d$, $a + 5\,d$, $a + 4\,d$, &c., to 13 terms.

7. $x + 9\,y$, $x + 7\,y$, $x + 5\,y$, &c., to 10 terms.

8. $3\frac{6}{7}$, $4\frac{4}{21}$, $4\frac{1}{21}$, &c., to 14 terms.

9. Insert 12 arithmetic means between 2 and 41.

10. Find the sum of 12 terms of the series whose 5th term is 14, and 10th term 31.

11. The sum of 12 terms of an A. P. is 123, and the common difference is $1\frac{1}{2}$; find the first term.

12. Given m the middle term of an A. P., and n the number of terms ; show that the sum of the series is mn.

13. If P, Q be the pth and qth terms respectively, find the first term and the common difference.

14. If S_r represent the sum of r terms of an A. P., show that

$$S_{2n+1} = S_{n+2} + S_{n-1} + (n^2 + n - 2)\,d.$$

15. If P, Q, R be respectively the pth, qth, rth terms, show that

$$P\,(q - r) + Q\,(r - p) + R\,(p - q) = 0.$$

16. There are m arithmetical progressions, the first term of each of which is 1, and the common differences respectively 1, 2, 3, &c., m, Prove that the sum of the nth terms

$$= \frac{\overline{n-1}\,.\,m^2 + \overline{n+1}\,.\,m}{2}.$$

17. If $\phi\,(n)$ represent the sum of n terms of an A. P., show that

$$\phi\,(3\,n) = 3\left\{\phi\,(2\,n) - \phi\,(n)\right\}.$$

18. Show that $\phi\,(n) + \phi\,(n+1) + \phi\,(n+2) + \ldots\ldots \phi\,(2\,n)$

$$= \frac{n\,(3\,n-1)}{1\,.\,2}\,a + \frac{n\,(n-1)\,(7\,n-2)}{1\,.\,2\,.\,3}\,d.$$

19. Show also that

$$\frac{\phi(n)}{n} + \frac{\phi(n-1)}{n-1} + \text{\&c., to } n \text{ terms} = na + \tfrac{1}{4} n(n-1) d.$$

20. Find the sum of n terms of the series whose rth term is $3r - 5$.

21. The sum of three numbers in A. P. is 15, and the sum of their squares is 83. Find the numbers.

22. The sum of four numbers in A. P. is 10, and the sum of their products taken two together is 35. Find the numbers.

23. The result in Ex. 15 may be arranged thus:

$$p(Q - R) + q(R - P) + r(P - Q) = 0.$$

What A. P. does this suggest; and what is the relation between the common difference of this series and the one in Ex. 15?

24. How many terms of the series 4, 7, 10, &c., amount to 144?

Geometrical Progression.

24. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *common ratio.*

Thus 2, 6, 18, 54, &c., is a G. P., and so is 12, 4, $1\frac{1}{3}$, &c., the common ratios being 3 and $\frac{1}{3}$ respectively.

It is evident that the common ratio may be found by dividing any term by the *preceding term.*

Let a be the first term, and r the common ratio.

The G. P. would then be

$$a, ar, ar^2, ar^3, \text{\&c.}$$

If there be n terms, the last or nth term $= ar^{n-1}$.

Hence we have in G. P., $l = ar^{n-1}$..... (1).

25. *To find the sum of a Geometrical Progression.*

Let a be the first term, r the common ratio, and S the sum of n terms; then we have

$$S = a + ar + ar^2 + \text{\&c., } + ar^{n-1},$$

or multiplying each side by r,

$$rS = \quad ar + ar^2 + \text{\&c., } + ar^{n-1} + ar^n.$$

Hence, subtracting the upper line from the lower, we have

$$r\,S - S = ar^n - a, \text{ or } S(r-1) = a(r^n - 1);$$

$$\therefore S = a \cdot \frac{r^n - 1}{r - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since $a \cdot \dfrac{r^n - 1}{r-1} = \dfrac{ar^n - a}{r-1} = \dfrac{r \cdot ar^{n-1} - a}{r-1} = \dfrac{rl - a}{r-1}$,

We have also, $S = \dfrac{rl - a}{r-1}$ ..... (3).

When r is less than unity, it is usual to write these results as follows, being obtained by changing the signs of numerator and denominator in each case and reversing the order:

$$S = a \cdot \frac{1 - r^n}{1 - r} = \frac{a - rl}{1 - r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

26. *To find the limit of an infinite* G. P. *whose common ratio is less than unity.*

DEF. The **Limit** of a series is that quantity to which the sum of the series continually approaches as the number of terms increases.

We have (Art. 25), $S = a \cdot \dfrac{1 - r^n}{1 - r}$.

But since $r < 1$, and n indefinitely large, the value of r^n is indefinitely small, and becomes smaller and smaller the larger n is. Indeed, the value of r^n continually approaches *zero*, as n increases indefinitely.

Let Σ represent the limit of the sum of the series, then we have

$$\Sigma = a \cdot \frac{1 - 0}{1 - r} = \frac{a}{1 - r}.$$

27. *To insert* m *geometrical means between* a *and* b.

Since there are m *means*, there must be $m + 2$ *terms*.

Hence (Art. 24) $l = ar^{\overline{m+2}-1} = ar^{m+1}$.

And therefore, since here $l = b$, we have

$$ar^{n+1} = b, \text{ or } r^{n+1} = \frac{b}{a}.$$

$$\therefore r = \sqrt[n+1]{\frac{b}{a}}.$$

This gives the common ratio, and the means can therefore be easily inserted.

Ex. 1. Sum the series 2, 4, 8, 16, &c., to 10 terms.

Generally, $S = a \cdot \frac{r^n - 1}{r - 1}$.

Here $a = 2$, $r = 2$, $n = 10$ $\Big\}$ $\therefore S = 2 \cdot \frac{2^{10} - 1}{2 - 1} = 2(2^{10} - 1)$.

Ex. 2. Given the first term of a G. P. to be 5, the last term 160, and the number of terms 6. Find the common ratio.

By Art. 24, $l = ar^{n-1}$.

Here $l = 160$, $a = 5$, $n = 6$ $\Big\}$ Hence, substituting, $160 = 5r^{6-1} = 5r^5$

$\therefore r^5 = 160 \div 5 = 32$, or $r = \sqrt[5]{32} = 2$.

Ex. 3. Find the value of the recurring decimal ·666 &c.

The value required is the limit of the G.P. $\frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3}$ + &c., to infinity.

Now (Art. 26) $\Sigma = \frac{a}{1 - r}$.

Here $a = \frac{6}{10}$, $r = \frac{1}{10}$ $\Big\}$ $\therefore \Sigma = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{\frac{6}{10}}{\frac{9}{10}} = \frac{6}{9} = \frac{2}{3}$.

The meaning of this result is that $\frac{2}{3}$ is the quantity to which the value of ·666 &c. approaches as the number of decimal figures taken is increased.

Ex. 4. Find the sum to n terms, and the limit to infinity, of the series:

$a + (a + b)r + (a + 2b)r^2 +$ &c., when $r < 1$.

We have

$$S = a + (a+b)r + (a+2b)r^2 + \&c. + (a + \overline{n-1}.b)\, r^{n-1}$$

$$\text{And } \therefore Sr = \quad ar + (a+b)\, r^2 + \&c. + (a + \overline{n-2}.b)\, r^{n-1}$$

$$+ (a + \overline{n-1}.b)\, r^n.$$

Therefore, subtracting,

$$S(1-r) = a + br + br^2 + \&c. + br^{n-1} - (a + \overline{n-1}.b)r^n.$$

But (Art. 25) the sum of the G. P.

$$br + br^2 + \&c. + br^{n-1} = br \cdot \frac{1 - r^{n-1}}{1-r}.$$

$$\therefore S(1-r) = a + br \cdot \frac{1-r^{n-1}}{1-r} - (a + \overline{n-1}.b)\, r^n.$$

$$\therefore S = \frac{a}{1-r} + br \cdot \frac{1-r^{n-1}}{(1-r)^2} - (a + \overline{n-1}.b) \cdot \frac{r^n}{1-r} \ldots (1).$$

If r be a proper fraction, the limits of r^{n-1} and r^n, as n indefinitely increases, are evidently *zero*.

Hence we have

$$\Sigma = \frac{a}{1-r} + br \cdot \frac{1-0}{(1-r)^2} - (a + \overline{n-1}.b) \cdot \frac{0}{1-r}$$

$$= \frac{a}{1-r} + \frac{br}{(1-r)^2}.$$

Ex. V.

1. Find the 6th term of the series 3, 6, 12, &c.
2. Find the 5th term of the series 35, 7, $1\frac{2}{5}$, &c.
3. Sum the series 81, − 27, 9, &c., to 6 terms.
4. Sum the series $2\frac{4}{5}$, 7, $17\frac{1}{2}$, &c., to 12 terms.
5. Sum the series 6, − 2, $\frac{2}{3}$, &c., to n terms.
6. Find the sum of 10 terms of the series whose first term is $\frac{1}{2}$, and third term $\frac{1}{8}$.

Sum to infinity the following series:

7. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c.$
8. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} - \&c.$
9. $10 - 9 + 8{\cdot}1 - 7{\cdot}29 + \&c.$
10. $5\frac{1}{3} + 1\frac{1}{3} + \frac{1}{3} + \frac{1}{11\frac{1}{2}} + \&c.$
11. $\cdot 6 - \cdot 06 + \cdot 006 - \&c.$

12. $a^2b - a + \frac{1}{b} -$ &c.

13. $(a^2 - b^2) + (a - b) + \frac{a - b}{a + b} +$ &c.

14. $(\sqrt{5} + 2) + 1 + (\sqrt{5} - 2) + (9 - 4\sqrt{5}) +$ &c.

15. $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} +$ &c.

16. $\frac{3}{4} + \frac{4}{3} + \frac{1}{4} + \frac{1}{3} + 1\frac{1}{2} + 1\frac{1}{3} +$ &c.

17. $\frac{\sqrt{3} + 1}{\sqrt{3} - 1} + 1 + \frac{\sqrt{3} - 1}{\sqrt{3} + 1} +$ &c.

18. $\frac{1}{3} + \frac{2}{2\,5} + \frac{3}{1\,2\,5} +$ &c.

19. If the sum of 10 terms of G. P. is 513, and common ratio is -2, find the series.

20. Find the 5th term of the G. P. whose 3rd and 6th terms are respectively a, b.

21. If P, Q, R be respectively the pth, qth, rth terms of a geometrical series, then

$$P^{q-r} \cdot Q^{r-p} \cdot R^{p-q} = 1.$$

22. Show that the sum of 2^n terms of a G. P. series may be expressed in the form

$$a(r + 1)(r^2 + 1)(r^4 + 1) \ldots . (r^{2^n - 1} + 1).$$

23. If s_1 be the sum of an infinite series, and s_2 the sum of the squares of the terms of the series, find the series.

24. If s_p be the sum of p terms of a geometric progression, show that

$$s_n - s_m = r^m s_n - r^n s_m.$$

25. If s_m be the sum of the mth powers of an infinite geometric series, then

$$\frac{1}{s_1} \pm \frac{1}{s_2} \pm \frac{1}{s_3} \pm \ldots . \textit{ad. inf.} = \frac{1}{a \pm 1} - \frac{r}{a \pm r}.$$

26. If $\frac{ma + nb}{ma - nb} = \frac{mb + nc}{mb - nc} = \frac{mc + nd}{mc - nd} =$ &c., then *a, b, c, d, &c., are in G. P.*

27. If s_p be the sum of p terms of a G. P., show that the sum of the products of every two terms $= \frac{r}{r+1} s_p \cdot s_{p-1}$.

28. Show also that

$$ns_n + (s_1 + s_2 + s_3 + \text{\&c., to } n \text{ terms}) = \frac{s_1}{r-1}\left\{n(r^n - 2) + \frac{r}{r-1}(r^n - 1)\right\}.$$

If a, b, c, d are in geometrical progression, show that:

29. $(a^2 + b^2)(c^2 + d^2) = b^2(b + d)^2$.

30. $a^2 + b^2 + c^2 = (a + b + c)(a - b + c)$.

31. $3(a - b + c)^2 - (a^2 + b^2 + c^2) = 2(a - b + c)(a - 2b + c)$.

32. $(a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(b + c)^2$.

33. $a^2(a - 2b + c)^2 = (a - b)^4$.

34. $a^4(a - 3b + 3c - d)^2 = (a - b)^6$.

35. Find the sum of n terms of the series whose nth term is $2^n + n$.

36. If a, b, c, d are in geometrical progression, show that $(a - b)^2$, $(b - c)^2$, $(c - d)^2$ are in G. P.

Harmonical Progression.

28. Quantities are said to be in **Harmonical Progression** when any three consecutive terms being taken, the first is to the third as the difference between the first and second is to the difference between the second and third.

Thus, if a, b, c be in H. P., we must have

$$a : c :: a - b : b - c.$$

We cannot make a formula for the sum of n terms of an H. P., as in the cases of Arithmetical and Geometrical Progression.

29. *The reciprocals of the terms of an Harmonical Progression form a series in Arithmetical Progression.*

Let a, b, c be in H. P.

Then (Art. 28), $a : c :: a - b : b - c$,

$$\therefore\ a(b - c) = c(a - b),$$

or dividing each side by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

that is, the difference between the first and second of the terms $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ is equal to the difference between the second and third.

Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.

30. *To insert* m *harmonical means between* a *and* b.

Taking the reciprocals $\frac{1}{a}$ and $\frac{1}{b}$, we must first find m arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, and afterwards invert them.

Let d be the common difference of the arithmetical series.

Then (Art. 23), $$d = \frac{\frac{1}{b} - \frac{1}{a}}{m + 1} = \frac{a - b}{(m + 1)\,ab}.$$

The arithmetical means are

$$\frac{1}{a} + \frac{a - b}{(m + 1)\,ab}, \frac{1}{a} + \frac{2(a - b)}{(m + 1)\,ab}, \text{ &c., } \frac{1}{a} + \frac{m(a - b)}{(m+1)\,ab}, \text{ or}$$

$$\frac{a + mb}{(m + 1)\,ab}, \frac{2a + \overline{m - 1}\,.\,b}{(m + 1)\,ab}, \text{ &c., } \frac{ma + b}{(m + 1)\,ab}.$$

Hence, inverting, the harmonical means are

$$\frac{(m + 1)\,ab}{a + mb}, \frac{(m + 1)\,ab}{2a + \overline{m - 1}\,.\,b}, \text{ &c., } \frac{(m + 1)\,ab}{ma + b}.$$

31. *If* A, G, H *respectively represent the Arithmetical, Geometrical, and Harmonical means between* a *and* c, *show that* A : G :: G : H.

We have $A - a = c - A$, and $\therefore A = \frac{1}{2}(a + c)$ ..... (1).

Again, $$\frac{G}{a} = \frac{c}{G}, \text{ and } \therefore G = \sqrt{ac} \text{ ..... (2).}$$

Also, $a : c :: H - a : c - H$ and $\therefore H = \frac{2ac}{a + c}$ ..... (3).

Multiplying (1) and (3), we have

$$AH = \tfrac{1}{2}(a + c) \cdot \frac{2ac}{a + c} = ac \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And from (2) we get $G^2 = ac$ ..... (5).

(4) = (5), then $AH = G^2$, or $A : G :: G : H$.

Cor. $A > G$, and $G > H$.

For, (1) − (2), $A - G = \tfrac{1}{2}(a + c) - \sqrt{ac}$

$= \tfrac{1}{2}(a - 2\sqrt{ac} + c)$

$= \tfrac{1}{2}(\sqrt{a} - \sqrt{c})^2$, a positive quantity.

$\therefore A > G$.

And hence, since A, G, H form a continued proportion, it must follow that $G > H$. *Q.E.D.*.

Ex. 1. The 6th term of an H. P. is $\frac{1}{13}$, and the 1st term is $\frac{1}{3}$, find the intermediate terms.

Inverting, we have 3 and 13 as the 1st and 6th terms of the corresponding A. P.

Hence $13 = 3 + 5d$, or $d = 2$.

Therefore the intermediate terms of the A. P. are 5, 7, 9, 11.

Hence the required terms are $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, $\frac{1}{11}$.

Ex. 2. If the first three of four numbers are in A. P., and the last three in H.P., then the product of the means equals the product of the extremes.

Let a, b, c, d be the four numbers.

Then, since a, b, c are in A. P., we have, by Art. 31,

$$b = \tfrac{1}{2}(a + c) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And, since b, c, d are in H. P., we have also, by Art. 31,

$$c = \frac{2bd}{b + d} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Substituting in (2) for the value of b in (1), we have

$$c = \frac{(a + c)d}{b + d}, \text{ or } bc + cd = ad + cd;$$

$$\text{or } bc = ad. \quad Q.E.D.$$

Ex. VI.

1. Insert 7 harmonic means between 1 and $\frac{1}{15}$.

2. Insert 6 harmonic means between 2 and $\frac{2}{3}$.

3. Continue to 4 terms each way the series 2, $1\frac{1}{2}$, $1\frac{1}{3}$.

4. If P, Q, R are respectively the pth, qth, and rth terms of an H. P., show that

$$QR(q - r) + RP(r - p) + PQ(p - q) = 0.$$

5. Given P, Q, the pth and qth terms respectively, find the nth term.

6. What quantity must be added to each of the quantities a, b, c in order to form an H. P.?

7. If s, s' be the sums of two infinite series whose first term is 1 and common ratios r and r', then will s, s', $\frac{1}{r + 1 - 2r'}$, be in H. P.

8. If a, b, c are in H. P., show that

$$a - b : a :: a - c : a + c.$$

9. If G, H be the geometric and harmonic means between a and b, show that

$$b = G \cdot \frac{\sqrt{G + H} - \sqrt{G - H}}{\sqrt{G + H} + \sqrt{G - H}}.$$

10. The sum of three numbers in H. P. is $1\frac{2}{3}$, and the sum of their reciprocals is $8\frac{1}{5}$. Find the numbers.

11. If H be the harmonic mean between a and b, and H_1 the harmonic mean between a and H, and also H_2 the harmonic mean between H and b, show that

$$H_1 : H_2 :: a + 3b : 3a + b.$$

12. If A_1, A_2, be the arithmetic means corresponding to the harmonic means H_1, H_2 in the last example, show that

$$A_1 H_1 = A_2 H_2.$$

CHAPTER V.

THE BINOMIAL THEOREM.

32. We have already shown (Art. 32, Vol. I.) how to develop by multiplication the positive integral powers of $a + b$, and indeed by *assuming* the truth of an apparent general law, shown how to develop these powers by inspection. We shall now prove this law, which goes by the name of the **Binomial Theorem**, and may be expressed as follows:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \text{\&c.}$$

where n may be integral or fractional, positive or negative.

Positive Integral Index.

33. *To prove the Binomial Theorem for a positive integral index.*

By actual multiplication, we have

$$(a + b_1)(a + b_2) = a^2 + (b_1 + b_2)\, a + b_1 b_2.$$

$$(a + b_1)(a + b_2)(a + b_3) = a^3 + (b_1 + b_2 + b_3) a^2 + (b_1 b_2 + b_1 b_3 + b_2 b_3)\, a + b_1 b_2 b_3.$$

$$(a + b_1)(a + b_2)(a + b_3)(a + b_4) = a^4 + (b_1 + b_2 + b_3 + b_4)\, a^3 + (b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4)\, a^2 + (b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4)\, a + b_1 b_2 b_3 b_4.$$

We observe—as on Page 164, (1), (2), Vol. I.—the following laws:

1. The coefficient of the *first* term is unity; that of the *second* term the sum of the letters b_1, b_2, b_3, &c., taken *singly;* that of the *third*, the sum of their products taken *two* together; that of the *fourth*, the sum of their products taken *three* together; and so on, the last term being the product of all the letters b_1, b_2, b_3, &c.

2. The index of a is, in the first term, that of the number of binomial factors, and it *sinks* one every term.

3. The number of terms is *one more* than the number of binomial factors.

We will assume this law to be true for r factors. We then have

$$(a + b_1)(a + b_2)(a + b_3) \ldots . (a + b_r) = a^r + p_1 a^{r-1} + p_2 a^{r-2} + p_3 a^{r-3} + \&c. + p_r \ldots (1),$$

Where p_1 = the sum of b_1, b_2, b_3, &c., b_r taken *singly*,
p_2 = „ the products taken *two* together,
p_3 = „ „ *three* „
&c. = &c.
p_r = the product of all the letters b_1, b_2, b_3, &c., b_r.

Multiply each side of (1) by $a + b_{r+1}$, then

$$(a + b_1)(a + b_2)(a + b_3) \ldots . (a + b_r)(a + b_{r+1})$$

$$= a^{r+1} + (p_1 + b_{r+1}) a^r + (p_2 + p_1 b_{r+1}) a^{r-2} + (p_3 + p_2 b_{r+1}) a^{r-1} + \&c. + p_r b_{r+1} \ldots (2).$$

Now on the first side of (2) we have the product of $(r + 1)$ binomial factors.
And on the second side,

$p_1 + b_{r+1}$ $= b_1 + b_2 + b_3 + \&c. + b_r + b_{r+1}$
$= $ the sum of the letters b_1, b_2, b_3, &c., b_{r+1}, taken singly;

$p_2 + p_1 b_{r+1}$ = the sum of the products of the letters b_1, b_2, b_3, &c., b_{r+1} taken *two* together.
$p_3 + p_2 b_{r+1}$ = " " " " *three* together.
&c. = &c.
$p_r b_{r+1}$ = product of all the $(r + 1)$ letters b_1, b_2, b_3, &c., b_{r+1}.

Hence in (2) we have shown that by assuming the above law to hold for r binomial factors, it also holds for *one factor more.* Now we know by actual multiplication that it holds for 2, 3, 4 binomial factors.

It therefore holds for 5, and therefore also for 6, 7, &c. Hence it is generally true for any number of binomial factors.

Let us now suppose each of the quantities b_1, b_2, b_3, &c., to be equal to b, and let there be n binomial factors $a + b$.

Then we have for their product $(a + b)(a + b)(a + b)\ldots$ to n factors or $(a + b)^n$.

Hence

$$(a + b)^n = a^n + p_1 a^{n-1} + p_2 a^{n-2} + p_3 a^{n-3} + \&c. + p_n \quad \ldots\ldots(3).$$

Now $p_1 = b + b + b +$ &c. to n terms $= nb$.

$p_2 = bb + bb + bb +$ &c. to as many terms as there are combinations of n things taken two together, and therefore, by Art. 25, to $\frac{n(n-1)}{1 \cdot 2}$ terms.

$$\therefore p_2 = \frac{n(n-1)}{1 \cdot 2} bb = \frac{n(n-1)}{1 \cdot 2} b^2.$$

And so $p_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} b^3$ &c.

Also $p_n = b \,.\, b \,.\, b \,.\,.\,.\,.$ to n factors $= b^n$.

Hence, substituting in (3), we have $(a+b)^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3 + \&c. + b^n. \quad \dot{Q}.E.D.$$

For exercises upon this formula, the student is referred to Ex. X., Vol I.

Cor. 1. If $a=1, b=x$, we have, since a^n, a^{n-1}, &c., each $= 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\,.\,2}x^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3 + \&c. + x^n.$$

$$= 1 + C_1x + C_2x^2 + C_3x^3 + \&c. + C_nx^n.$$

where C_1, C_2, C_3, &c., C_n represent respectively the number of combinations of n things taken 1, 2, 3, &c., n together.

Cor. 2. Since the coefficient of the $(r+1)$th term is C_r, we have, by Art. 25,

The $(r+1)$th term of the expansion of $(1+x)^n$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}x^r, \text{ or, as it may be written,}$$

$$= \frac{\lfloor n}{\lfloor r \quad \lfloor n-r}x^r.$$

34. *The coefficients of the terms equidistant from the beginning and end are equal.*

For (Art. 33, Cor. 1) the coefficient of the $(r+1)$th term from the beginning is C_r, and that of the $(r+1)$th term from the end is C_{n-r}.

Now, in Art. 26, we have shown that $C_r = C_{n-r}$, and hence the truth of the proposition.

35. *To find the middle term of the expansion of* $(1+x)^n$.

(1.) Let n be even.

Then, as there are $(n + 1)$ terms in the expansion, the middle term will be the $\left(\frac{n}{2} + 1\right)$th term.

Hence (Art. 33, Cor. 2) putting $\frac{n}{2}$ for r, we have

Middle term

$$= \frac{n\,(n-1)\,(n-2)\,\ldots\,\left(n - \frac{n}{2} + 1\right)}{1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}}\,x^{\frac{n}{2}}.$$

$$= \frac{n\,(n-1)\,(n-2)\,\ldots\,\left(\frac{n}{2} + 1\right)}{1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}}\,x^{\frac{n}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

We may transform this expression thus:

Middle term

$$= \frac{n\,(n-1)\,(n-2)\,\ldots\,\left(\frac{n}{2} + 1\right).\frac{n}{2}\left(\frac{n}{2} - 1\right)\ldots\,2\,.\,1}{1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}\,.\,1\,.\,2\,\ldots\,\frac{n}{2}}\,x^{\frac{n}{2}}$$

$$= \frac{n\,(n-1)\,(n-2)\,\ldots\,4\,.\,3\,.\,2\,.\,1}{\left(1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}\right)^2}\,x^{\frac{n}{2}};$$

or, separating the even and odd factors,

$$= \frac{\{n\,(n-2)\,\ldots\,4\,.\,2\} \times \{(n-1)\,\ldots\,3\,.\,1\}}{\left(1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}\right)^2}\,x^{\frac{n}{2}}$$

$$= \frac{2^{\frac{n}{2}}\left\{\frac{n}{2}\left(\frac{n}{2} - 1\right)\ldots\,2\,.\,1\right\}\,.\,\left\{1\,.\,3\,.\,5\,\ldots\,(n-1)\right\}}{\left(1\,.\,2\,.\,3\,\ldots\,\frac{n}{2}\right)^2}$$

or, striking out common factors from numerator and denominator,

$$= 2^{\frac{n}{2}} \frac{1 \quad 3 \cdot 5 \ldots . (n-1)}{1 \cdot 2 \cdot 3 \ldots . \frac{n}{2}} x^{\frac{n}{2}}.$$

(2.) Let n be odd.

As there are $(n + 1)$ or an even number of terms in the expansion, there will evidently be two middle terms. These are the $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2} + 1\right)$th terms

The coefficient of each may be easily shown to be

$$\frac{n\,(n-1)\,(n-2) \ldots \frac{n+1}{2}}{1 \cdot 2 \cdot 3 \ldots \frac{n+1}{2}}, \text{ or } 2^{\frac{n-1}{2}} \cdot \frac{1 \cdot 3 \cdot 5 \ldots . \quad n}{1 \cdot 2 \cdot 3 \ldots . \frac{n+1}{2}}.$$

36. *To find the greatest coefficient in the expansion of* $(1 + x)^n$, n *being an integer.*

Since (Art. 31, Cor. 1) C_r represents the coefficient of the $(r + 1)$th term, and we have shown that C_r is greatest, if n be even, when $r = \frac{n}{2}$; and if n be odd, when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ (*see* Art. 20), it follows that:

(1.) When n is *even*, the greatest coefficient is the $\left(\frac{n}{2} + 1\right)$th;

(2.) When n is *odd*, the greatest coefficients are the $\left(\frac{n-1}{2} + 1\right)$th and $\left(\frac{n+1}{2} + 1\right)$th, or the $\left(\frac{n+1}{2}\right)$th and the $\left(\frac{n+3}{2}\right)$th coefficients.

These evidently correspond to the coefficients of the middle *terms*.

The Binomial Theorem for any Index.

37. Let $f(m)$ represent the algebraical expression

$$1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \&c.,$$

where m may be either integral or fractional, positive or negative.

Hence we have

$$f(m) = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \&c. \ldots\ldots\ldots\ldots\ldots (1),$$

and, putting n for m,

$$f(n) = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c. \ldots\ldots\ldots\ldots\ldots (2).$$

Now, multiplying together (1) and (2) and arranging, we have

$$f(m)\ f(n) = 1 + (m+n)x + \frac{(m+n)(m+n-1)}{1.2}x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1.2.3}x^3 + \&c.$$

But, putting $(m + n)$ for m in (1), the expression on the right side of this equation is equal to $f(m + n)$.

Hence we have $$f(m) \cdot f(n) = f(m+n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Hence, also,

$$f(m) \cdot f(n) \cdot f(p) = f(m+n) \cdot f(p), \text{ or, by (3), } = f(m+n+p).$$

And generally,

$$f(m) \cdot f(n) \cdot f(p) \ldots . f(k) = f(m + n + p + \ldots . + k) \ldots\ldots\ldots\ldots\ldots (4),$$

when m, n, p, ... k may be either positive or negative, integral or fractional.

Let $m = n = p = \&c. = k = \frac{r}{s}$, where r and s *are positive integers.*

And suppose there are s of the quantities m, n, p, &c., k.

Then $$m + n + p + \&c. + k = s \text{ times } \frac{r}{s} = r,$$

And $$f(m) \cdot f(n) \cdot f(p) \ldots . f(k) = \left\{ f\left(\frac{r}{s}\right) \right\}^s.$$

Hence (4) becomes

$$\left\{ f\left(\frac{r}{s}\right) \right\}^s = f(r), \text{ or, taking the } s\text{th root of each side,}$$

$$f\left(\frac{r}{s}\right) = \left\{ f(r) \right\}^{\frac{1}{s}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

But, by (1),

$$f(r) = 1 + rx + \frac{r(r-1)}{1 \cdot 2} x^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3} x^3 + \&c.,$$

And hence, by Art. 35, since r is a positive integer,

$$f(r) = (1 + x)^r$$

We have therefore

$$\{f(r)\}^{\frac{1}{s}} = \{(1 + x)^r\}^{\frac{1}{s}} = (1 + x)^{\frac{r}{s}},$$

And therefore from (5)

$$f\left(\frac{r}{s}\right) = (1 + x)^{\frac{r}{s}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(6)}.$$

But from (1), putting $\frac{r}{s}$ for m,

$$f\left(\frac{r}{s}\right) = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s} - 1\right)}{1 \cdot 2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s} - 1\right)\left(\frac{r}{s} - 2\right)}{1 \cdot 2 \cdot 3}x^3 + \text{\&c.} \ldots\ldots\ldots\ldots \text{(7)}.$$

Hence, equating (6) and (7),

$$(1 + x)^{\frac{r}{s}} = 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s} - 1\right)}{1 \cdot 2}x^2 + \frac{\frac{r}{s}\left(\frac{r}{s} - 1\right)\left(\frac{r}{s} - 2\right)}{1 \cdot 2 \cdot 3}x^3 + \text{\&c.} \ldots\ldots\ldots\ldots \text{(A)}.$$

which proves the Binomial Theorem for a positive fractional index.

It is therefore true for a positive index, whether integral or fractional.

We shall now show it is true for a *negative* index.

We have in (3), $f(m) \cdot f(n) = f(m + n)$,

when there is no restriction as to the value of m and n.

Let $n = -m$, when m is a positive quantity.

We then have $f(m) \cdot f(-m) = f(m - m) = f(0)$.

But from (2), putting $n = 0$, we have $f(0) = 1$, and hence

$$f(m) \cdot f(-m) = 1$$

$$\therefore f(-m) = \frac{1}{f(m)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Now we have just shown that whether m be integral or fractional, so that it be positive, $f(m) = (1 + x)^m$

And therefore $$\frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1+x)^{-m}.$$

We have therefore from (8)

$$f(-m) = (1+x)^{-m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

But from (2), putting $-m$ for n, we have

$$f(-m) = 1 + (-m)x + \frac{-m(-m-1)}{1.2}x^2 + \frac{-m(-m-1)(-m-2)}{1.2.3}x^3 + \&c. \ldots (10).$$

Hence, equating (9) and (10), we have

$$(1+x)^{-m} = 1 + (-m)x + \frac{-m(-m-1)}{1\ \ 2}x^2 + \frac{-m(-m-1)(-m-2)}{1.2.3}x^3 + \&c.,$$

which proves the Binomial Theorem for any negative index.

We have therefore shown it to be true for any index.

38. *To find the sum of the coefficients of the expansion of* $(1 + x)^n$.

Allowing C_1, C_2, C_3, &c., to represent the coefficients of the *first*, *second*, *third*, &c., powers of x, we have

$$(1 + x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Put $x = 1$; then, since the powers of x are each $= 1$, we have

$$2^n = 1 + C_1 + C_2 + C_3 + C_4 + \&c.$$
$$\therefore 1 + C_1 + C_2 + C_3 + C_4 + \&c. = 2^n.$$

Cor. 1. Since, when n is a positive integer, C_1, C_2, C_3, &c., represent the number of combinations of n things when taken 1, 2, 3, &c., together respectively, it follows that

The total number of combinations of n things $= 2^n - 1$.

Cor. 2. Putting $x = -1$, we have

$$(1 - 1)^n \text{ or } 0 = 1 - C_1 + C_2 - C_3 + \&c, \text{ or}$$
$$C_1 + C_3 + C_5 + \&c. = 1 + (C_2 + C_4 + C_6 + \&c.)$$

Hence the number of *odd* groups which can be made of n things exceeds by *unity* the number of *even* groups.

39. *To find the* GREATEST TERM *in the expansion of* $(1 + x)^n$, *no regard being had to the sign of the term.*

By Art. 33, Cor. 2, $(r + 1)$th term $= \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots r} x^r$.

$$= \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3 \quad \ldots \quad r-1} x^{r-1} \cdot \frac{n-r+1}{r} x \ldots\ldots (1).$$

$$\text{And } r\text{th term} = \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3 \quad \ldots \quad r-1} x^{r-1} \ldots\ldots\ldots (2).$$

Hence, (1) ÷ (2), we have

$$\frac{(r+1)\text{th term}}{r\text{th term}} = \frac{n-r+1}{r} x = \left(\frac{n+1}{r} - 1\right) x \ldots\ldots\ldots (3).$$

We may therefore obtain the $(r+1)$th term from the rth by multiplying the latter by $\left(\frac{n+1}{r} - 1\right) x$.

And as long as $\left(\frac{n+1}{r} - 1\right) x$ continues to be greater than unity, the $(r+1)$th term is greater than the rth term.

Now, as r is *increased*, $\left(\frac{n+1}{r} - 1\right) x$ *diminishes*, and eventually *changes its sign*.

And if r be indefinitely increased, we may make the value of $\left(\frac{n+1}{r} - 1\right) x$ approach *as near to* $-x$ *as we please*.

Hence, if none of the values of $\left(\frac{n+1}{r} - 1\right) x$ is *zero*, we may, by taking r large enough, make any term as nearly equal to x times the preceding term as we please. (It must be remembered that we are not considering here the *sign* of the term).

Now $\left(\frac{n+1}{r} - 1\right)x = 0$, when $r = n + 1$, and this cannot be except when n is integral and positive. From this we learn that there are only $(n + 1)$ terms in the expansion of $(1 + x)^n$ when n is an integer and positive, and that there is no limit to the number of terms when n is negative or fractional.

It follows therefore that *when* n *is fractional or negative, there cannot possibly be a greatest term if* x *be* > 1, since by increasing r sufficiently any term may approach as nearly to x times the preceding as we please, and there is no limit to the value of r.

We will now consider separately the cases in which a greatest term is possible.

It will be necessary only to consider x as positive; for when x is negative, we may put $x = -y$, and find the greatest term in the expansion of $(1 + y)^n$, and this will be the greatest term required.

I. *Let* n *be integral and positive.*

Now when $\left(\frac{n+1}{r} - 1\right)x = 1$, we have $r = \frac{(n+1)x}{x+1}$.

(*a.*) *Let* $\frac{(n+1)x}{x+1}$ *be integral.*

Then there is a term corresponding to the value of this fraction such that it and the next succeeding term have the same magnitude, and are the greatest terms of the expansion.

Hence the rth and $(r + 1)$th terms are the greatest terms when $r = \frac{(n+1)x}{x+1}$ *an integer.*

(*b.*) *Let* $\frac{(n+1)x}{x+1}$ *be a fraction.*

Then $\left(\frac{n+1}{r} - 1\right)x < 1$, when the value of r is the in-

teger next greater than $\frac{(n+1)x}{x+1}$.

Hence the rth term is the greatest term for the value of r corresponding to the integer next greater than $\frac{(n+1)x}{x+1}$.

II. *Let* n *be fractional and positive.*

It has been shown that *when* x *is greater than unity there cannot be a greatest term.*

When x *is less than unity* the conditions are exactly as in I. above.

III. *Let* n *be negative.*

We have shown that here, as in II., *there is no greatest term when* x *is greater than unity.*

When x *is less than unity*, let $n = -m$.

Then $\left(\frac{n+1}{r} - 1\right)x = -\left(\frac{m-1}{r} + 1\right)x$.

When $\left(\frac{m-1}{r} + 1\right)x = 1$, we have $r = \frac{(m-1)x}{1-x}$.

(a.) *Let* $\frac{(m-1)x}{1-x}$ *be integral and positive.*

Then, reasoning as before, the rth and $(r+1)$th are the greatest terms when $r = \frac{(m-1)x}{1-x}$ a positive integer.

(b.) *Let* $\frac{(m-1)x}{1-x}$ *be fractional and positive.*

Then, as above, the rth is the greatest term for that value of r corresponding to the integer next greater than $\frac{(m-1)x}{1-x}$.

(c.) *Let* $\frac{(m-1)x}{1-x}$ *be negative.*

In this case, *since* x *is less* than unity, m is also less than *unity.*

Hence $\left(\frac{m-1}{r}+1\right)x$ is always less than x, and therefore *less than unity for every value of* r.

It follows therefore that when $\frac{(m-1)x}{1-x}$ is negative—that is, when x is less than unity, and m is less than unity—*the first term of the expansion is the greatest.*

Cor. 1. Since, when $\frac{(n+1)x}{x+1}$ is < 1, we have $nx < 1$,

And when $\frac{(m-1)x}{1-x}$ is < 1, x being less than unity, we have $mx < 1$,

It follows that the conditions that the *first* shall not be the greatest term are:

(1) When n is *a positive integer*, nx must be > 1.
(2) When n is *fractional or negative.*
(a.) If x be > 1, this condition is sufficient.
(b.) If x be < 1, nx must be > 1.

Cor. 2. Hence, also, in order that the *last* term shall not be the greatest term, since we need only consider the case of n being a positive integer, the condition is that $\frac{n}{x}$ is > 1, or $n > x$.

Approximate Values.

40. *To find the* nth *root of* N, *when* N *is very nearly an* nth *power of a rational quantity.*

Let $N = a^n + b$, where b is small compared with a^n .....(1).

Then $N^{\frac{1}{n}} = (a^n + b)^{\frac{1}{n}} = a\left(1 + \frac{b}{a^n}\right)^{\frac{1}{n}}$.

$$= a\left\{1 + \frac{1}{n} \cdot \frac{b}{a^n} + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)}{1.2} \cdot \left(\frac{b}{a^n}\right)^2 + \frac{\frac{1}{n}\left(\frac{1}{n} - 1\right)\left(\frac{1}{n} - 2\right)}{1.2.3} \left(\frac{b}{a^n}\right)^3 + \&c.\right\}$$

$$= a\left\{1 + \frac{1}{n} \cdot \frac{b}{a^n} - \frac{n-1}{1.2.n^2} \cdot \frac{b^2}{a^{2n}} + \frac{(n-1)(2n-1)}{1.2.3.n^3} \cdot \frac{b^3}{a^{3n}} - \&c.\right\}$$

Now, as b is small compared with a^n, the values of the successive terms of this expansion diminish rapidly, and hence we may approximate as nearly as we please to the value of $N^{\frac{1}{n}}$.

Ex. 1. Find the 7th root of 2192.

We have $N = 2192 = 2187 + 5 = 3^7 + 5 = 3^7\left(1 + \frac{5}{3^7}\right)$.

$\therefore N^{\frac{1}{7}} = 3\left(1 + \frac{5}{3^7}\right)^{\frac{1}{7}}$, from which by expansion, &c., we get

$$\sqrt[7]{2192} = 3\left\{1 + \frac{1}{7} \cdot \frac{5}{3^7} - \frac{6}{1.2.7^2} \cdot \left(\frac{5}{3^7}\right)^2 + \frac{6.13}{1.2.3.7^3} \cdot \left(\frac{5}{3^7}\right)^3 - \&c.\right\}.$$

Ex. 2. Find the 5th root of 1021.

Here $N = 1021 = 1024 - 3 = 4^5 - 3 = 4^5\left(1 - \frac{3}{4^5}\right)$.

$$\therefore N^{\frac{1}{5}} = 4\left(1 - \frac{3}{4^5}\right)^{\frac{1}{5}}$$

$$= 4\left\{1 - \frac{1}{5}\cdot\frac{3}{4^5} - \frac{4}{1\,.\,2\,.\,5^2}\cdot\left(\frac{3}{4^5}\right)^2 - \frac{4\,.\,9}{1\,.\,2\,.\,3\,.\,5^3}\cdot\left(\frac{3}{4^5}\right)^3 - \text{\&c.}\right\}$$

41. *If* c *is large compared with* a *or* b, *show that*

$$(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}}$$

$$= 2c^{-(m+1)} - (m+1)c^{-(m+3)}\left\{a^2 - (m+2)b^2\right\} \text{ nearly.}$$

We have $(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}}$

$$= c^{-(m+1)}\left\{1 - \left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2}\right)\right\}^{-\frac{m+1}{2}}; \text{ expanding, \&c., then}$$

$$= c^{-(m+1)}\left\{1 + \frac{m+1}{2}\left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\left(\frac{2b}{c} - \frac{a^2+b^2}{c^2}\right)^2 + \text{\&c.}\right\}$$

$$= c^{-(m+1)}\left\{1 + \frac{m+1}{2}\left(\frac{2b}{c} - \frac{a^2 + b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\cdot\frac{4b^2}{c^2}\right\} \ldots\ldots\ldots (1),$$

where we retain no terms involving higher powers of c than the second.

And similarly

$$(a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}} = c^{-(m+1)}\left\{1 - \frac{m+1}{2}\left(\frac{2b}{c} + \frac{a^2+b^2}{c^2}\right) + \frac{(m+1)(m+3)}{8}\cdot\frac{4b^2}{c^2}\right\} \ldots (2).$$

(1) + (2), then

$$(a^2 + b^2 - 2bc + c^2)^{-\frac{m+1}{2}} + (a^2 + b^2 + 2bc + c^2)^{-\frac{m+1}{2}}$$

$$= c^{-(m+1)}\left\{2 - (m+1)\frac{a^2 + b^2}{c^2} + (m+1)(m+3) \cdot \frac{b^2}{c^2}\right\}$$

$$= 2c^{-(m+1)} - (m+1)c^{-(m+3)}\left\{a^2 - (m+2)b^2\right\}. \quad Q.E.D.$$

This is an important result in the mathematical theory of magnetism.

42. *When* n + 1 *figures of a square root have been obtained by the ordinary method,* n *figures more may be found by dividing the remainder by the number formed by taking twice the quotient already obtained, provided that the whole number of figures in the root is* 2n + 1.

Let N be the number whose square root is required, a the part already obtained, and b the part required.

Then $\sqrt{N} = a + b$

$$\therefore N = a^2 + 2ab + b^2 \text{ or } N - a^2 = 2ab + b^2.$$

$$\therefore \frac{N - a^2}{2a} = b + \frac{b^2}{2a}.$$

Now $N - a^2$ is the remainder after obtaining the part a of the square root by the ordinary method.

And we see that $N - a^2$, when divided by $2a$, gives the remaining part b of the square root, together with the quantity $\frac{b^2}{2a}$.

We have then to show that $\frac{b^2}{2a}$ is a proper fraction, and therefore has no influence upon the integral value of the remaining figures of the root.

We assume that b contains n digits; and therefore b^2 contains not more than $2n$ digits.

But a, since it contains the $n + 1$ figures of the root already found, and also n ciphers, occupying the place of those already to be found, must contain $2n + 1$ digits.

Hence $\frac{b^2}{2a}$ is less than unity, and the proposition is proved.

This result has been assumed and illustrated by an example in Vol. I., p. 186.

43. *When* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *more figures may be found by dividing the remainder by the next trial divisor, provided that the whole number of figures in the root is* 2 n + 2.

Let N be the number whose cube root is required, a the part already obtained, and b the part required.

Then
$$\sqrt[3]{N} = a + b$$
$$\therefore N = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$\text{or} \quad N - a^3 = 3a^2b + 3ab^2 + b^3$$
$$\therefore \frac{N - a^3}{3a^2} = b + \frac{b^2}{a} + \frac{b^3}{3a^2}.$$

Now $N - a^3$ represents the remainder after obtaining the part a of the cube root, and $3a^2$ is the next trial divisor.

Hence we see that on dividing the remainder by the next trial divisor we get the required quotient b, together with the quantity $\frac{b^2}{a} + \frac{b^3}{3a^2}$.

Now, in order that this may not affect the integral value of the remaining figures of the cube root, we must show that $\frac{b^2}{a} + \frac{b^3}{3a^2}$ is less than unity.

Since b contains n figures, we must have $b < 10^n$.

And since a contains $2n + 2$ figures, a cannot be less *than* 10^{2n+1}, *but may* be equal to or greater than 10^{2n+1}.

$$\therefore \frac{b^2}{a} + \frac{b^3}{3a^2} \text{ is } < \frac{(10^n)^2}{10^{2n+1}} + \frac{(10^n)^3}{3(10^{2n+1})^2}$$

$$< \frac{1}{10} + \frac{1}{3 \cdot 10^{n+2}}, \text{ and } \therefore < 1.$$

Hence the truth of the proposition.

This result has been assumed and illustrated in Vol. I, p. 193.

44. We shall now work out a few examples on the Binomial Theorem.

Ex. 1. Expand to four terms the expression $(1 + 3x)^{-\frac{5}{3}}$.

$$(1 + 3x)^{-\frac{5}{3}} = 1 + \left(-\frac{5}{3}\right) \cdot 3x + \frac{-\frac{5}{3}\left(-\frac{5}{3} - 1\right)}{1 \cdot 2}(3x)^2$$

$$+ \frac{-\frac{5}{3}\left(-\frac{5}{3} - 1\right)\left(-\frac{5}{3} - 2\right)}{1 \cdot 2 \cdot 3}(3x)^3 + \&c.$$

$$= 1 - 5x + 20x^2 - \frac{220}{3}x^3 + \&c.$$

Ex. 2. Find the $(r + 1)$th term of the expansion of $(2 - 5x)^{-4}$.

$(r + 1)$th term

$$= \frac{-4(-4-1)(-4-2)\ldots\ldots(-4-r+1)}{1 \cdot 2 \cdot 3 \quad \ldots\ldots \quad r} 2^{-4-r}(-5x)^r$$

$$= \frac{(-1)^r \cdot 4 \cdot 5 \cdot 6 \ldots\ldots (r+3)}{1 \cdot 2 \cdot 3 \ldots\ldots \quad r} 2^{-4-r}(-1)^r \cdot (5x)^r$$

$$= (-1)^{2r} \cdot \frac{4 \cdot 5 \cdot 6 \ldots\ldots r(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots\ldots r} 2^{-4-r}(5x)^r;$$

or, since $(-1)^{2r} = 1$,

$$= \frac{(r+1)(r+2)(r+3)}{1 \cdot 2 \cdot 3} \cdot \frac{(5x)^r}{2^{r+4}}.$$

Ex. 3. If x be very small, show that

$$\frac{(1 + 2x)^{\frac{1}{2}} + (1 + 3x)^{\frac{1}{2}}}{2 + 5x - (1 + 4x)^{\frac{1}{2}}} = 2 - 4x, \text{ very nearly.}$$

Expanding and retaining terms containing the *first powers* of x only, we have

$$\frac{(1 + 2x)^{\frac{1}{2}} + (1 + 3x)^{\frac{1}{3}}}{2 + 5x - (1 + 4x)^{\frac{1}{2}}}$$

$$= \frac{(1 + \tfrac{1}{2} \cdot 2x) + (1 + \tfrac{1}{3} \cdot 3x)}{2 + 5x - (1 + \tfrac{1}{2} \cdot 4x)}$$

$$= \frac{2 + 2x}{1 + 3x} = (2 + 2x)\cdot(1 + 3x)^{-1};$$

or, expanding as before,

$$= (2 + 2x)(1 - 3x);$$

or multiplying out, and neglecting the term involving x^2,

$$= 2 - 4x. \quad Q.E.D.$$

Ex. 4. If p and q be very nearly equal, and n an integer, show that

$$\left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{(n + 1)p + (n - 1)q}{(n - 1)p + (n + 1)q} \text{ nearly.}$$

We have $\dfrac{p}{q} = \dfrac{\dfrac{2p}{p + q}}{\dfrac{2q}{p + q}} = \dfrac{\dfrac{(p + q) + (p - q)}{p + q}}{\dfrac{(p + q) - (p - q)}{p + q}}$

$$= \frac{1 + \dfrac{p - q}{p + q}}{1 - \dfrac{p - q}{p + q}}.$$

$$\therefore \left(\frac{p}{q}\right)^{\frac{1}{n}} = \frac{\left(1 + \dfrac{p - q}{p + q}\right)^{\frac{1}{n}}}{\left(1 - \dfrac{p - q}{p + q}\right)^{\frac{1}{n}}};$$

expanding and retaining only two terms, since $p - q$ is very small,

$$= \frac{1 + \dfrac{1}{n} \cdot \dfrac{p - q}{p + q}}{1 - \dfrac{1}{n} \cdot \dfrac{p - q}{p + q}} = \frac{n(p + q) + (p - q)}{n(p + q) - (p - q)}$$

$$= \frac{(n + 1)p + (n - 1)q}{(n - 1)p + (n + 1)q}. \quad Q.E.D.$$

Ex. 5. Find the limit of $\left(1 + \frac{1}{x}\right)^x$, when x increases without limit.

Expanding by the Binomial Theorem, we have

$$\left(1 + \frac{1}{x}\right)^x = 1 + \frac{x}{1} \cdot \frac{1}{x} + \frac{x(x-1)}{1 \cdot 2} \cdot \frac{1}{x^2} + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{x^3} + \&c.$$

$$= 1 + \frac{1}{1} + \frac{1 - \frac{1}{x}}{1 \cdot 2} + \frac{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)}{1 \cdot 2 \cdot 3} + \&c.$$

When x increases without limit, each of the quantities $\frac{1}{x}, \frac{2}{x}, \frac{3}{x}$, &c., decreases without limit, and becomes ultimately zero.

Hence, when x is infinite,

$$\left(1 + \frac{1}{x}\right)^x = 1 + \frac{1}{1} + \frac{1 - 0}{1 \cdot 2} + \frac{(1 - 0)(1 - 0)}{1 \cdot 2 \cdot 3} + \&c.$$

$$= 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \&c.$$

It is usual to denote this series by the symbol e; it is the base of the Napierian system of logarithms.

We therefore have

$$\text{Limit of } \left(1 + \frac{1}{x}\right)^x \text{ when } x \text{ is infinite } = e.$$

The approximate value of e is 2·71828.

Ex. VII.

Expand to four terms the six following expressions:

1. $(1 + x)^7$.
2. $(3 + 2x)^{\frac{1}{2}}$.
3. $(4 - 5x)^{-\frac{2}{3}}$.
4. $(a + bx^2)^{-1}$.
5. $(1 - 2x + 3x^2)^{-2}$.
6. $\left(\frac{a}{x} - \frac{b}{y}\right)^{-5}$.

Find the middle terms of:

7. $(a + b)^9$.
8. $(3 + 5x)^{10}$.
9. $(a^2 - b^2)^6$.
10. $\left(x + \frac{1}{x}\right)^{2n}$.
11. $\left(3 + \frac{x}{4}\right)^5$.
12. $(4a^{\frac{1}{2}} - 3b^{\frac{1}{3}})^7$.

Find the general or $(r + 1)$th terms of:

13. $(3 - 5x)^{-\frac{1}{5}}$. 14. $\left(ax + \frac{b}{x}\right)^n$. 15. $\left(a^{\frac{1}{p}} - b^{\frac{1}{q}}\right)^{-1}$.

16. If ${}^mC_p$ represent the pth coefficient of the expansion of $(a + x)^m$, show that ${}^mC_p + {}^mC_{p+1} = {}^{m+1}C_{p+1}$.

17. If P be the sum of the odd terms of the expansion of the binomial $(a + x)^m$, and Q the sum of the even terms, then will $P^2 - Q^2 = (a^2 - x^2)^m$.

18. In the last example, $4PQ = (a + x)^{2m} - (a - x)^{2m}$.

19. Show that $1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{4^4} + \&c. = 4$.

20. If A, B are two consecutive coefficients of the expansion of $(1 + x)^n$, show that the coefficients after B are

$$B \cdot \frac{nB - A}{(n+2)A + B}, \quad B \cdot \frac{(nB - A)(\overline{n-1} \cdot B - 2A)}{(\overline{n+2} \cdot A + B)(\overline{n+3} \cdot A + 2B)}, \text{ \&c.}$$

21. Find the greatest term of the expansion of $\left(2 + \frac{1}{3}\right)^5$.

22. Find the greatest term of the expansion of $\left(1 - \frac{1}{4}\right)^{-3}$.

23. Show that

$$\sqrt[3]{a + x} = a^{\frac{1}{3}}\left\{1 + \frac{1}{3} \cdot \frac{x}{a + x} + \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{x^2}{(a + x)^2} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot \frac{x^3}{(a + x)^3}, \text{ \&c.}\right\}$$

24. If n be a positive integer, prove that

$$1 - 2n + \frac{2n(2n-1)}{\lfloor 2} - \ldots + (-1)^{n-1} \cdot \frac{2n(2n-1)\ldots(n+2)}{\lfloor n-1} = (-1)^{n-1} \cdot \frac{\lfloor 2n}{2(\lfloor n)^2}$$

25. If n be an odd number, show that

$$1 + \frac{2n(2n-1)}{\lfloor 2} + \frac{2n(n-1)(2n-2)(2n-3)}{\lfloor 4} + \&c. + \frac{2n}{\lfloor n-1 \cdot \lfloor n+1} = 2^{2n-2},$$

26. Show that

$$\frac{1 \cdot 2 \cdot 3 \ldots . p \cdot 2^{p+q+1}}{(q+1)(q+2)\ldots.(p+q+1)} = \frac{1}{p+1} + \frac{q}{(p+1)(p+2)}$$
$$+ \frac{q(q-1)}{(p+1)(p+2)(p+3)} + \&c. + \frac{q(q-1)\ldots.2 \cdot 1}{(p+1)\ldots.(p+q+1)}$$
$$+ \frac{1}{q+1} + \frac{p}{(q+1)(q+2)} + \frac{p(p-1)}{(q+1)(q+2)(q+3)}$$
$$+ \&c. + \frac{p(p-1)\ldots.2 \cdot 1}{(q+1)\ldots.(p+q+1)}.$$

27. If a, b, n are positive integers, and b less than $2a - 1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

28. Show that the integral part of $(1 + \sqrt{3})^{2m+1}$ contains 2^{m+1} as a factor.

29. Find the coefficient of x^m in the expansion of

$$\frac{p + qx}{q + rx}$$

in a series of ascending powers of x, and show that if P and Q be the coefficients of x^m, according as it is expanded according to ascending or descending powers respectively; then $PQ = -\frac{(pr - q)^2}{q^2 r^2}$.

30. Prove that the coefficient of x^r in

$$(1 + 2x + 3x^2 + \&c. \text{ ad inf.})^2$$

is $\frac{1}{6}(r+1)(r+2)(r+3)$.

31. Show that if R is the remainder, after taking n terms of the expansion of $(1 - x)^{-2}$, then

$$R = \frac{(n+1)x^n - nx^{n+1}}{(1-x)^2}.$$

32. Find the middle term of $\left(x + \frac{1}{x}\right)^{2n}$, and show that the numerator contains 2^n as a factor.

33. Show that $\frac{(1-x)^n}{\lfloor n} = \frac{1}{\lfloor n} + \frac{1}{\lfloor n-1} \cdot \frac{x}{1-x}$

$+ \frac{1}{\lfloor 2 \,.\, \lfloor n-2} \cdot \frac{x^2}{(1-x)^2}$ + &c.

34. Prove that $\left(x + \frac{1}{x}\right)^{2n}$

$$= \left(x^{2n} + \frac{1}{x^{2n}}\right) + 2n\left(x^{2n-2} + \frac{1}{x^{2n-2}}\right)$$

$$+ \frac{2n(2n-1)}{1:2}\left(x^{2n-4} + \frac{1}{x^{2n-4}}\right) + \&c. + 2^n \cdot \frac{1.3.5\ldots(2n-1)}{\lfloor n}$$

35. When x is small, show that $\frac{(1+x)^{\frac{1}{2}} + (1-2x)^{\frac{1}{2}}}{(1-3x)^{\frac{1}{3}}}$

$= 2 + \frac{3}{2}x$, nearly.

36. If x be greater than a, prove that the sum of all the terms of the expansion of $(x+a)^n$ but the first two is less than $(2^n - n - 1)ax^{n-1}$.

CHAPTER VI.

INDETERMINATE COEFFICIENTS AND PARTIAL FRACTIONS.

Indeterminate Coefficients.

45. *If* $a + bx + cx^2 + dx^3 + \&c. = 0$ *for all values of* x, *then each of the quantities* a, b, c, d, &c., *must be zero.*

Since for *all values* of x it is true that

$$a + bx + cx^2 + dx^3 + \&c. = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

it must be true when $x = 0$.

Putting, then, $x = 0$, we have $a = 0$.

Hence, from (1), $bx + cx^2 + dx^3 + \&c. = 0$;

or, dividing each side by x,

$$b + cx + dx^2 + \&c. = 0,$$

Since this is also true for all values of x, we may put $x = 0$
We then get $b = 0$.
And so we may show that $c = 0$, $d = 0$, &c.

Cor. *If for all values of* x *we have*

$$A + Bx + Cx + Dx^3 + \&c. = a + bx + cx^2 + dx^3 + \&c.;$$

then we must have $A = a$, $B = b$, $C = c$, &c.

For, transposing,

$$(A - a) + (B - b)x + (C - c)x^2 + (D - d)x^3 + \&c. = 0.$$

Hence, by the above proposition, since this is true for all values of x, we have

$$A - a = 0, \text{ or } A = a;$$
$$B - b = 0, \text{ or } B = b;$$
$$C - c = 0, \text{ or } C = c; \&c.$$

In many algebraical transformations the above proposition is especially useful.

Ex. 1. Given $x = y - 3y^2$, find y in a series of ascending powers of x.

Here $$x = y - 3y^2 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Assume $$y = ax + bx^2 + cx^3 + dx^4 + \&c. \ldots\ldots\ldots (2).$$

We have not introduced a constant term into the series for y, for from (1) we see that, when $x = 0$, $y = 0$.

Squaring (2), we have

$$\begin{aligned} y^2 &= a^2x^2 + b^2x^4 + \&c. \\ &\quad + 2abx^3 + 2acx^4 + \&c. \\ &= a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \&c. \ldots\ldots\ldots (3). \end{aligned}$$

Hence, substituting in (1) from (2) and (3),

$$\begin{aligned} x &= (ax + bx^2 + cx^3 + dx^4 + \&c.) \\ &\quad - 3\{a^2x^2 + 2abx^3 + (b^2 + 2ac)x^4 + \&c.\} \\ &= ax + (b - 3a^2)x^2 + (c - 6ab)x^3 + (a - 3b^2 - 6ac)x^4 + \&c. \end{aligned}$$

Equating the coefficients of like powers of x on each side of this equation, we have

$$a = 1.$$
$$b - 3a^2 = 0, \text{ or } b = 3 \times 1^2 = 3.$$
$$c - 6ab = 0, \text{ or } c = 6 \times 1 \times 3 = 18.$$
$$d - 3b^2 - 6ac = 0, \text{ or } d = 3 \times 3^2 + 6 \times 1 \times 18 = 135.$$
&c.

Substituting these values of a, b, c, d, &c., in (2), we get

$$y = x + 3x + 18x^3 + 135x^4 + \&c.$$

Ex. 2. Express the quotient of $\frac{1 + 5x}{1 - 2x + 7x^2}$ in ascending powers of x.

Let $\frac{1 + 5x}{1 - 2x + 7x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c.....(1).$

Clearing of fractions, we have

$$\begin{aligned} 1 + 5x = {} & a + bx + cx^2 + dx^3 + ex^4 + \&c. \\ & - 2ax - 2bx^2 - 2cx^3 - 2dx^4 - \&c. \\ & + 7ax^2 + 7bx^3 + 7cx^4 + \&c. \end{aligned}$$

$$= a + (b - 2a)x + (c - 2b + 7a)x^2 + (d - 2c + 7b)x^3 + (e - 2d + 7c)x^4 + \&c.$$

Hence, equating coefficients of like powers of x, we have

$$a = 1$$
$$b - 2a = 5, \text{ or } b = 5 + 2 \times 1 = 5.$$
$$c - 2b + 7a = 0, \text{ or } c = 2 \times 7 - 7 \times 1 = 7.$$
$$d - 2c + 7b = 0, \text{ or } d = 2 \times 7 - 7 \times 7 = -35.$$
$$e - 2d + 7c = 0, \text{ or } e = 2(-35) - 7 \times 7 = -119.$$
&c.

Hence, by substitution in (1), we have

$$\frac{1 + 5x}{1 - 2x + 7x^2} = 1 + 7x + 7x^2 - 35x^3 - 119x^4 - \&c.$$

Ex. 3. Find the square root of $1 - x + x^2$.

Let $\sqrt{1 - x + x^2} = a + bx + cx^2 + dx^3 + ex^4 + \&c.(1).$

$$\therefore 1 - x + x^2 = a^2 + b^2x^2 + c^2x^4 + \&c.$$
$$+ 2abx + 2acx^2 + 2adx^3 + 2aex^4 + \&c.$$
$$+ 2bcx^3 + 2bdx^4 + \&c.$$

Hence, equating coefficients of like powers of x, we have

$$a^2 = 1, \quad \therefore a = 1.$$

$$2ab = -1, \quad \therefore b = -\frac{1}{2a} = -\frac{1}{2}.$$

$$b^2 + 2ac = 1, \quad \therefore c = \frac{1 - b^2}{2a} = \frac{3}{8}.$$

$$2ad + 2bc = 0, \quad \therefore d = -\frac{bc}{a} = -\frac{3}{16}.$$

$$c^2 + 2ae + 2bd = 0, \quad \therefore e = -\frac{c^2 + 2bd}{2a} = -\frac{21}{64}, \text{ &c.}$$

Hence, substituting in (1),

$$\sqrt{1 - x + x^2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \frac{3}{16}x^3 - \frac{21}{64}x^4 + \text{ &c.}$$

Partial Fractions.

46. When the denominator of an algebraical fraction can be broken up into factors it is always possible to separate the fraction into two or more fractions having denominators of lower dimensions. The limits of this work will only allow space for one or two examples. We shall not enter upon the general theory.

Ex. 1. Resolve $\dfrac{3x + 7}{x^2 + x - 30}$ into partial fractions.

Since $x^2 + x - 30 = (x - 5)(x + 6)$,

assume $$\frac{3x + 7}{x^2 + x - 30} = \frac{A}{x - 5} + \frac{B}{x + 6} \quad \ldots\ldots\ldots\ldots (1).$$

Clearing of fractions, then

$$3x + 7 = A(x + 6) + B(x - 5) \quad \ldots\ldots\ldots\ldots (2).$$

We may now proceed by either of the following methods:

I. *First Method.*

Collecting like terms, we have

$$3x + 7 = (A + B)x + (6A - 5B).$$

Equating coefficients of like powers of x, then

$$\left.\begin{aligned} A + B &= 3 \\ \text{and } 6A - 5B &= 7 \end{aligned}\right\}, \text{ from which } A = 2,\ B = 1.$$

Hence, from (1) by substitution,

$$\frac{3x + 7}{x^2 + x - 30} = \frac{2}{x - 5} + \frac{1}{x + 6}.$$

II. *Second Method.*

To find A, we shall in (2) make the original denominator of A, viz. $x - 5$, equal to *zero*. And similarly for B.

Thus we have in (2)

$$3x + 7 = A(x + 6) + B(x - 5).$$

Put $x - 5 = 0$, or $x = 5$, then

$$3 \times 5 + 7 = A(5 + 6), \text{ or } A = 2.$$

Put $x + 6 = 0$, or $x = -6$, then

$$3(-6) + 7 = B(-6 - 5), \text{ or } B = 1.$$

Hence, as before,

$$\frac{3x + 7}{x^2 + x - 10} = \frac{2}{x - 5} + \frac{1}{x + 6}.$$

It is often more convenient to follow the second method.

Ex. 2. Resolve $\dfrac{1}{(x - 1)^2 (x + 2) (x + 3)}$ into partial fractions.

Assume

$$\frac{1}{(x-1)^2 (x+2) (x+3)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2} + \frac{D}{x+3} \ldots\ldots (1).$$

Then, clearing of fractions,

$$1 = A(x + 2)(x + 3) + B(x - 1)(x + 2)(x + 3) + C(x - 1)^2 (x + 3) + D(x - 1)^2 (x + 2) \ldots\ldots (2).$$

Put $x - 1 = 0$, or $x = 1$, then

$$1 = A(1 + 2)(1 + 3), \text{ or } A = \tfrac{1}{12}.$$

Put $x + 2 = 0$, or $x = -2$, then

$$1 = C(-2 - 1)^2 (-2 + 3) = 9C, \text{ or } C = \tfrac{1}{9}.$$

Put $x + 3 = 0$, or $x = -3$, then

$1 = D(-3-1)^2(-3+2) = -16D$, or $D = -\frac{1}{16}$.

We have thus determined A, C, D.

We cannot thus find B, but shall proceed as follows:

In (1) substitute the values of A, C, D just found, and transpose; then $\frac{B}{x-1}$

$$= \frac{1}{(x-1)^2(x+2)(x+3)} - \frac{1}{12(x-1)^2} - \frac{1}{9(x+2)} + \frac{1}{16(x+3)};$$

or, simplifying,

$$= \frac{-7(x-1)(x+2)(x+3)}{144(x-1)^2(x+2)(x+3)} = -\frac{7}{144(x-1)}.$$

$$\therefore B = -\frac{7}{144}.$$

We have therefore from (2), by substituting in (1) the values of A, B, C, D, obtained

$$\frac{1}{(x-1)^2(x+2)(x+3)} =$$

$$\frac{1}{12(x-1)^2} - \frac{7}{144(x-1)} + \frac{1}{9(x+2)} - \frac{1}{16(x+3)}.$$

Ex. 3. Find the partial fractions of $\frac{3x+5}{x^4+x^2+1}$.

The factors of the denominator are $x^2 + x + 1$ and $x^2 - x + 1$.

We shall assume the numerators of the required partial fractions to be of the form $Ax + B$ and $Cx + D$ respectively.

We then have

$$\frac{3x+5}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \ldots\ldots(1).$$

Or $3x+5 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1) \ldots(2)$.

Let $x^2 + x + 1 = 0$, or $x^2 = -x - 1$, then from (2)

$$3x + 5 = (Ax+B)(-x-1-x+1) = -2Ax^2 - 2Bx$$
$$= -2A(-x-1) - 2Bx$$
$$= (2A - 2B)x + 2A.$$

Equating coefficients of like powers of x, then

$$2A = 5, \text{ or } A = \tfrac{5}{2};$$

and $2A - 2B = 3$, or $2B = 5 - 3 = 2$, or $B = 1$.

And similarly we find $C = -\tfrac{5}{2}$, $D = 4$.

Hence, from (1),

$$\frac{3x + 5}{x^4 + x^2 + 1} = \frac{5x + 2}{2(x^2 + x + 1)} - \frac{5x - 8}{2(x^2 - x + 1)}.$$

Ex. VIII.

1. Find x in a series of ascending powers of y, when $y = 2x - 5x^2$.

2. Find by the method of indeterminate coefficients the quotient of $\dfrac{1 - 2x}{(1 - x)^3}$.

3. Find the cube root of $1 + 3x + 4x^2$.

4. Express by a series the difference between $(1 - 3x)^{-1}$ and $(1 + 2x)^{-1}$.

5. Find the sum of the series 1^2, 2^2, 3^2, &c., to n terms.

6. Show that $1^3 + 2^3 + 3^3 + 4^3 + \&c. + x^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$.

Resolve into partial fractions:

7. $\dfrac{x + 9}{(x + 3)(x + 5)}$.

8. $\dfrac{1}{(x^2 - 6)(x^2 + 8)}$.

9. $\dfrac{1}{x^3 + 6x^2 + 11x + 6}$.

10. $\dfrac{1}{x(1 + x)^2(1 + x + x^2)}$.

11. $\dfrac{4x^2 + x - 1}{3x^3 - 4x^2 + x}$.

12. $\dfrac{3}{1 + x^3}$.

13. $\dfrac{1}{(x^4 + x)(x^4 - 1)}$.

14. $\dfrac{1}{(x + a)(x + b)(x + c)}$.

15. $\dfrac{x}{(x + a)(x + b)(x + c)}$.

16. $\dfrac{x^2}{(x + a)(x + b)x + c)}$.

17. $\dfrac{1}{x^3 + 6x^2 + 16x - 21}$.

18. $\dfrac{x^2 + mx + n}{(x + a)(x + b)(x + c)}$.

CHAPTER VII.

THE EXPONENTIAL THEOREM.—LOGARITHMIC SERIES.

Exponential Theorem.

47. *To expand* a^x *in a series of ascending powers of* x.

$$a^x = \{1 + (a - 1)\}^x$$

$$= 1 + x(a - 1) + \frac{x(x - 1)}{1 \cdot 2}(a - 1)^2 + \frac{x(x - 1)(x - 2)}{1 \cdot 2 \cdot 3}(a - 1)^3 + \&c.$$

$$= 1 + x(a - 1) + \frac{x^2 - x}{2}(a - 1)^2 + \frac{x^3 - 3x^2 + 2x}{6}(a - 1)^3 + \&c.$$

Collecting the terms involving the first power only of x, we have

$$a^x = 1 + \left\{(a - 1) - \tfrac{1}{2}(a - 1)^2 + \tfrac{1}{3}(a - 1)^3 - \&c.\right\} x + \text{terms in } x^2, x^3, \&c.$$

Assume then

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \&c. \ldots\ldots\ldots (1),$$

Where $A = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$ (2).

From (1), squaring and arranging, we have

$$a^{2x} = 1 + 2Ax + (A^2 + 2B)x^2 + (2C + 2AB)x^3 + \&c. \ldots (3).$$

But from (1), putting therein $2x$ for x, we also get

$$a^{2x} = 1 + A(2x) + B(2x)^2 + C(2x)^3 + \&c.$$

$$= 1 + 2Ax + 4Bx^2 + 8Cx^3 + \&c. \ldots\ldots\ldots (4).$$

Hence the series in (3) and (4) are identical, and we may therefore equate the coefficients of like powers of x.

Hence $4B = A^2 + 2B$, or $2B = A^2$. Therefore $B = \frac{A^2}{1 \cdot 2}$.

$8\,C = 2\,C + 2\,AB$, or $6\,C = 2\,A \cdot \frac{A^2}{1\,.\,2}$. $\therefore C = \frac{A^3}{1\,.\,2\,.\,3}$. &c.

Hence, substituting in (1),

$$a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1\,.\,2} + \frac{A^3x^3}{1\,.\,2\,.\,3} + \text{&c.}\ldots\ldots\ldots(5).$$

Since this equation is true for all values of x, we may put

$$Ax = 1, \text{ and } \therefore x = \frac{1}{A}.$$

We then have $a^{\frac{1}{A}} = 1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \text{&c.}$

Hence, by Art. 42, Ex. 5,

$$a^{\frac{1}{A}} = e, \text{ or } a = e^A\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Taking the logarithm of each side of the equation to base e, we get

$$A = \log_e a\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Hence, substituting in (5) for this value of A, we get

$$a^x = 1 + \frac{x \log_e a}{1} + \frac{(x \log_e a)^2}{1\,.\,2} + \frac{(x \log_e a)^3}{1\,.\,2\,.\,3} + \text{&c.}$$

The $(n + 1)$th term of this series is $\frac{(x \log_e a)^n}{\lfloor n}$, and the formula itself is known by the name of the **Exponential Theorem.**

Cor. 1. Put $a = e$, then since $\log_e e = 1$, we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \text{&c.}$$

Cor. 2. Since in (7) we have $A = \log_e a$, it follows at once from (2) that

$$\log_e a = (a - 1) - \tfrac{1}{2}(a - 1)^2 + \tfrac{1}{3}(a - 1)^3 - \text{&c.}$$

Logarithmic Series.

48. *To show that* $\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \text{&c.}$

and that $\log_e (1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \text{&c.}$

In Art. 47, Cor. 1, we have

$$\log_e a = (a - 1) - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.$$

Put $1 + x$ for a, and therefore x for $a - 1$, then

$$\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now put $-x$, then

$$\log_e (1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These series are not suitable for the calculation of logarithms. We proceed to obtain from them more convergent series.

49. *To show that* $\log_e (n + 1)$

$$= \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \&c.\right\}.$$

Now $\log_e \dfrac{1+x}{1-x} = \log_e (1 + x) - \log_e (1 - x)$

$$= \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \&c.\right)$$

$$- \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \&c.\right)$$

$$= 2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c.\right) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since this is true for all values of x, we may put

$$\frac{1+x}{1-x} = \frac{m}{n}, \text{ and therefore also } x = \frac{m-n}{m+n}.$$

We then have $\log_e \dfrac{m}{n}$

$$= 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\} (2).$$

Now put $m = n + 1$, and therefore

$$m - n = 1, \text{ and } m + n = 2n + 1 ;$$

Then we have $\log_e \frac{n+1}{n}$

$$= 2\left\{ \frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \&c. \right\}.$$

Hence, transposing, $\log_e (n+1)$

$$= \log_e n + 2\left\{ \frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \&c. \right\};$$

an important series, by which the logarithm of any number may be easily computed when that of the next lower is known.

50. *To show that* $\log_e (x+1)$

$$= 2\log_e x - \log_e (x-1) - 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

We have in (2) of the last Art., putting

$m = x^2$, and $n = x^2 - 1$, and therefore

$m - n = 1$, and $m + n = 2x^2 - 1$,

$$\log_e \frac{x^2}{x^2-1} = 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

But $\log_e \frac{x^2}{x^2-1} = \log_e \frac{x^2}{(x+1)(x-1)}$

$$= 2\log_e x - \log_e (x+1) - (x-1).$$

Hence $2\log_e x - \log_e (x+1) - \log_e (x-1)$

$$= 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

Or, transposing, $\log_e (x+1)$

$$= 2\log_e x - \log_e (x-1) - 2\left\{ \frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \&c. \right\}.$$

This is a very useful formula, by which the logarithm of any number is found from the logarithms of the two numbers next preceding.

51. *To show that, when* x *is large, whatever be the base,*

$$2\log x = \log (x+1) + \log (x-1) + \frac{1}{2x}\left\{ 1 + \frac{1}{6x^2} + \frac{1}{90x^4} + \&c. \right\} \log \frac{x+1}{x-1}.$$

$$\text{Now, } \log_e \frac{x^2}{x^2-1} = -\log_e \frac{x^2-1}{x^2} = -\log_e \left(1 - \frac{1}{x^2}\right)$$

$$= -\left(-\frac{1}{x^2} - \frac{1}{2}\cdot\frac{1}{x^4} - \frac{1}{3}\cdot\frac{1}{x^6} - \&c.\right)$$

$$= \frac{1}{x^2} + \frac{1}{2}\cdot\frac{1}{x^4} + \frac{1}{3}\cdot\frac{1}{x^6} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\text{And } \log_e \frac{x+1}{x-1} = \log_e \frac{1+\frac{1}{x}}{1-\frac{1}{x}}, \text{ and therefore, by Art. 49 (1),}$$

$$= 2\left\{\frac{1}{x} + \frac{1}{3}\cdot\frac{1}{x^3} + \frac{1}{5}\cdot\frac{1}{x^5} + \&c.\right\} \ldots\ldots(2).$$

Dividing (1) by (2), we have, after performing the division,

$$\log_e \frac{x^2}{x^2-1} \div \log_e \frac{x+1}{x-1} = \frac{1}{2}\left\{\frac{1}{x} + \frac{1}{6x^3} + \frac{7}{90x^5} + \&c.\right\} \ldots(3).$$

$$\text{Now, } \log_e \frac{x^2}{x^2-1} = 2\log_e x - \log_e (x+1) - \log_e (x-1);$$

Hence, multiplying each side of (3) by $\log_e \frac{x+1}{x-1}$, and transposing, &c.

$$2\log_e x = \log_e (x+1) + \log_e (x-1)$$

$$+ \frac{1}{2x}\left\{1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c.\right\} \log_e \frac{x+1}{x-1} \ldots\ldots\ldots(4).$$

Now it is shown (Art. 50) that

$$\log_e x = \log_e a \,.\, \log_a x.$$

Hence we may replace the base e in (4) by the base a, if we introduce the factor $\log_e a$ into every term. Dividing each side of the newly-formed equation by this factor, we then get the formula in (4) expressed thus:

$$2\log_a x = \log_a (x+1) + \log_a (x-1)$$

$$+ \frac{1}{2x}\left\{1 + \frac{1}{6x^2} + \frac{7}{90x^4} + \&c.\right\} \log_a \frac{x+1}{x-1}.$$

Logarithms to base e are called Napierian logarithms, from their inventor.

It is evident that, by the series of the last articles, and by similar series, Napierian logarithms may be calculated. It will be necessary to use each series however for prime numbers only, since composite numbers may be broken up into their prime factors, and their logarithms easily obtained from those of their factors.

Ex. 1. Find $\log_e 2$, $\log_e 3$.

We have

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c.\right\}.$$

Put $m = 2$, $n = 1$, and $\therefore \frac{m-n}{m+n} = \frac{2-1}{2+1} = \frac{1}{3}$.

We then have

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \&c.\right\};$$

Or, reducing to decimals and adding

$$= \cdot 6931471\ldots$$

Similarly, putting $m = 3$, $n = 2$, and $\therefore \frac{m-n}{m+n} = \frac{1}{5}$, we have

$$\log_e \frac{3}{2} = 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \&c.\right\};$$

or, since

$$\log_e \frac{3}{2} = \log_e 3 - \log_e 2,$$

we have,

$$\log_e 3 = \log_e 2 + 2\left\{\frac{1}{5} + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \&c.\right\};$$

or, performing the necessary reductions, &c.,

$$= 1 \cdot 0986122\ldots$$

Ex. 2. Find $\log_e 5$, $\log_e 10$.

Put $m = 5$, $n = 4$, and $\therefore \frac{m-n}{m+n} = \frac{1}{9}$, then since

$\log_e \frac{5}{4} = \log_e 5 - 2\log_e 2$, we have, by transposing,

$$\log_e 5 = 2\log_e 2 + 2\left\{\frac{1}{9} + \frac{1}{3}\left(\frac{1}{9}\right)^3 + \frac{1}{5}\left(\frac{1}{9}\right)^5 + \text{\&c.}\right\};$$

or, reducing, &c.,

$$= 1{\cdot}6094379\ldots$$

Again,

$$\log_e 10 = \log_e (2 \times 5) = \log_e 2 + \log_e 5$$
$$= {\cdot}6931471\ldots + 1{\cdot}6094379\ldots$$
$$= 2{\cdot}3025850\ldots$$

Ex. 3. Find $\log_e 16205$.

$$\log_e 16205 = \log_e (2^3 \times 3^4 \times 5^2 + 5)$$
$$= \log_e \left\{2^3 \times 3^4 \times 5^2 \left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right)\right\}$$
$$= 3\log_e 2 + 4\log_e 3 + 2\log_e 5 + \log_e\left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right).$$

By substituting the values of $\log_e 2$, $\log_e 3$, $\log_e 5$, already found, and by developing $\log_e\left(1 + \frac{1}{2^3 \times 3^4 \times 5}\right)$, the required logarithm is known.

Calculation of Common Logarithms.

52. To show that $\log_b N = \frac{1}{\log_a b} \cdot \log_a N$.

Let $$N = a^x = b^y \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Then we have, from the definition of a logarithm,

$$x = \log_a N \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$
$$y = \log_b N \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, we have in (1), $a^x = b^y$, or taking logarithms to base a,

$$x = y \log_a b;$$

or, substituting from (2) and (3),

$$\log_a N = \log_b N \cdot \log_a b$$

$$\therefore \log_b N = \frac{1}{\log_a b} \cdot \log_a N.$$

COR. Put $b = 10$, and $a = e$, then

$$\log_{10} N = \frac{1}{\log_e 10} \cdot \log_e N.$$

Hence common logarithms may be found from Napierian logarithms by multiplying the latter by $\frac{1}{\log_e 10}$.

We call this quantity the *modulus* of the common system of logarithms; and if we represent the modulus by μ, we have

$$\mu = \frac{1}{\log_e 10} = \frac{1}{2{\cdot}30258509\ldots} = {\cdot}43429448\ldots$$

53. By multiplying by μ the series obtained in Arts. 48, 49, and putting $\mu \log_e = \log_{10}$ we have, omitting the suffix 10.

$$\log(1 + x) = \mu \left\{ x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \&c. \right\} \ldots (1).$$

$$\log(1 - x) = \mu \left\{ -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \&c. \right\} \ldots\ldots (2).$$

$$\log \frac{1 + x}{1 - x} = 2\mu \left\{ x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \&c. \right\} \ldots\ldots (3).$$

$$\log \frac{m}{n} = 2\mu \left\{ \frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \&c. \right\} (4).$$

$$\log(n + 1)$$

$$= \log n + 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \&c. \right\} (5).$$

$$\log(x + 1)$$

$$= 2\log x - \log(x - 1) - 2\mu \left\{ \frac{1}{2x^2-1} + \frac{1}{3}\left(\frac{1}{2x^2+1}\right)^3 + \&c. \right\} (6).$$

Proportional Parts.

54. *To find the logarithm of a number containing* (n + 1) *digits, when the table contains the logarithm of numbers containing* n *digits.*

We have

$$\log(n+\delta) - \log n = \log\frac{n+\delta}{n} = \log\left(1 + \frac{\delta}{n}\right)$$

$$= \mu\left(\frac{\delta}{n} - \frac{1}{2}\cdot\frac{\delta^2}{n^2} + \frac{1}{3}\cdot\frac{\delta^3}{n^3} - \&c.\right).$$

Suppose n to be an integer containing 5 figures, and δ a quantity less than unity.

We then have, since μ (= ·434...) is $< \frac{1}{2}$,

$$\mu\cdot\frac{\delta^2}{2\,n^2} < \frac{1}{4}\cdot\frac{1}{10000^2} \text{ and } \therefore < \cdot000000003\,;$$

$$\mu\cdot\frac{\delta^3}{3\,n^3} < \frac{1}{6}\cdot\frac{1}{10000^3} \text{ and } \therefore < \cdot000000000002, \&c.$$

Hence, at least as far as the *seventh* place of decimals, the omission of all the terms of the above series after the first will not affect the result.

We then have

$$\log(n + \delta) - \log n = \mu\cdot\frac{\delta}{n} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

And, similarly,

$$\log(n + 1) - \log n = \mu\cdot\frac{1}{n} = d \text{ suppose, } \ldots\ldots (2).$$

Then we have,

$$\log(n + \delta) - \log n = \delta d \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now d is the difference between the logarithms of two consecutive numbers;

And δ is the difference (less than unity) of two numbers, the logarithm of the greater of which is required.

Hence we have the following rule:

To find the logarithm of a number which lies between two consecutive numbers, multiply the difference of the given

number and the next lower in the tables by the difference of the logarithms of the numbers next above and below the given number, and add this result to the logarithm of the smaller given number.

This rule has been assumed in Vol. I. of this work, to which the student is referred for practical applications.

REMARK. Since the mantissæ of all numbers, having the same digits, are the same, it follows that the above applies equally to numbers the integral part of which contains more than five digits.

55. *To prove that*

$$n^r - n(n-1)^r + \frac{n(n-1)}{\lfloor 2}(n-2)^r - \frac{n(n-1)(n-2)}{\lfloor 3}(n-3)^r$$

$$+ \&c. = \lfloor n, \textit{if } r = n, \textit{and } = 0, \textit{if } r < n.$$

We have

$$(e^x - 1)^n = e^{nx} - n \cdot e^{(n-1)x} + \frac{n(n-1)}{\lfloor 2} e^{(n-2)x}$$

$$- \frac{n(n-1)(n-2)}{\lfloor 3} e^{(n-3)x} + \&c.$$

If we now expand each of the quantities e^{nx}, $e^{(n-1)x}$, &c., by the Exponential Theorem, and write down the coefficient of x^r in the whole expression, we have

Coefficient of x^r in $(e^x - 1)^n$

$$= \frac{n^r}{\lfloor r} - n \cdot \frac{(n-1)^r}{\lfloor r} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{(n-2)^r}{\lfloor r}$$

$$- \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{(n-3)^r}{\lfloor r} + \&c. \ldots\ldots\ldots\ldots (1).$$

Again, $(e^x - 1)^n$

$$= \left(x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \&c.\right)^n$$

$$= x^n + \text{terms involving higher powers of } x \ldots\ldots\ldots (2).$$

Hence the coefficient of x^n is *unity;* and since the expansion contains no terms with lower powers of x than the nth, we may say that

The coefficient of x^r in $(e^x - 1)^n = 1$, when $r = n$,
and $= 0$, when $r < n$, } ...(3).

Hence from (1) and (3), equating the coefficients of x^r,

$$\frac{n^r}{\lfloor r} - n\frac{(n-1)^r}{\lfloor r} + \frac{n(n-1)}{\lfloor 2} \cdot \frac{(n-2)^r}{\lfloor r} - \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \frac{(n-3)^r}{\lfloor r} + \&c.$$

$= 1$, when $r = n$,
and $= 0$, when $r < n$.

$$\therefore n^r - n(n-1)^r + \frac{n(n-1)}{\lfloor 2}(n-2)^r - \frac{n(n-1)(n-2)}{\lfloor 3}(n-3)^r + \&c.$$

$= \lfloor n$, if $r = n$.
and $= 0$, if $r < n$.

Ex. IX.

Show that

1. $\dfrac{e^{2x} - 1}{xe^x} = 2\left\{1 + \dfrac{x^2}{\lfloor 3} + \dfrac{x^4}{\lfloor 5} + \&c.\right\}$

2. $\dfrac{e^{2x} + 1}{e^x} = 2\left\{1 + \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} + \&c.\right\}$

3. $m^n = m + \dfrac{m(m-1)}{1 \cdot 2}(2^n - 2)$

$$+ \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}(3^n - 3 \cdot 2^n + 3) + \&c.$$

4. $n - n(n-1) + \dfrac{n(n-1)(n-2)}{2} - \&c. = 0.$

5. $n^{n-1} - n(n-1)^{n-1} + \dfrac{n(n-1)}{1 \cdot 2}(n-2)^{n-1} - \&c. = 0.$

6. $n^n - n(n-1)^n + \dfrac{n(n-1)}{1 \cdot 2}(n-2)^n - \&c. = \lfloor n.$

7. $n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{1 \cdot 2}(n-2)^{n+1} - \&c.$

$$= \frac{1}{2} n \underline{\lfloor n+1}.$$

8. $1^n - n \cdot 2^n + \frac{n(n-1)}{1 \cdot 2} 3^n - \&c. + (-1)^n \cdot (n+1)^n$

$$= (-1)^n \underline{\lfloor n}.$$

9. $1^m + 2^m + 3^m + \&c. + n^m = \frac{n^{m+1}}{m+1} + \frac{n}{2} + m \cdot \frac{n^{m-1}}{1 \cdot 2} + \&c.$

10. If p, q are each less than 1, show that $\frac{\log(1-p)}{\log(1-q)}$ is less than $\frac{p}{q - pq}$, and greater than $\frac{p - pq}{q}$.

11. If $u_r = r^p$, show that

$$u_1 - nu_2 + \frac{n(n-1)}{1 \cdot 2} u_3 + \&c. + (-1)^n u_{n+1} = (-1)^n \underline{\lfloor n}.$$

12. Show that the limit of $\left(1 + \frac{1}{nx}\right)^x$, when x is infinite, is $e^{\frac{1}{n}}$.

13. Having given that $1 \cdot 2 \cdot 3 \ldots n = \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}$, when n is increased without limit, show that

$$1 \cdot 3 \cdot 5 \ldots (2n-1) = 2^{n+\frac{1}{2}} \cdot n^n \cdot e^{-n},$$

when n is increased without limit.

14. Hence show that the limit of

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n} \cdot \frac{1}{2n-1} \text{ is } n^{-\frac{3}{2}},$$

when n is increased without limit.

15. Show that

$$\left(1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19x^4}{720} + \&c.\right) \log_e(1+x) = x.$$

16. If $a = b - x$, show that $\log_e a$

$$= \log_e b - \frac{1}{2b - x}\left\{\left(2 - \frac{1}{2}\right)x + \left(\frac{2}{2} - \frac{1}{3}\right)\frac{x^2}{b} + \left(\frac{2}{3} - \frac{1}{4}\right)\frac{x^3}{b^2} + \&c.\right\}.$$

17. If $a^2 + 1 = 0$, show that

(1.) $e^{ax} + e^{-ax} = 2\left\{1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \&c.\right\}$

(2.) $e^{ax} - e^{-ax} = 2a\left\{x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \&c.\right\}$

18. If a, b, c are consecutive numbers, show that

$$\log b = \frac{1}{2}(\log a + \log c) + \mu\left\{\frac{1}{2ac+1} + \frac{1}{3}\cdot\frac{1}{(2ac+1)^2} + \frac{1}{5}\frac{1}{(2ac+1)^5} + \&c.\right\}$$

CHAPTER VIII.

THE MULTINOMIAL THEOREM.

56. It is often required to expand the powers of algebraical expressions containing more than two terms. This may be done gradually by means of the Binomial Theorem, but the method is often cumbersome and inconvenient. By means of the Multinomial Theorem we are able to do this more concisely.

57. *To find the general term of the expansion of*

$$(a + b + c + d + \&c.)^n.$$

Put $b_1 = b + c + d + \&c.$,

and $n = p + q_1$.

Then the $(q_1 + 1)$th term of $(a + b_1)^n$ is

$$\frac{n(n-1)\ldots(n-q_1+1)}{\lfloor q_1}\, a^{n-q_1} b_1^{q_1}$$

$$= \frac{n(n-1)\ldots(p+1)}{\lfloor q_1}\, a^p b_1^{q_1} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

when q_1 is a positive integer.

Put $c_1 = c + d + e + \&c.$,

and $q_1 = q + r_1$.

Then the $(r_1 + 1)$th term of $b_1^{q_1}$, or $(b + c_1)^{q_1}$ is

$$\frac{q_1 (q_1 - 1) \ldots (q_1 - r_1 + 1)}{\lfloor r_1} b^{q_1 - r_1} c_1^{r}$$

$$= \frac{q_1 (q_1 - 1) \ldots (q + 1)}{\lfloor r_1} b^q c_1^{r_1} = \frac{\lfloor q_1}{\lfloor q \lfloor r_1} b^q c_1^{r_1} \ldots\ldots\ldots (2.)$$

where q_1, q, r, are positive integers.

Hence, combining this with the expression in (1), where for $b_1^{q_1}$ we may substitute its general term, we find

The general term of the expansion of $(a + b + c_1)^n$

$$= \frac{n (n - 1) \ldots (p + 1)}{\lfloor q_1} a^p : \frac{\lfloor q_1}{\lfloor q \lfloor r_1} b^q c_1^{r_1}$$

$$= \frac{n (n - 1) \ldots (p + 1)}{\lfloor q \lfloor r} a^p b^q c_1^{r_1} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Proceeding in this way we shall find

The general term of the expansion of $(a + b + c + d + \&c.)^n$

$$= \frac{n (n - 1) \ldots (p + 1)}{\lfloor q \lfloor r \ \&c.} \cdot a^p b^q c^r \ \&c.$$

where q, r, &c., *are positive integers.*

Since $n = p + q_1$, $q_1 = q + r_1$, $r_1 = r + s_1$, &c., we must have

$p + q + r + \&c. = n$, and q, r, s, &c., *positive integers.*

Cor. When n is a positive integer, we have

General term of $(a + b + c + d + \&c.)^n$

$$= \frac{n (n - 1) \ldots (p + 1) \cdot p (p - 1) \ldots 1}{\lfloor p \lfloor q \lfloor r \ \&c.} a^p b^q c^r \ \&c.$$

$$= \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \ \&c.} a^p b^q c^r \ \&c.$$

where $p + q + r + \&c. = n$.

58. *To expand* $(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.)^n$.

In the result of Art. 57, put $a = a_0$, $b = a_1 x$, $c = a_2 x^2$, &c., then

General term of $(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$

$$= \frac{n(n-1)\ldots(p+1)}{\lfloor q \quad \lfloor r \quad \lfloor s \quad \&c.} a_0^p (a_1x)^q (a_2x^2)^r (a_3x^3)^s \ \&c.$$

$$= \frac{n(n-1)\ldots(p+1)}{\lfloor q \quad \lfloor r \quad \lfloor s \quad \&c.} a_0^p a_1^q a_2^r a_3^s \ldots x^{q+2r+3s+\&c.} \text{(A)}.$$

where $p + q + r + s + \&c. = n$, and q, r, s, &c., *positive integers.*

When n is a positive integer, this result, by Art. 56, Cor., may be thus expressed:

General term of $(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^n$

$$= \frac{\lfloor n}{\lfloor p \quad \lfloor q \quad \lfloor r \quad \lfloor s \quad \&c.} a_0^p a_1^q a_2^r a_3^s \ldots x^{q+2r+3s+\&c.} \ldots \text{(B)}.$$

where $p + q + r + s + \&c. = n$, and p, q, r, s, &c., are positive integers.

Suppose now we are required to find the coefficient of the term involving x^m.

Taking the formula in (A) above, we have the conditions

$$p + q + r + s + \&c. = n,$$
$$\text{and } q + 2r + 3s + \&c. = m.$$

where q, r, s, &c., *are positive integers.*

Now, by assigning to q, r, s, &c., *all possible values*, subject to the condition that they are positive integers, and that

$$q + 2r + 3s + \&c. = m,$$

we may obtain sets of values of p, q, r, &c., which, when substituted in (A), will give the partial coefficients of x^m; and the sum of these will give the whole coefficient.

Ex. 1. Find the coefficient of x^6 in the expansion of $(1 - 3x + 4x^2 - 2x^3)^5$.

As $n = 5$, a positive integer, we may use the formula (B).

We have

$$p + q + r + s = 5 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$
$$q + 2r + 3s = 6 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

We shall find it convenient to proceed as follows:

1. Commence with the highest possible value of the letter

at the right of (2), viz., in this case, of s; and take each lower value until we arrive at $s = 0$.

2. Then with the resulting equation—here with the equation containing $q + 2r$—take first the highest possible value of r, and so on.

3. Find the value of p from (1) by means of the values of q, r, s thus obtained.

Thus :—

Take $s = 2$, and $\therefore q + 2r = 0$, where we may have

$$r = 0, q = 0, p = 3.$$

Take $s = 1$, and $\therefore q + 2r = 3$, where we may have

$$r = 1, q = 1, p = 2,$$

or, $$r = 0, q = 3, p = 1.$$

Take $s = 0$, and $\therefore q + 2r = 6$, where we may have

$$r = 3, q = 0, p = 2,$$

$$r = 2, q = 2, p = 1,$$

$$r = 1, q = 4, p = 0.$$

These are all the possible solutions, and they are conveniently arranged thus :

p	q	r	s
3	0	0	2
2	1	1	1
1	3	0	1
2	0	3	0
1	2	2	0
0	4	1	0

Hence, from (B), since $a_0 = 1$, $a_1 = -3$, $a_2 = 4$, $a_3 = -2$, substituting the values of p, q, r, s, and omitting from the formula every term containing a letter whose value

may happen to be zero, and remembering that $\lfloor 1 = 1$, we have

Coefficient of x^6

$$= \frac{\lfloor 5}{\lfloor 3 \ \lfloor 2}(1)^3(-2)^2 + \frac{\lfloor 5}{\lfloor 2}(1)^2(-3)(4)(-2)$$

$$+ \frac{\lfloor 5}{\lfloor 3}(1)(-3)^3(-2) + \frac{\lfloor 5}{\lfloor 2 \ \lfloor 3}(1)^2(4)^2$$

$$+ \frac{\lfloor 5}{\lfloor 2 \ \lfloor 2}(1)(-3)^2(4)^2 + \frac{\lfloor 5}{\lfloor 4}(-3)^4(4)$$

$$= 40 + 1440 + 1080 + 640 + 4320 + 1620 = 9140.$$

Ex. 2. Find the coefficient of x^3 in $(1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}}$

Here $n = \frac{1}{3}$, a fraction; we shall therefore use the formula (A).

We have

$$p + q + r + s + \&c. = \tfrac{1}{3} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

and

$$q + 2r + 3s + \&c. = 3 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations (1) and (2) need not contain any letter beyond s; for if we take another letter, as t, we have

$$q + 2r + 3s + 4t = 3;$$

and the only value of t possible is $t = 0$, and so of any other letters. We shall therefore treat equations (1) and (2) as if they were written

$$p + q + r + s = \tfrac{1}{3}.$$
$$q + 2r + 3s = 3.$$

On trial we find the only solutions to be as in the annexed table:

p	q	r	s
$-\frac{2}{3}$	0	0	1
$-\frac{5}{3}$	1	1	0
$-\frac{8}{3}$	3	0	0

Hence the coefficient of x^3

$$= \frac{\frac{1}{3}}{1}(10) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1}(3)(6) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{\lfloor 3}(3)^3$$

$$= \frac{10}{3} - 4 + \frac{5}{3} = 1.$$

We may verify this result.

For $1 + 3x + 6x^2 + 10x^3 + \&c. = (1 - x)^{-3}$

$$\therefore (1 + 3x + 6x^2 + 10x^3 + \&c.)^{\frac{1}{3}} = \left\{(1 - x)^{-3}\right\}^{\frac{1}{3}}$$

$$= (1 - x)^{-1} = 1 + x + x^2 + x^3 + \&c.$$

Ex. X.

Find the coefficient of

1. x^3 in $(1 + x + x^2)^4$.
2. x^4 in $(1 - 2x + 3x^2 - 4x^3)^3$.
3. x^5 in $(4 - 5x + x^2)^{\frac{1}{2}}$.
4. x^3 in $(6 + 5x - 6x^2)^5$.
5. x^4 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.
6. x^3 in $\left(1 + \frac{1}{3}x + \frac{2}{27}x^2 + \frac{14}{81}x^3 + \&c.\right)^3$.
7. a^2bc^3d in $(a + b + c + d)^7$.

8. a^3b^2 in $(a + b + c + d)^5$.

9. abc^4d^2 in $(a + b + c + d + e)^8$.

10. x^6 in $(a_0 + a_1x + a_2x^2 + a_3x^3)^4$.

11. x^4 in $(1 + 2x + 3x^2 + 4x^3 + \&c.)^{-\frac{3}{2}}$.

12. x^5 in $(1 + 4x + 10x^2 + 20x^3 + 35x^4 + 42x^5 + \&c.)^{-4}$.

13. Expand $(1 - 3x + 4x^2)^{-\frac{1}{3}}$ to five terms.

14. Expand $(1 + 7x^2 + x^3)^{\frac{1}{2}}$ to four terms.

15. Show that the coefficient of the middle term of the expansion of $(1 + x + x^2)^m$

$$= 1 + \frac{m(m-1)}{\lfloor 1} + \frac{m(m-1)(m-2)(m-3)}{(\lfloor 2)^2}$$

$$+ \frac{m(m-1)\ldots(m-5)}{(\lfloor 3)^2} + \&c.$$

$$+ \frac{m(m-1)\ldots(m-2r+1)}{(\lfloor r)^2} + \&c.$$

16. If c_r denote the coefficient of x^r in the expansion of $(1 + 2x + 2x^2 + 2x^3 + 2x^4 + \&c.)^n$, show that

$$(m + 1)c_{m+1} - 2nc_m - (m - 1)c_{m-1} = 0.$$

17. If c_0, c_1, c_2, &c., be the successive coefficients of the expansion of $(1 + 3x + 5x^2 + \&c. + \overline{2p+1}\ x^p)^n$, show that

$$c_0 + c_1 + c_2 + \&c. + c_{np} = (p + 1)^{2n}.$$

18. Show that the sum of the coefficients of the expansion of

$$\{1 - 3x - 5x^2 + 7x^3 + 9x^4 \ldots + (-1)^m(4m-1)x^{2m-1}\}^n$$

$= 0$, or $(-2)^n$, according as m is of the form $2p$ or $2p + 1$.

CHAPTER IX.

INTEREST AND ANNUITIES.

Interest.

59. *To find the amount of a given sum for a given number of years at a given rate, simple interest.*

Let P = the principal in pounds.
r = the interest of £1 for 1 year.
n = number of years.
M = the amount.

We have

Interest of P for 1 year = Pr.

$\therefore$,, n years = Pnr.

Hence $M = P + Pnr = P(1 + nr)$ .....(1).

$\therefore$ also $P = \dfrac{M}{1 + nr}$ .....(2).

and $r = \dfrac{M - P}{nP}$ .....(3).

Cor. If interest be allowed for *fractions* of years the above formulæ hold.

60. *To find the amount of a given sum for a given number of years at a given rate, compound interest.*

Let R = the amount of £1 for 1 year.
$\therefore$ $R = 1 + r$.

Also, the amount of P for 1 year = PR.
$\therefore$,, P ,, 2 ,, = $PR \cdot R = PR^2$.
,, P ,, 3 ,, = $PR^2 \cdot R = PR^3$.
&c. = &c.

Hence also ,, P for n years = PR^n.

$\therefore$ $M = PR^n$
and $\therefore$ $\log M = \log P + n \log R$ } .....(1).

$$\left.\begin{aligned} \text{We have also} \quad P &= \frac{M}{R^n} \\ \text{and} \quad \log P &= \log M - n \log R \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\left.\begin{aligned} \text{And again,} \quad R &= \sqrt[n]{\frac{M}{P}} \\ \text{and} \quad \therefore \log R &= \frac{1}{n}(\log M - \log P) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$\left.\begin{aligned} \text{We have also,} \quad R^n &= \frac{M}{P} \\ \text{and} \quad \therefore n &= \frac{\log M - \log P}{\log R} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Cor. If n be fractional, and interest be allowed for the fractional part of a year at the same rate, the above formulæ require modification.

For, let $n = m + \frac{1}{p}$, where $\frac{1}{p}$ is a proper fraction, and m an integer.

We have, by (1), amount for m years $= PR^m$.

Now amount of £1 for $\frac{1}{p}$ year $= 1 + \frac{r}{p}$.

Hence, amount of PR^m for $\frac{1}{p}$ year, and

$$\therefore \text{Amount of P for } \left(m + \frac{1}{p}\right) \text{ years}$$

$$= PR^m\left(1 + \frac{r}{p}\right).$$

It is easy to see here that interest is really reckoned in two ways, for whole years, and for a fraction of a year. If it be agreed that interest is to be reckoned for every mth part of a year, we obtain a different formula, as shown in the next article.

61. *To find the amount of a given sum at compound interest, interest being reckoned* m *times a year.*

In n *years there are* mn distinct periods, for which com-

pound interest is to be reckoned. We therefore have to find the amount of a sum P for mn periods of time, when $\frac{r}{m}$ is the interest of £1 for *one such period.*

Hence we have simply to apply formula (1) of last article, where we must put mn for n, and $\frac{r}{m}$ for r. We then get

$$\mathrm{M} = \mathrm{P}\left(1 + \frac{r}{m}\right)^{mn}.$$

Cor. Let m be infinite, that is, suppose interest to be considered due every instant, then

M = limit of $\mathrm{P}\left(1 + \frac{r}{m}\right)^{mn}$, when $m = \infty$: or, by Art. 42,

$$\mathrm{M} = \mathrm{P}e^{rn}.$$

Annuities.

62. *To find the amount of an annuity for a given time, reckoning simple interest.*

Let A be the annuity, n the number of years, r the interest of £1 for one year, M the amount.

Then, amount due at the end of

1 year = A.
2 ,, = A + (1 + r)A.
3 ,, = A + (1 + r)A + (1 + 2 r)A.
&c. = &c.
n ,, = A + (1 + r)A + (1 + 2 r)A + &c. + $(1 + \overline{n-1}\,.\,r)$A.

Or, $\mathrm{M} = \{1 + (1 + r) + (1 + 2r) + \&c. + (1 + \overline{n-1}\,.\,r)\}\,\mathrm{A}$.

$$= \left\{2 \times 1 + \overline{n-1}\,.\,r\right\}\frac{n}{2}\,.\,\mathrm{A} = n\mathrm{A} + \tfrac{1}{2}n(n-1)r\mathrm{A}.$$

63. *To find the present worth of an annuity to continue for a given time, reckoning simple interest.*

I. Let P be the present worth of an annuity A, which is to continue for n years.

Then if P be put out to simple interest for n years, its amount will be equal to the amount of the annuity for the *same time.*

Hence, $P(1 + nr) = nA + \frac{1}{2} n(n-1) rA$. Therefore

$$P = \frac{nA + \frac{1}{2} n(n-1) rA}{1 + nr} = \frac{nA}{2} \cdot \frac{2 + (n-1)r}{1 + nr} \ldots\ldots\ldots (A).$$

II. There is another way of solving this problem, which brings out a different result. It proceeds on the principle that the present worth of the annuity is the sum of the present worth of the separate annual payments.

Thus, present worth of A due 1 year hence $= \frac{A}{1 + r}$

" " 2 " $= \frac{A}{1 + 2r}$

" " 3 " $= \frac{A}{1 + 3r}$

&c. = &c.

" " n " $= \frac{A}{1 + nr}$.

Now, taking P as the sum of these, we have

$$P = \left\{ \frac{1}{1 + r} + \frac{1}{1 + 2r} + \frac{1}{1 + 3r} + \text{&c.} + \frac{1}{1 + nr} \right\} A \ldots\ldots (B).$$

If this sum be put out to simple interest for n years, we shall find that its amount is not the same as that of the annuity for the same time. The reason appears to be that in reality we are reckoning both *simple* and *compound* interest.

Thus, for any specified year, as the pth, the present value of a payment $A = \frac{A}{1 + pr}$.

It is true that at *simple interest* this sum amounts to A in p years, and we may allow this sum A to remain till the n years are completed. But this is equivalent to *interest upon the interest* gained in the first p years, in addition to the interest upon the principal merely.

We shall presently find that when *compound interest* is reckoned throughout, the present worth of an annuity is the same by either method of procedure.

64. ***To find the amount of an annuity, reckoning compound interest.***

We have

Amount due at the end of

$$\begin{aligned}
1 \text{ year} &= A \\
2 \;\;\text{,,} &= A + AR \\
3 \;\;\text{,,} &= A + AR + AR^2 \\
\&c. &= \&c. \\
n \;\;\text{,,} &= A + AR + AR^2 + \&c. + AR^{n-1} \\
&= A \cdot \frac{R^n - 1}{R - 1} = \frac{A}{r}(R^n - 1).
\end{aligned}$$

Hence $M = \frac{A}{r}(R^n - 1)$.

65. ***To find the present worth of an annuity, reckoning compound interest.***

I. We shall proceed on the principle that if the present worth P be put out to compound interest for n years, it ought to amount to the same as the annuity for that time.

Now, amount of P in n years $= PR^n$;

And, amount of annuity for n years $= \frac{A}{r}(R^n - 1)$.

Equating these we have

$$PR^n = \frac{A}{r}(R^n - 1),$$

$$\therefore P = \frac{A}{r}(1 - R^{-n}).$$

II. We will now proceed on the principle that the present worth P is the sum of the present worths of the respective annual payments.

We have, Art. 60,

$$\begin{aligned}
\text{Present worth of A due 1 year hence} &= \frac{A}{R}. \\
\text{,,} \quad \text{,,} \quad 2 \quad \text{,,} &= \frac{A}{R^2} \\
\text{,,} \quad \text{,,} \quad 3 \quad \text{,,} &= \frac{A}{R^3} \\
\&c. &= \&c. \\
\text{,,} \quad \text{,,} \quad n \quad \text{,,} &= \frac{A}{R^n}
\end{aligned}$$

$$\therefore P = \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \&c. + \frac{A}{R^n} = \frac{A}{r} \cdot \frac{1 - \frac{1}{R^n}}{1 - \frac{1}{R}} = \frac{A}{r}(1 - R^{-n}).$$

We thus see that when compound interest is reckoned, we arrive at the same result, by finding the present worth on either principle.

66. *To find the present worth of an annuity to continue for ever.*

I. *Reckoning compound interest.*

We have $$P = \frac{A}{r}(1 - R^{-n}).$$

When $n = \infty$, the limit of $R^{-n} = 0$; and then $P = \frac{A}{r}$.

Hence, the present worth of an annuity A to continue for ever $= \frac{A}{r}$.

II. *Reckoning simple interest.*

Taking the formula (A) in Art. 63, we have

$$P = \frac{nA}{2} \cdot \frac{2 + (n-1)r}{1 + nr} = \frac{nA}{2} \cdot \frac{\frac{2}{n} + \left(1 - \frac{1}{n}\right)r}{\frac{1}{n} + r}.$$

Now, when $n = \infty$, the limit of

$$\frac{\frac{2}{n} + \left(1 - \frac{1}{n}\right)r}{\frac{1}{n} + r} = \frac{0 + (1 - 0)r}{0 + r} = 1.$$

Hence, the limit of P, when $n = \infty$, $= \infty \frac{A}{2} = \infty$.

This result shows that, reckoning so-called simple interest, an *infinite sum* of money is required to be left, in order to insure an equal annual payment for ever.

It indicates therefore that the only correct method of computing annuities is on the compound interest principle.

We should arrive at the same result by taking formula (B) of Art. 63.

67. *To find the present worth of an annuity to commence in* p *years, and to continue for* q *years.*

Amount of annuity A to continue for q years

$$= \frac{A}{r}(R^q - 1) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now, if P be the present worth, it will amount in $(p + q)$ years to exactly as much money as the annuity, if left to accumulate for q years.

And the amount of P in $(p + q)$ years $= P \cdot R^{p+q} \ldots$ (2).

Hence, equating (2) and (1), we have

$$P \cdot R^{p+q} = \frac{A}{r}(R^q - 1);$$

$$\therefore P = \frac{A}{r} \cdot \frac{R^q - 1}{R^{p+q}} = \frac{A}{r}(R^{-p} - R^{-p-q}).$$

Cor. If the annuity is to continue for ever, we have $q = \infty$, and the limit of $R^{-p-q} = 0$.

Hence the present worth of a perpetual annuity which is to be entered upon in p years $= \frac{A}{r} \cdot R^{-p}$.

Ex. 1. A sum of £a is borrowed for a period of m years, to be repaid by equal annual instalments, the first payment to be made after one year. Find the amount of the annual instalment.

Let A be the annual instalment.

Then the amount of this annual payment in m years

$$= \frac{A}{r}\{R^m - 1\}.$$

Again, if the sum a be allowed to accumulate for m years at compound interest, its amount $= a \cdot R^m$.

Now, these two amounts ought to be equal.

Hence we have

$$\frac{A}{r}\{R^m - 1\} = a\ R^m;$$

$$\therefore A = \frac{a}{r} \cdot \frac{R^m}{R^m - 1} = \frac{a}{r} \cdot \frac{1}{1 - R^{-m}}.$$

Ex. 2. A debt a now due, is paid off in n years by a series of n payments in arithmetical progression; reckoning compound interest show that, if b be the first of the n payments, the common difference of the arithmetical progression is

$$\frac{aR^n r^2 - b(R^n - 1)r}{R(R^{n-1} - 1) - (n-1)r}.$$

If we suppose the debt and the whole of the payments to be allowed to accumulate for the n years, the two amounts should be equal.

Now, amount of the debt a in n years $= aR^n$.....(1).

And, amount of the n payments in the same time

$= b\,R^{n-1} + (b+d)\,R^{n-2} + (b+2d)\,R^{r-3} + \&c. + (b + \overline{n-1}\cdot d)$.

where d is the common difference of the A. P.

But (Art. 27, Ex. 4) the sum of this series

$$= \frac{bR^{n-1}}{1 - \frac{1}{R}} + \frac{dR^{n-1}}{R} \cdot \frac{1 - \frac{1}{R^{n-1}}}{\left(1 - \frac{1}{R}\right)^2} - (b + \overline{n-1}\,.\,d)R^{n-1} \cdot \frac{\frac{1}{R^n}}{1 - \frac{1}{R}}$$

$$= \frac{b}{r} \cdot R^n + \frac{d}{r^2} \cdot (R^n - R) - \frac{b + \overline{n-1}\,.\,d}{r} \quad \ldots\ldots\ldots\ldots (2).$$

(2) = (1), then

$$\frac{b}{r} \cdot R^n + \frac{d}{r^2}(R^n - R) - \frac{b + \overline{n-1}\,.\,d}{r} = a\,.\,R^n;\ \text{or,}$$

$$\left\{\frac{R^n - R}{r^2} - \frac{n-1}{r}\right\} d = aR^n - \frac{b}{r}(R^n - 1)$$

$$\therefore d = \frac{aR^n r^2 - b\,(R^n - 1)r}{R\,(R^{n-1} - 1) - (n-1)r}.$$

Ex. XI.

1. Find the compound interest on £530 for 12 years at $3\frac{1}{2}$ per cent.

2. In what time will a sum of money double itself at 5 per cent., compound interest?

3. Find the amount of an annuity of £50 for 10 years at 4 per cent.

4. A *corporation borrows* £3,769 at 4 per cent., to be re-

paid in 30 years by equal annual instalments. What will be the annual payment?

5. Find the equated time for £320, £390, £450, due respectively in 6, 7, 8 months, simple interest being reckoned.

6. In how many years will £1 amount to £5 at 6 per cent. per annum, compound interest?

7. A freehold bringing in £120 a year is sold for £1,920. Find the rate of interest.

8. A debt of £1,200 is to be paid out in 20 years by annual payments, increasing in A. P. If the first payment is £70, what is the last payment?

9. If interest be payable every instant, in how many years would £1 amount to £6?

10. The present value of a freehold to be entered upon in 5 years is £1,600. Find the rent, interest being reckoned at 6 per cent.

11. A property is let out on lease for a years at an annual rental of £b, and after c years the lease is renewed on paying a fine of £d. What is the additional rent equivalent to this fine?

12. An annuity a is found to amount in n years to the same sum as when b is put out for the same time at compound interest. Express n in terms of a, b, r, where r is the interest of one pound for one year.

CHAPTER X.

CONTINUED FRACTIONS.

68. Let x be any quantity, rational or irrational; and suppose that $x = a_0 + \frac{1}{x_1}$, $x_1 = a_1 + \frac{1}{x_2}$, $x_2 = a_2 + \frac{1}{x_3}$, &c.; where a_0, a_1, a_2, a_3, &c, are respectively the greatest integers in x, x_1, x_2, x_3, &c.

We then have

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \&c.}}}, \text{ or, as often written,}$$

$$x = a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{\&c.}$$

Such an expression is called a *continued fraction*, and it is *rational* or *irrational* according as the quotients a_1, a_2, a_3, &c., terminate or not.

The quantities a_0, a_1, a_2, a_3, &c., are called *incomplete quotients;* the first of them, a_0, may be zero, but each of the others must at least be equal to unity.

The quantities x_1, x_2, x_3, &c., are called *complete quotients;* and when the continued fraction is made to terminate by a *complete* quotient, the resulting expression is identically equal to the given quantity x.

When, however, it is made to terminate by the successive *incomplete* quotients, we get successive approximations to the true value of x.

Thus—

(1.) We may express the true value of the given quantity x in either of the following ways:

$$a_0 + \frac{1}{x_1}, \; a_0 + \frac{1}{a_1 +} \frac{1}{x}, \; a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{x_3}, \; \&c.$$

(2.) We may express the corresponding approximations as follows:

$$a_0, \; a_0 + \frac{1}{a_1}, \; a_0 + \frac{1}{a_1 +} \frac{1}{a_2}, \; \&c.$$

Cor. When the approximation to the given quantity x is expressed by terminating the continued fraction by an incomplete quotient, we may obtain the true value of x by substituting the corresponding *complete quotient* for the last incomplete quotient.

69. *Conversion of a given fraction into a continued fraction.*

Let $\frac{P}{Q}$ be the given fraction, and let the ordinary operation of finding the G. C. M. of P and Q be performed.

We will suppose the operation to stand thus:

$$\begin{array}{l} Q)\ P\ (a_0 \\ \quad a_0Q \\ \hline p)\ Q\ (a_1 \\ \quad a_1p \\ \hline q)\ p\ (a_2 \\ \quad a_2q \\ \hline r)\ \&c. \end{array}$$

Then we have

$$\frac{P}{Q} = a_0 + \frac{p}{Q};\ \frac{Q}{p} = a_1 + \frac{q}{p};\ \frac{p}{q} = a_2 + \frac{r}{q};\ \&c$$

By successive substitutions we therefore get

$$\frac{P}{Q} = a_0 + \frac{1}{a_1 +}\ \frac{1}{a_2 +}\ \frac{1}{a_3 +}\ \ldots\ \frac{1}{a_r +}\ \ldots$$

It will be presently seen (Art. 72) that the approximations obtained by terminating the continued fraction by the successive *incomplete quotients* are alternately less and greater than the given fraction; and further, that they gradually approach nearer and nearer to the true value. For this latter reason these approximations are called *converging fractions*, or *convergents*.

The convergents of $\frac{P}{Q}$ may be written as follows:

$$\frac{a_0}{1},\ \frac{a_0a_1 + 1}{a_1},\ \frac{a_0a_1a_2 + a_2 + a_0}{a_1a_2 + 1},\ \&c.$$

If we look at the third of these, we notice:

(1.) Its numerator is found by multiplying the numerator

of the *second* convergent by the *third* incomplete quotient, and adding the numerator of the *first* convergent.

Thus, $a_0a_1a_2 + a_2 + a_0 = (a_0a_1 + 1)\,a_2 + a_0$.

(2.) Its *denominator* is similarly formed.

Thus, $a_1a_2 + 1 = a_1(a_2) + 1$.

The existence of a general law for forming these convergents may therefore be suspected. This law is proved in the next article, and may be expressed thus:

Law of formation of the convergents.

(1.) To find the numerator of the $(n + 1)$th convergent, multiply the numerator of the nth convergent by the next incomplete quotient, and add to the product the numerator of the $(n - 1)$th convergent.

(2.) To find the denominator pursue a similar method.

In order, however, to apply this rule practically, it is evident that we require to have found at least *two convergents* by ordinary vulgar fractions. But if we make use of a *fictitious convergent* for *the first*, we may then apply the above rule when we know only one other.

Thus, *assuming* $\frac{1}{0}$ *as the first convergent*, we may form all the convergents by the above law, if we only know $\frac{a_0}{1}$, the first approximation, and the successive incomplete quotients. The accompanying table exhibits this to the eye at once.

Incomplete Quotients.	a_0	a_1	a_2	a_3	&c.
Convergents,....	$\frac{1}{0}$	$\frac{a_0}{1}$	$\frac{a_0a_1 + 1}{a_1}$	$\frac{a_0a_1a_2 + a_2 + a_0}{a_1a_2 + 1}$	&c.
Order,.....	1st	2nd	3rd	4th	&c.

Cor. If we designate by P_0, P_1, P_2, P_3, &c., the numera-

tors, and by Q_0, Q_1, Q_2, Q_3, &c., the denominators, of the successive convergents, we have

$$P_0 = 1,\ P_1 = a_0,\ P_2 = a_0a_1 + 1,\ P_3 = a_0a_1a_2 + a_2 + a_0,\ \&c.$$
$$Q_0 = 0,\ Q_1 = 1,\ Q_2 = a_1,\qquad Q_3 = a_1a_2 + 1,\ \&c.$$

70. *To show that* $P_n = a_nP_{n-1} + P_{n-2}$,
and that $Q_n = a_nQ_{n-1} + Q_{n-2}$.

We shall *assume* the rth convergent to be formed according to this law, and then show that the $(r + 1)$th is formed according to the law.

By the last Art. we represent the rth convergent by $\frac{P_{r-1}}{Q_{r-1}}$.

We *assume*, then, that

$$\left.\begin{aligned} P_{r-1} &= a_{r-1}P_{r-2} + P_{r-3}, \text{ and} \\ Q_{r-1} &= a_{r-1}Q_{r-2} + Q_{r-3}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

$$\therefore \frac{P_{r-1}}{Q_{r-1}} = \frac{a_{r-1}P_{r-2} + P_{r-3}}{a_{r-1}Q_{r-2} + Q_{r-3}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Now, Art. 68, Cor., we can from this expression obtain the true value of $\frac{P}{Q}$, if we replace the incomplete quotient a_{r-1} by the corresponding complete quotient x_{r-1}.

We then have $\dfrac{P}{Q} = \dfrac{x_{r-1}P_{r-2} + P_{r-3}}{x_{r-1}Q_{r-2} + Q_{r-3}}$.

But, Art. 68, $x_{r-1} = a_{r-1} + \dfrac{1}{x_r}$; we therefore have

$$\frac{P}{Q} = \frac{(a_{r-1} + \frac{1}{x_r})P_{r-2} + P_{r-3}}{(a_{r-1} + \frac{1}{x_r})Q_{r-2} + Q_{r-3}} = \frac{x_r(a_{r-1}P_{r-2} + P_{r-3}) + P_{r-2}}{x_r(a_{r-1}Q_{r-2} + Q_{r-3}) + Q_{r-2}},$$

or, by (1), $$\frac{P}{Q} = \frac{x_rP_{r-1} + P_{r-2}}{x_rQ_{r-1} + Q_{r-2}}.$$

Now, replacing x_r by the corresponding incomplete

quotient a_r, we get the $(r + 1)$th convergent. Thus we find—

$$\text{The numerator, } P_r = a_r P_{r-1} + P_{r-2};$$
$$\text{The denominator, } Q_r = a_r Q_{r-1} + Q_{r-2}.$$

Hence, on the assumption that the law is true for the numerator and denominator of the rth convergent, we have shown it to be true for that of the $(r + 1)$th convergent.

Now, we know, Art. 69, it to be true for the 3rd and 4th convergents; it is therefore true for the 5th; and hence for the 6th, &c. We conclude, then, that it is generally true.

71. *To show that* $P_n Q_{n-1} - Q_n P_{n-1} = (-1)^n$.

We have

$$P_n = a_n P_{n-1} + P_{n-2}$$
$$Q_n = a_n Q_{n-1} + Q_{n-2}$$

We easily get

$$\frac{P_n - P_{n-2}}{Q_n - Q_{n-2}} = \frac{P_{n-1}}{Q_{n-1}}; \text{ or,}$$

$$P_n Q_{n-1} - Q_{n-1} P_{n-2} = Q_n P_{n-1} - P_{n-1} Q_{n-2}; \text{ or,}$$
$$-(P_n Q_{n-1} - Q_n P_{n-1}) = P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}$$

Or, multiplying each side by $(-1)^{n-1}$, we have

$$(-1)^n (P_n Q_{n-1} - Q_n P_{n-1})$$
$$= (-1)^{n-1} (P_{n-1} Q_{n-2} - Q_{n-1} P_{n-2}).$$

We learn from this equation that, *whatever be the value of* r, the quantity $(-1)^r (P_r Q_{r-1} - Q_r P_{r-1})$ has *a constant value.*

$$\text{When } r = 1, \text{ it becomes } (-1)(P_1 Q_0 - Q_1 P_0)$$
$$= -(a_0 \times 0 - 1 \times 1) = 1.$$

We then have

$$\text{Constant value of } (-1)^r (P_r Q_{r-1} - Q_r P_{r-1}) = 1.$$

Put $r = n$, then we get

$$P_n Q_{n-1} - Q_n P_{n-1} = \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n. \quad \textit{Q.E.D.}$$

Cor. 1. Hence, dividing each side by Q_nQ_{n-1}, we have

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = (-1)^n \cdot \frac{1}{Q_nQ_{n-1}}.$$

Cor. 2. *All converging fractions are in their lowest terms.*

For we have $P_nQ_{n-1} - Q_nP_{n-1} = (-1)^n$.

Hence, if P_n and Q_n have a common measure, this measure must divide *unity*. And the same may be said of the common measure of P_{n-1} and Q_{n-1}, if any.

72. *Of two consecutive convergents, the one is greater and the other less than the true value of the continued fraction, and each convergent approaches more nearly to the true value than the preceding.*

Let $\frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be consecutive convergents to a continued fraction whose true value is x.

Then we have

$$x = \frac{x_nP_n + P_{n-1}}{x_nQ_n + Q_{n-1}};$$ from which we have

$$x_n = \frac{xQ_{n-1} - P_{n-1}}{P_n - xQ_n}, \text{ or } \frac{Q_n}{Q_{n-1}}x_n = \frac{x - \frac{P_{n-1}}{Q_{n-1}}}{\frac{P_n}{Q_n} - x} \quad \ldots\ldots\ldots\ldots(1).$$

Now Q_n, Q_{n-1}, x_n are positive quantities.

Hence, $x - \frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n} - x$ are of the same sign.

Therefore x lies between $\frac{P_{n-1}}{Q_{n-1}}$ and $\frac{P_n}{Q_n}$, which proves the first part of the proposition.

Again (Art. 65), x_n (a complete quotient) is greater than unity.

Also, $\frac{Q_n}{Q_{n-1}}$ *must* be greater than unity.

Hence, from (1), $$\frac{x - \frac{P_{n-1}}{Q_{n-1}}}{\frac{P_n}{Q_n} - x} > 1;$$

that is, the difference between x and $\frac{P_{n-1}}{Q_{n-1}}$ is greater than that between $\frac{P_n}{Q_n}$ and x.

Hence $\frac{P_n}{Q_n}$ is a nearer approximation to the true value of x than $\frac{P_{n-1}}{Q_{n-1}}$.

COR. Since the *first* convergent is $\frac{1}{0}$, and the *second* $\frac{a_0}{1}$, it follows that:

Convergents of an *odd* order form a decreasing series, and convergents of an *even* order form an increasing series, each series gradually approaching to the true value of the continued fraction.

73. *To find the limits of the error in taking* $\frac{P_{n-1}}{Q_{n-1}}$ *as an approximation to the true value of* x.

By Art. 68, we have

$$x = \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}}; \text{ hence}$$

$$x - \frac{P_{n-1}}{Q_{n-1}} = \frac{x_n P_n + P_{n-1}}{x_n Q_n + Q_{n-1}} - \frac{P_{n-1}}{Q_{n-1}}$$

$$= \frac{x_n(P_n Q_{n-1} - Q_n P_{n-1})}{Q_{n-1}(x_n Q_n + Q_{n-1})}, \text{ or, by Art. 70,}$$

$$= (-1)^n \cdot \frac{x^n}{Q_{n-1}(x_n Q_n + Q_{n-1})}$$

$$= (-1)^n \cdot \frac{1}{Q_{n-1}\left(Q_n + \frac{1}{x^n} Q_{n-1}\right)}.$$

Now x_n is > 1, and therefore $\frac{1}{x_n} > 0$ and < 1.

Hence

$$x - \frac{P_{n-1}}{Q_{n-1}} < (-1)^n \cdot \frac{1}{Q_{n-1}Q_n}, \text{ and } > \frac{1}{Q_{n-1}(Q_n + Q_{n-1})}.$$

Cor. 1. Since Q_{n-1} is $< Q_n$, and $\therefore Q^2_{n-1} < Q_{n-1}Q_n$, and $Q_{n-1}Q_n < Q_n^2$; we have

$$x - \frac{P_{n-1}}{Q_{n-1}} < \frac{(-1)^n}{Q^2_{n-1}}, \text{ and } > \frac{(-1)^n}{2\,Q_n^2}.$$

Cor. 2. Hence if it be required to find a convergent $\frac{P_{n-1}}{Q_{n-1}}$ which differs from x by a quantity less than any given quantity $\frac{1}{a}$, we must find the successive convergents until we reach one whose denominator Q_{n-1} is $\equiv \sqrt{a}$.

74. *Any convergent* $\frac{P_n}{Q_n}$ *approaches more nearly to the value of* x *than any other fraction whose denominator is* $< Q_n$.

Let $\frac{p}{q}$ be any fraction which lies between $\frac{P_n}{Q_n}$ and x.

If $\frac{p}{q}$ be a convergent, then (Art. 72) q is necessarily $> Q_n$.

But if $\frac{p}{q}$ be not one of the convergents, then, since x lies between $\frac{P_n}{Q_n}$ and $\frac{P_{n-1}}{Q_{n-1}}$, and $\frac{p}{q}$ lies between $\frac{P_n}{Q_n}$ and x, it follows that

$$\frac{p}{q} \text{ lies between } \frac{P_n}{Q_n} \text{ and } \frac{P_{n-1}}{Q_{n-1}}.$$

Hence $\frac{p}{q} \approx \frac{P_{n-1}}{Q_{n-1}} < \frac{P_n}{Q_n} \approx \frac{P_{n-1}}{Q_{n-1}} < \frac{P_nQ_{n-1} \approx Q_nP_{n-1}}{Q_nQ_{n-1}}$.

But (Art. 68) neglecting the sign, $P_nQ_{n-1} \approx Q_nP_{n-1} = 1$;

$$\therefore \frac{p}{q} \sim \frac{P_{n-1}}{Q_{n-1}} < \frac{1}{Q_n Q_{n-1}}, \text{ or}$$

$$pQ_{n-1} \sim qP_{n-1} < \frac{q}{Q_n} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now $pQ_{n-1} \sim qP_{n-1}$ cannot be equal to *zero*, for then $\frac{p}{q} = \frac{P_{n-1}}{Q_{n-1}}$, which is contrary to our hypothesis. And hence, since p, q, P_{n-1}, Q_{n-1} are all integers,

$pQ_{n-1} \sim qP_{n-1}$ cannot be less than *unity*.

Therefore the condition in (1) is impossible, if $q < Q_n$.

Hence no fraction $\frac{p}{q}$ can approach more nearly to the value of x, than the convergent $\frac{P_n}{Q_n}$, if q is $< Q_n$.

75. ***To solve the indeterminate equation***

$$ax + by = c$$

by means of a continued fraction.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding the last $\left(\text{viz., } \frac{a}{b}\right)$. Then we have (Art. 71),

$$aq - bp = \pm 1, \text{ or } a(\pm cq) - b(\pm cp) = c.$$

Or, adding and subtracting abt, we have

$$a(bt \pm cq) + b(\pm cp - at) = c.$$

Comparing this with the original equation, we see that the given equation is solved by putting

$$x = bt \pm cq$$
$$y = \mp cp - at.$$

Ex. Reduce $\frac{314159}{100000}$ to a continued fraction, and find the convergents.

100000	314159 300000	3
14159	100000 99113	7
887	14159 887	15
	5289 4435	
854	887 854	1
33	854 66	25
	194 165	
29	33 29	1
4	29 28	7
1	4 4	4

The continued fraction is

$$3 + \cfrac{1}{7+} \cfrac{1}{15+} \cfrac{1}{1+} \cfrac{1}{25+} \cfrac{1}{1+} \cfrac{1}{7+} \cfrac{1}{4}.$$

And the convergents

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{9208}{2931}, \frac{9563}{3044}, \frac{76149}{24239}, \frac{314159}{100000}$$

Ex. XII.

Reduce to continued fractions :

1. $\frac{3}{1}\frac{2}{3}\frac{6}{5}$. 2. $\frac{6}{1}\frac{3}{3}\frac{7}{9}\frac{1}{5}$ 3. $\frac{7}{1}\frac{1}{2}\frac{0}{5}\frac{2}{7}$. 4. $\frac{2}{1}\frac{1}{3}\frac{7}{6}\frac{3}{4}$.

5. Why is $\frac{3}{1}\frac{5}{1}\frac{5}{3}$ a nearer approximation to π than $\frac{2}{7}^2$?

6. Show that $\frac{P_n}{Q_n} = \frac{P_{n+1} - P_{n-1}}{Q_{n+1} - Q_{n-1}}$.

7. If a_{n-1} be the nth incomplete quotient, prove that

$$a_n = \sqrt{\frac{(P_n - P_{n-2})\,(Q_n - Q_{n-2})}{P_{n-1} \cdot Q_{n-1}}}.$$

8. If $\frac{P_r}{Q_r}$ represents the $(r + 1)$th convergent, show that

$(P_nQ_{n-1} - Q_nP_{n-1})\,(P_{n-1}Q_{n-2} - Q_{n-1}P_{n-2}) \ldots$ to n factors

$= (-1)^{\frac{1}{2}n(n+1)}$.

9. If $\frac{P_r}{Q_r}$ represents the $(r + 1)$th convergent to the fraction $\frac{P}{Q}$, show that the $(n + 1)$th complete quotient

$$= \frac{PQ_{n-1} - QP_{n-1}}{QP_n - PQ_n}.$$

10. Prove that $\frac{P_{2n}}{Q_{2n}} - \frac{P_n}{Q_n} =$

$$\frac{1}{Q_{2n}Q_{2n-1}} - \frac{1}{Q_{2n-1}Q_{2n-2}} + \frac{1}{Q_{2n-2}Q_{2n-3}} - \&c. +$$

$$(-1)^{n-1} \cdot \frac{1}{Q_{n+1}Q_n}.$$

11. If $\frac{p}{q}$ be a convergent to a given quantity x, show that no fraction having a less denominator than q can be a nearer approximation to x than $\frac{p}{q}$.

12. Solve in positive integers the indeterminate equation $3x - 5y = 11$.

Reduction of Quadratic Irrational Quantities to Continued Fractions.

76. Every quadratic equation with rational coefficients may be reduced to the form

$$ax^2 + 2bx + c = 0,$$

where a, b, c are integers, either positive or negative.

Solving this equation, we get

$$x = \frac{\pm\sqrt{b^2 - ac} - b}{a}, \text{ which may be also}$$

written $$x = \frac{\sqrt{b^2 - ac} \mp b}{\pm a}.$$

If we put $N = b^2 - ac$, we have

$$\frac{N - b^2}{a} = -c, \text{ and } x = \frac{\sqrt{N} \mp b}{\mp a}.$$

Hence, since for the purposes of continued fractions we need consider only the *magnitude* of these roots, and since, too, we may so choose the sign of a that x shall be always positive, we arrive at the following result:

The roots (neglecting the sign) of a quadratic equation having rational coefficients may, when irrational, be always referred to the form

$$x = \frac{\sqrt{N} + p}{q},$$

where

(1.) p and q are integers either positive or negative, the sign of q being so taken that x shall be positive;

(2.) $\frac{N - p^2}{q}$ is an integer.

Continued fractions developed from such irrational quantities *are* called irrational continued fractions of the second *degree.*

77. *To express* $\frac{\sqrt{N} + p_0}{q_0}$ *as a continued fraction,* p_0 *and* q_0 *being integers, and the sign of* q_0 *being so taken that the expression shall be positive, and where also* $\frac{N - p_0^2}{q_0}$ *is an integer.*

Let a_0 be the greatest integer in $\frac{\sqrt{N} + p_0}{q_0} = x$, suppose.

Then $x = a_0 + \left(\frac{\sqrt{N} + p_0}{q_0} - a_0 \right) = a_0 + \frac{\sqrt{N} - (a_0 q_0 - p_0)}{q_0}$

$$= a_0 + \frac{N - (a_0 q_0 - p_0)^2}{q_0 \{ \sqrt{N} + (a_0 q_0 - p_0) \}}$$

$$= a_0 + \frac{\frac{N - (a_0 q_0 - p_0)^2}{q_0}}{\sqrt{N} + (a_0 q_0 - p_0)}$$

$$= a_0 + \frac{q_1}{\sqrt{N} + p_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $q_1 = \frac{N - (a_0 q_0 - p_0)^2}{q_0}$,

and $p_1 = a_0 q_0 - p_0$.

We then have $x = a_0 + \frac{1}{\frac{\sqrt{N} + p_1}{q_1}} = a_0 + \frac{1}{x_1} \ldots\ldots\ldots (2)$,

where $x_1 = \frac{\sqrt{N} + p_1}{q_1}$.

We may notice that x_1 has the *same form* as the given quantity x; and we may therefore in the same manner obtain as complete quotients x_2, x_3, &c., expressed in the *same form* as x.

We may then write $x = \frac{\sqrt{N} + p_0}{q_0} = a_0 + \frac{1}{x_1}$,

$$x_1 = \frac{\sqrt{N} + p_1}{q_1} = a_1 + \frac{1}{x_2},$$

$$x_2 = \frac{\sqrt{N} + p_2}{q_2} = a_2 + \frac{1}{x_3},$$
$$\&c. = \quad \&c.$$
$$x_n = \frac{\sqrt{N} + p_n}{q_n} = a_n + \frac{1}{x_{n+1}},$$
$$\&c. = \quad \&c.$$

where a_0, a_1, a_2, &c., are all positive integers; and where x_1, x_2, x_3, &c., are also positive quantities greater than unity.

We hence have

$$x = a_0 + \frac{1}{a_1 +} \quad \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \quad \ldots \quad \frac{1}{a_n +} \quad \ldots$$

the required continued fraction.

Ex. 1. Reduce $\frac{\sqrt{17} + 3}{4}$ to a continued fraction.

The expression is positive, p_0 and q_0 are integers, and $\frac{N - p_0^2}{q_0} = \frac{17 - 9}{4} = 4$.

It therefore answers the required conditions.

We have, since *unity* is the greatest integer in the given surd,

$$\frac{\sqrt{17} + 3}{4} = 1 + \left(\frac{\sqrt{17} + 3}{4} - 1\right) = 1 + \frac{\sqrt{17} - 1}{4}$$
$$= 1 + \frac{17 - 1}{4(\sqrt{17} + 1)} = 1 + \frac{4}{\sqrt{17} + 1}$$
$$= 1 + \frac{1}{\frac{\sqrt{17} + 1}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\frac{\sqrt{17} + 1}{4} = 1 + \left(\frac{\sqrt{17} + 1}{4} - 1\right) = 1 + \frac{\sqrt{17} - 3}{4}$$
$$= 1 + \frac{17 - 9}{4(\sqrt{17} + 3)} = 1 + \frac{2}{\sqrt{17} + 3}$$
$$= 1 + \frac{1}{\frac{\sqrt{17} + 3}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\frac{\sqrt{17}+3}{2} = 3 + \frac{\sqrt{17}-3}{2} = 3 + \frac{1}{\frac{\sqrt{17}+3}{4}} \ldots\ldots (3).$$

We have thus arrived at a quantity $\frac{\sqrt{17}+3}{4}$ exactly the same as that with which we started. The quotients will therefore recur.

We then have, by successive substitution,

$$\frac{\sqrt{17}+3}{4} = 1 + \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{3+}\ \ldots.$$

Ex. 2. Reduce $\sqrt{26}$ to a continued fraction.

Here $p_0 = 0$, $q_0 = 1$, and the given expression answers the required conditions.

$$\sqrt{26} = 5 + (\sqrt{26} - 5) = 5 + \frac{26-25}{\sqrt{26}+5} = 5 + \frac{1}{\sqrt{26}+5}.$$

Hence $\sqrt{26} + 5 = 10 + \frac{1}{\sqrt{26}+5}$.

And so, by successive substitutions, we get

$$\sqrt{26} = 5 + \frac{1}{10+}\ \frac{1}{10+}\ \frac{1}{10+}\ \ldots.$$

78. *To show that* (i.) $p_n = a_{n-1}q_{n-1} - p_{n-1}$

(ii.) $q_n q_{n-1} = N - p_n^2$.

We have, in the last Art.

$$x_n = \frac{\sqrt{N}+p_n}{q_n} = a_n + \frac{1}{x_{n+1}} \ldots\ldots (1).$$

Putting $n-1$ for n, then

$$\frac{\sqrt{N}+p_{n-1}}{q_{n-1}} = a_{n-1} + \frac{1}{x_n}\text{; from which}$$

$$x_n = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})} \ldots\ldots (2).$$

(1) = (2), then

$$\frac{\sqrt{N}+p_n}{q_n} = \frac{q_{n-1}}{\sqrt{N} - (a_{n-1}q_{n-1} - p_{n-1})}\text{; or, reducing,}$$

$$N - p_n(a_{n-1}q_{n-1} - p_{n-1}) - (a_{n-1}q_{n-1} - p_{n-1} - p_n)\sqrt{N} = q_nq_{n-1}.$$

Equating the rational and irrational terms, then

$(a_{n-1}q_{n-1} - p_{n-1} - p_n)\sqrt{N} = 0$, and $\therefore p_n = a_{n-1}q_{n-1} - p_{n-1}$;

and $N - p_n(a_{n-1}q_{n-1} - p_{n-1}) = q_nq_{n-1}$, or

$$q_nq_{n-1} = N - p_n \cdot p_n = N - p_n^2.$$

The expressions (i.) and (ii.) just proved contain the law of formation of the complete quotients, and hold for all positive integral values of n.

79. *The quantities* p_1, p_2, p_3, &c., *and* q_1, q_2, q_3, &c., *are all integers, and moreover, after a certain value of* n, p_n *and* q_n *are positive integers.*

We have (Art. 72) $x_n = \dfrac{a_nP_n + P_{n-1}}{a_nQ_n + Q_{n-1}}$.

Hence also, replacing the incomplete quotient a_n by the complete quotient x_n, we get

$$x = \frac{x_nP_n + P_{n-1}}{x_nQ_n + Q_{n-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But $x = \dfrac{\sqrt{N} + p_0}{q_0}$, $x_n = \dfrac{\sqrt{N} + p_n}{q_n}$; then we have from (1)

$$\frac{\sqrt{N} + p_0}{q_0} = \frac{\dfrac{\sqrt{N} + p_n}{q_n} \cdot P_n + P_{n-1}}{\dfrac{\sqrt{N} + p_n}{q_n} \cdot Q_n + Q_{n-1}} = \frac{P_n \cdot \sqrt{N} + P_np_n + P_{n-1}q_n}{Q_n\sqrt{N} + Q_np_n + Q_{n-1}q_n}.$$

Reducing, then

$$Q_n \cdot N + p_0(Q_np_n + Q_{n-1}q_n) + (Q_np_0 + Q_np_n + Q_{n-1}q_n)\sqrt{N} = P_nq_0 \cdot \sqrt{N} + q_0(P_np_n + P_{n-1}q_n).$$

Equating the rational and irrational parts, we have

$$Q_np_0 + Q_np_n + Q_{n-1}q_n = P_nq_0 \quad \ldots\ldots\ldots\ldots (2).$$

$$Q_n \cdot N + p_0(Q_np_n + Q_{n-1}q_n) = q_0(P_np_n + P_{n-1}q_n) \ldots (3).$$

From (2) we have

$$Q_np_n + Q_{n-1}q_n = P_nq_0 - Q_np_0;$$ and from (3)

$$(P_nq_0 - Q_np_0)p_n + (P_{n-1}q_0 - Q_{n-1}p_0)q_n = Q_nN.$$

Solving for q_n we get

$$(P_nQ_{n-1} - Q_nP_{n-1})q_0 \cdot q_n = (P_nq_0 - Q_np_0)^2 - Q_n^2 \cdot N.$$

Hence, since (Art. 70) $P_nQ_{n-1} - Q_nP_{n-1} = (-1)^n$, we have

$$q_n = \frac{1}{(-1)^n q_0}\{(P_nq_0 - Q_np_0)^2 - Q_n^2N\};$$

$$\text{but } \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n;$$

$$\therefore q_n = \frac{(-1)^n}{q_0}\{(P_nq_0 - Q_np_0)^2 - Q_n^2N\} \ldots\ldots\ldots (4).$$

And so p_n

$$= \frac{(-1)^n}{q_0}\{Q_{n-1}Q_nN - (P_{n-1}q_0 - Q_{n-1}p_0)(P_nq_0 - Q_np_0)\} \ldots (5).$$

Now the expressions in (4) and (5) may be easily expressed thus:

$$q_n = (-1)^n\left\{P_n^2q_0 - 2P_nQ_np_0 - Q_n^2 \cdot \frac{N - p_0^2}{q_0}\right\}, \text{ and}$$

$$p_n =$$

$$(-1)^n\left\{Q_{n-1}Q_n \cdot \frac{N - p_0^2}{q_0} - P_{n-1}P_nq_0 - P_nQ_{n-1}p_0 - Q_nP_{n-1}P_0\right\}.$$

Now $\frac{N - p_0^2}{q_0}$ is by hypothesis integral; and hence, since all the other quantities in these expressions are integers, it follows that p_n and q_n *are integers.* Moreover, after n has reached a certain limit, they are *positive* integers.

For from (2)

$$Q_{n-1}q_n = P_nq_0 - Q_np_0 - Q_np_n;$$

$$\therefore \frac{Q_{n-1}}{Q_n} \cdot q_n = \left(\frac{P_n}{Q_n} \cdot q_0 - p_0\right) - p_n;$$

$$\text{but } x_n = \frac{\sqrt{N} + p_n}{q_n}, \text{ or } q_n = \frac{\sqrt{N} + p_n}{x_n}.$$

Therefore, substituting,

$$\frac{Q_{n-1}}{Q_n} \cdot \frac{\sqrt{N} + p_n}{x_n} = \left(\frac{P_n}{Q_n}q_0 - p_0\right) - p_n.$$

$$\therefore \frac{Q_{n-1}}{Q_n} = x_n \cdot \frac{\left(\frac{P_n}{Q_n} \cdot q_0 - p_0\right) - p_n}{\sqrt{N} + p_n}.$$

Now $\frac{P_n}{Q_n}$ is the $(n+1)$th convergent to x, or $\frac{\sqrt{N}+p_0}{q_0}$; and it differs the less from x, the greater the value of n (Art. 70).

Hence by taking n large enough $\frac{P_n}{Q_n} \cdot q_0 - p_0$ may be made to differ from $\sqrt{N}$ by as small a quantity as we please.

Hence, when n has attained a certain limit,

$\frac{Q_{n-1}}{Q_n}$ differs from $x_n \cdot \frac{\sqrt{N}-p_n}{\sqrt{N}+p_n}$ by as small a quantity as we please.

Now if p_n is negative, $x_n \cdot \frac{\sqrt{N}-p_n}{\sqrt{N}+p_n}$ is greater than unity,

and therefore $\frac{Q_{n-1}}{Q_n} > 1$, or $Q_{n-1} > Q_n$.

But this is impossible, and therefore p_n cannot be negative.

Hence p_n is a *positive* integer, after n has reached a certain limit.

And so we may show that q_n is a positive integer, after n has reached a certain limit.

80. *The continued fraction is periodic after a certain number of quotients.*

For (Art. 78) we have $p_n + p_{n-1} = a_{n-1}q_{n-1}$ .....(1).

and $q_nq_{n-1} = N - p_n^2$ .....(2).

Now, by the last Art., after a certain limit, p_n and q_n are positive integers.

We shall consider only the portion of the continued fraction beyond this limit.

We have then q_n, q_{n-1}, each positive integers.

Therefore from (2) $p_n < \sqrt{N}$;

and therefore, also, $p_{n-1} < \sqrt{N}$.

Hence from (1) we have $a_{n-1}q_{n-1} < 2\sqrt{N}$.

Now a_{n-1} is a positive integer, and q_{n-1} is such.

It must therefore follow that $a_{n-1} < 2\sqrt{N}$,

and $q_{n-1} < 2\sqrt{N}$.

Or, putting n for $n - 1$, we find that for all values of a_n, p_n, q_n, beyond a certain limit, we must have

$$p_n < \sqrt{N},\ a_n < 2\sqrt{N},\ q_n < 2\sqrt{N}.$$

Hence the values of a_n, p_n, q_n are limited, and the complete quotient $x_n = \dfrac{\sqrt{N} + p_n}{q_n}$ must have a *finite number of different values*, as must also the incomplete quotient a_n.

Therefore commencing with the first complete quotient in which p_n and q_n are positive, the values of a_n, p_n, q_n will recur, and the number of quotients in the period cannot exceed $2\sqrt{N} \times \sqrt{N}$, or $2N$.

Cor. Since from (2) we have $q_n q_{n-1} < N$,
we have also $q_{n-1} q_{n-2} < N$.

Hence, if $q_{n-1} > \sqrt{N}$, we must have $q_n < \sqrt{N}$, and $q_{n-2} < \sqrt{N}$.

Hence, if any denominator of a complete quotient is $> \sqrt{N}$, *the denominators immediately preceding and following are each* $< \sqrt{N}$.

Ex. 1. If $N = a^2 + 1$, find $\sqrt{N}$ in the form of a continued fraction.

$$\sqrt{N} = a + (\sqrt{a^2+1} - a) = a + \frac{1}{\sqrt{a^2+1} + a} = a + \frac{1}{\sqrt{N} + a}.$$

Hence, by successive substitution, we get

$$\sqrt{N} = a + \cfrac{1}{2a + \cfrac{1}{2a + \ldots}}.$$

Ex. 2. Develop $\dfrac{\sqrt{27} + 3}{2}$ as a continued fraction.

$$\frac{\sqrt{27}+3}{2}=4+\frac{\sqrt{27}-5}{2}=4+\frac{27-25}{2(\sqrt{27}+5)}=4+\frac{1}{\sqrt{27}+5}\ldots(1)$$

$$\frac{\sqrt{27}+5}{1}=10+\frac{\sqrt{27}-5}{1}=10+\frac{27-25}{\sqrt{27}+5}=10+\frac{2}{\sqrt{27}+5}\ldots(2)$$

$$\frac{\sqrt{27}+5}{2}=5+\frac{\sqrt{27}-5}{2}=5+\frac{27-25}{2(\sqrt{27}+5)}=5+\frac{1}{\sqrt{27}+5}\ldots(3)$$

Hence, by successive substitutions from (1), (2), (3), we have $\dfrac{\sqrt{27}+3}{2}=4+\dfrac{1}{10+}\,\dfrac{1}{5+}\,\dfrac{1}{10+}\,\dfrac{1}{5+\ldots}$.

81. *Every periodic continued fraction is the development of one of the roots of a quadratic equation.*

Suppose the period to commence after the quotient x_{m-1}, and to consist of x' terms.

Then we must have $x_m=x_{m+x'}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$.

Also we have $$x=\frac{x_mP_m+P_{m-1}}{x_mQ_m+Q_{m-1}}\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

And $$x=\frac{x_{m+x'}P_{m+x'}+P_{m+x'+1}}{x_{m+x'}Q_{m+x'}+Q_{m+x'+1}},$$

which, by (1), $$=\frac{x_mP_{m+x'}+P_{m+x'-1}}{x_mQ_{m+x'}+Q_{m+x'-1}}\ldots\ldots\ldots\ldots(3).$$

From (2), solving for x_m, we get

$$x_m=\frac{xQ_m-P_m}{P_{m-1}-xQ_{m-1}}.$$

And from (3), solving for x_m, we get

$$x_m=\frac{xQ_{m+x'}-P_{m+x'}}{P_{m+x'-1}-xQ_{m+x'-1}}.$$

Equating these, we have

$$\frac{xQ_m-P_m}{P_{m-1}-xQ_{m-1}}=\frac{xQ_{m+x'}-P_{m+x'}}{P_{m+x'-1}-xQ_{m+x'-1}}$$

from which we get x as one of the roots of a quadratic *equation.*

Ex. If $x = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{4 + \ldots}}}}}$, express x as the root of a quadratic equation with integral coefficients.

Put $y = \cfrac{1}{3 + \cfrac{1}{4 + \ldots}}$, then we have

$$x = 1 + \frac{1}{2 + y} \quad \ldots\ldots(1).$$

and $$x = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + y}}} \quad \ldots\ldots(2).$$

From (1) $y = \dfrac{3 - 2x}{x - 1}$; and from (2) $y = \dfrac{43 - 30x}{7x - 10}$.

Hence we have $\dfrac{3 - 2x}{x - 1} = \dfrac{43 - 30x}{7x - 10}$; or, simplifying,

$16x^2 - 32x + 13 = 0$, from which $x = \dfrac{4 + \sqrt{3}}{4}$.

Ex. XIII.

Develop as continued fractions:

1. $\sqrt{10}$. 2. $\sqrt{17}$. 3. $\sqrt{19}$. 4. $\sqrt{29}$. 5. $\dfrac{\sqrt{14} + 3}{5}$.

6. $\dfrac{\sqrt{22} + 4}{3}$. 7. $\dfrac{\sqrt{28} + 5}{3}$. 8. $\dfrac{\sqrt{3} + 1}{\sqrt{3} - 1}$.

Express as continued fractions the roots of the six following equations:

9. $3x^2 - 8x + 2 = 0$. 10. $5x^2 - 6x - 7 = 0$.

11. $2x^2 - 5x + 1 = 0$. 12. $\dfrac{3}{x} = \dfrac{4}{x + 2} + 1$.

13. $x + \sqrt{x + 2} = 3$. 14. $x^2 - 4ax + 3a^2 = 1$.

15. Find the value of $5 + \cfrac{1}{10 + \cfrac{1}{10 + \ldots}}$.

16. Show that the product of the infinite continued fractions

$$\frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{2+\ldots.}, \text{ and } 4+\frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{4+\ldots.},$$

is $1\frac{6}{7}$.

17. Show that

$$\left(\frac{1}{m+}\ \frac{1}{n+}\ \frac{1}{p+}\ \frac{1}{q+}\ \frac{1}{m+\ldots.}\right)\left(q+\frac{1}{p+}\ \frac{1}{n+}\ \frac{1}{m+}\ \frac{1}{q+\ldots.}\right)$$

$$= \frac{n+q+npq}{m+p+mnp}.$$

18. Show that

$$\left(a+\frac{1}{2a+}\ \frac{1}{2a+\ldots.}\right)\left(a-\frac{1}{2a-}\ \frac{1}{2a-\ldots.}\right)=\sqrt{a^4-1}.$$

19. Find the value of $2+\frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{3+\ldots.}$.

20. Find the first four convergents to the greater roots of the equations in examples 9, 10, 11.

21. Show that the roots of a quadratic which can be expressed as a recurring continued fraction are of contrary or the same sign, according as the recurring period commences with the first quotient or not.

22. When the two roots of a quadratic are developed as a continued fraction, the quotients of the recurring period occur in reverse order.

23. If $\frac{P_{n-2}}{Q_{n-2}}, \frac{P_{n-1}}{Q_{n-1}}, \frac{P_n}{Q_n}$ be successive convergents to the continued fraction

$$\frac{a_1}{b_1+}\ \frac{a_2}{b_2+}\ \frac{a_3}{b_3+\ldots.},$$

show that $\frac{P_n}{Q_n}=\frac{a_nP_{n-2}+b_nP_{n-1}}{a_nQ_{n-2}+b_nQ_{n-1}}$, and find whether the *successive convergents* are necessarily in their lowest terms.

24. If $\cfrac{1}{a+}\ \cfrac{1}{b+}\ \cfrac{1}{a+}\ \cfrac{1}{b+\ldots.}$ be a periodic continued fraction, show that

$$\frac{P_{n+2} + P_{n-2}}{P_n} = \frac{Q_{n+2} + Q_{n-2}}{Q_n} = ab + 2.$$

CHAPTER XI.

SERIES.

82. A series is said to be **convergent** or **divergent** according as the sum of n terms has or has not a *finite limit*, when n is increased indefinitely.

Thus, the series $1 + \frac{1}{3} + \frac{1}{9} +$ &c. is *convergent.*

$$\text{For } S_n = \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right).$$

And hence, when $n = \infty$, $\Sigma = \frac{3}{2}(1 - 0) = \frac{3}{2}$.

Again, the series $1 + 3 + 9 +$ &c. is *divergent.*

$$\text{For } S_n = \frac{3^n - 1}{3 - 1} = \frac{1}{2}(3^n - 1).$$

And, when $n = \infty$, we have $\frac{1}{2}(3^n - 1) = \infty$.

83. *If all the terms of an infinite series are of the same sign, and each term is always greater than some given finite quantity, the series is divergent.*

Let each of the terms be greater than a given quantity k.

Then, the *sum of n terms* $> nk$.

Now, if n be indefinitely increased, nk is greater than any assignable quantity.

Hence there is no limit to the sum of the series when n is infinite, and therefore the series is divergent.

84. *If the term of an infinite series be alternately positive and negative, and the terms continually decrease, the series is convergent.*

Let $a - b + c - d + e - f +$ &c. be the series, where the quantities a, b, c, d, &c., continually decrease.

We then have $\Sigma = (a-b) + (c-d) + (e-f) +$ &c., ... (1),

and $\Sigma = a - (b-c) - (d-e) -$ &c., ... (2).

From (1), since $a > b$, $c > d$, &c., we learn that $\Sigma > a - b$.

And from (2), since $b > c$, $d > e$, &c., we learn that $\Sigma < a$.

Hence Σ lies between a and $a - b$, and therefore is finite; and the series is convergent.

85. *When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically less than some proper fraction, the series is convergent.*

Let $a_1 + a_2 + a_3 +$ &c. be the portion of the infinite series, after a finite number of terms, in which the ratio of each term to the preceding is numerically less than a given proper fraction k.

Then we have $a_2 < ka_1$, $a_3 < ka_2$, &c.

Hence $\Sigma < a_1 + ka_1 + k^2a_1 +$ &c. $< \dfrac{a_1}{1-k}$.

This portion therefore of the infinite series has a finite limit. Now the sum of the finite number of terms before the terms a_1, a_2, a_3, &c. is necessarily finite. Hence the sum of the whole series is finite, and the series is convergent.

Cor. 1. In the same manner it may be shown that

When the terms of an infinite series are such that after some finite number of terms, the ratio of each term to the preceding is numerically greater than unity, the series is divergent.

COR. 2. So also we may prove that

When after some finite number, the ratio of each term to the preceding is unity, and the terms have all the same sign, the series is divergent.

Case when the Ratio of each Term to the Preceding approaches Unity.

86. *When the ratio of each term to the preceding is less than unity, but still approaches unity, the series* may *be convergent or divergent.*

87. *To show that the series* $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} +$ *&c. is convergent when* m *is greater than unity, and divergent when* m *is equal to or less than unity.*

The ratio of the nth to the $(n-1)$th term $= \frac{1}{n^m} \div \frac{1}{(n-1)^m}$

$$= \left(\frac{n-1}{n}\right)^m.$$

This ratio is less than unity, but continually approaches unity as n is indefinitely increased.

(1). Let m be *less than unity.*

Now, $S_n = \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots + \frac{1}{n^m}$.

And the *least* of these terms is $\frac{1}{n^m}$.

Hence, $S_n > n$ times $\frac{1}{n^m} > n^{1-m}$.

But since m is less than unity, $1 - m$ is positive, and the value of n^{1-m} increases indefinitely as n is increased.

Hence, as n is indefinitely increased, the sum of n terms becomes greater than any assignable quantity.

The series is therefore divergent.

(2.) Let n be *equal to unity.*

$$\text{Then } \Sigma = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16}\right) + \&c.$$

$$\therefore \quad \Sigma > 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) + \&c.$$

$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \&c.$$

Hence Σ is greater than any assignable quantity, and the series is *divergent.*

(3) Let n be *greater than unity.*

$$\text{We have } \Sigma = 1 + \left(\frac{1}{2^n} + \frac{1}{3^n}\right) + \left(\frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n}\right) + \left(\frac{1}{8^n} + \frac{1}{9^n} + \ldots + \frac{1}{15^n}\right) + \&c.$$

$$\therefore \Sigma < 1 + 2\left(\frac{1}{2^n}\right) + 4\left(\frac{1}{4^n}\right) + 8\left(\frac{1}{8^n}\right) + \&c.$$

$$< 1 + \frac{1}{2^{n-1}} + \frac{1}{2^{2n-2}} + \frac{1}{2^{3n-3}} + \&c.$$

Now this is a geometric series whose common ratio is $\frac{1}{2^{n-1}}$.

It has therefore a finite sum. Hence the given series has also a finite sum, and is convergent.

88. *If two series,* $u_0 + u_1 + u_2 + \ldots$, $v_0 + v_1 + v_2 + \ldots$, *be such that the limit of* $\frac{u_n}{v_n}$ *when* n *becomes infinite is a finite quantity, then the two series are both convergent or both divergent.*

Let $\frac{u_0}{v_0} = k_0$, $\frac{u_1}{v_1} = k_1$, $\ldots$, $\frac{u_n}{v_n} = k_n$, &c.

And let k be the limit to which k_n continually approaches *as n is increased* indefinitely.

We have

$$u_n + u_{n+1} + u_{n+1} + \ldots = k_n v_n + k_{n+1} v_{n+1} + k_{n+2} v_{n+2} + \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1).$$

Now, as k is the limit to which k_n continually approaches, all the ratios $k_n, k_{n+1}, \ldots$, must lie between $k + h$ and $k - h$ where h may be as small a quantity as we please.

Hence, $k_n v_n + k_{n+1} v_{n+1} + k_{n+2} v_{n+2} + \ldots$ lies between

$$(k + h)(v_n + v_{n+1} + v_{n+2} + \ldots) \text{ and}$$
$$(k - h)(v_n + v_{n+1} + v_{n+2} + \ldots),$$

where h is as small a quantity as we please.

Hence, from (1), $u_n + u_{n+1} + u_{n+2} + \ldots$ lies between

$$(k + h)(v_n + v_{n+1} + v_{n+2} + \ldots) \text{ and}$$
$$(k - h)(v_n + v_{n+1} + v_{n+2} + \ldots),$$

where h is as small as we please.

Hence the series $u_n + u_{n+1} + u_{n+2} + \ldots$ will have a finite limit according as the series $v_n + v_{n+1} + v_{n+2} + \ldots$ has a finite limit.

The two series, $u_n + u_{n+1} + u_{n+2} + \ldots$, and $v_n + v_{n+1} + v_{n+2} + \ldots$, are therefore both convergent or both divergent.

And since the finite number of terms preceding u_n and v_n cannot affect this conclusion, it follows that the given series are either both convergent or both divergent.

89. *The series* $\phi(1) + \phi(2) + \phi(3) + \phi(4) +$ &c., *and*

$$\phi(1) + \mathrm{m}\,\phi(\mathrm{m}) + \mathrm{m}^2\,\phi(\mathrm{m}^2) + \mathrm{m}^3\,\phi(\mathrm{m}^3) + \text{&c.},$$

are convergent and divergent together, when $\phi(\mathrm{x})$ *is positive, and continually diminishes as* x *increases,* m *being a positive integer.*

By extending the first series we may write it thus

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{&c.}$$
$$= \phi(1)$$
$$+ \{\phi(2) + \phi(3) + \ldots + \phi(m)\}$$
$$+ \{\phi(m+1) + \phi(m+2) + \ldots \phi(m^2)\}$$
$$+ \{\phi(m^2+1) + \phi(m^2+2) + \ldots \phi(m^3)\} + \text{&c.} \ldots\ldots (1).$$

But since the quantities $\phi(1)$, $\phi(2)$, $\phi(3)$, &c. are positive, and continually diminish, we have

$$\phi(2) + \phi(3) + \ldots\ldots + \phi(m) > (m-1)\,\phi(m), \text{ and}$$
$$< (m-1)\,\phi(1),$$
$$\phi(m+1) + \phi(m+2) + \ldots\ldots + \phi(m^2) > (m^2-m)\,\phi(m^2), \text{ and}$$
$$< (m^2-m)\,\phi(m),$$
$$\phi(m^2+1) + \phi(m^2+2) + \ldots\ldots + \phi(m^3) > (m^3-m^2)\,\phi(m^3), \text{ and}$$
$$< (m^3-m^2)\,\phi(m^2),$$
&c., &c.

Hence, adding these inequalities, we have, by (1),

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{&c.}$$
$$> \phi(1) + (m-1)\left\{\phi(m) + m\phi(m^2) + m^2\phi(m^3) + \text{&c.}\right\},$$
$$\text{and} > \phi(1) + (m-1)\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{&c.}\right\}.$$

Or, arranging, we have

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \text{&c.}$$
$$> \frac{1}{m}\phi(1)$$
$$+ \frac{m-1}{m}\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \text{&c.}\right\} \ldots\ (2).$$
$$\text{And} < \phi(1) + (m-1)\left\{\phi(1) + m\phi(m) + m^2\phi(m^2) + \text{&c.}\right\} \ldots\ (3).$$

From (2), we learn that when the second series is divergent, the first is divergent; and from (3), that the first is convergent when the second is convergent.

Ex. The series $u_1 + u_2 + u_3 + \ldots\ldots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when the limit is less than, or equal to, unity.

By what has just been proved,

The two series $u_1 + u_2 + u_3 + \ldots\ldots$, and $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots\ldots$ are both convergent or both divergent, if the limit, *when n is infinite*, of the ratio $u_n : \frac{1}{n^m}$ is a finite quantity.

Let k be this limit.

Then, when n is infinite, we have

$$u_n \cdot n^m = k, \text{ or } \log u_n + m \log n = \log k;$$

$$\text{or } \frac{-\log u_n}{\log n} = m - \frac{\log k}{\log n};$$

or, since the limit of $\frac{\log k}{\log n} = 0$, the limit of $\frac{-\log u_n}{\log n} = m$.

Now, by Art. 86, the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \ldots$ is convergent when m is greater than unity, and divergent when m is less than, or equal to, unity.

Hence, since m is the limit of $\frac{-\log u_n}{\log n}$, and in that case the convergency or divergency of the series $u_1 + u_2 + u_3 + \ldots$ depends upon that of the series $\frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m}$ + &c., it follows that the series $u_1 + u_2 + u_3 + \ldots$ is convergent when the limit of $\frac{-\log u_n}{\log n}$ is greater than unity, and divergent when equal to, or less than unity.

Ex. XIV.

Test the convergency or divergency of the following series :—

1. $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots$

2. $\frac{1}{\lfloor 2} + \frac{2}{\lfloor 3} + \frac{3}{\lfloor 4} + \ldots$

3. $\frac{1}{\lfloor 2} + \frac{1.3}{\lfloor 3} + \frac{1.3.5}{\lfloor 4} + \frac{1.3.5.7}{\lfloor 5} + \ldots$

4. $\frac{a}{\lfloor a+1} + \frac{2a^2}{\lfloor a+2} + \frac{3a^3}{\lfloor a+3} + \ldots$

5. $\frac{a}{a+b} + \frac{a^2}{(a+b)(a+2b)} + \frac{a^3}{(a+b)(a+2b)(a+3b)} + \ldots$

6. $\dfrac{1}{a(a+b)} + \dfrac{1}{(a+b)(a+2b)} + \dfrac{1}{(a+2b)(a+3b)} + \ldots.$

7. $1 + x + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \ldots.$

8. $1 + \dfrac{2n(2n-1)}{1.2} + \dfrac{2n(2n-1)(2n-2)(2n-3)}{1.2.3.4} + \ldots.$

9. $\dfrac{x}{2} + \dfrac{2^3x^2}{2^2+1} + \dfrac{3^3x^3}{3^2+1} + \ldots. + \dfrac{n^3x^n}{n^2+1} + \ldots.$

10. $1 + \dfrac{a}{b}x + \dfrac{a(a+1)}{b(b+1)}x^2 + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \ldots.$

11. Show that the series whose nth term is $\phi(n)x^n$ is convergent or divergent according as the limit of $\dfrac{\phi(n)}{\phi(n-1)}$, when n is infinite, is less or greater than unity.

If the limit be negative, or unity, what is the test you apply?

12. Show that the $\dfrac{1}{1} + \dfrac{1}{1.2} + \dfrac{1}{1.2.3} + \ldots.$ is convergent, and that the error in taking the first n terms for the series is less than $\dfrac{1}{(n-1)\lfloor n-1}$.

Recurring Series.

90. The series $a_0 + a_1x + a_2x + \ldots.$ **is called a recurring series,** when the relation existing between every set of (r) consecutive terms can be expressed by the equation

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \ldots. + p_{r-1}a_{n-r+1} = 0;$$

where p_1, p_2, p_3, $\ldots.$ p_{r-1} are constant quantities, and where n may have any integral value.

It is usual to call $1 + p_1x + p_2x^2 + \ldots. + p_{r-1}x^{r-1}$ *the scale of relation.*

A simple case is an ordinary geometric series.

Thus, in the series $1 + 2x + 4x^2 + 8x^3 + \ldots\ldots$, we have

$$a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ \&c.,\ a_n = 2^n,\ \&c.$$

Hence $\dfrac{a_n}{a_{n-1}} = \dfrac{2^n}{2^{n-1}} = 2$, or $a_n - 2\,a_{n-1} = 0$.

This equation expresses generally the relation existing between any *two* consecutive terms.

So we have $a_1 - 2\,a_0 = 0$, $a_2 - 2\,a_1 = 0$, $a_3 - 2\,a_2 = 0$, &c. And *the scale of relation* of the series is $1 - 2x$.

The following examples will illustrate the method of finding the scale of relation of a given series.

Ex. 1. Given the series $2 + 2x - 4x^2 - 22x^3 - 46x^4 - \&c.$, find the scale of relation.

We can see that the given series is not an ordinary G. P., and therefore its scale of relation will consist of more than *two* terms.

Assume it to be $1 + p_1x + p_2x^2$.

Then we have
$$-\ 4 + 2p_1 + 2p_2 = 0 \ldots\ldots\ldots\ldots\ldots(1),$$
$$-\ 22 - 4p_1 + 2p_2 = 0 \ldots\ldots\ldots\ldots\ldots(2),$$
$$-\ 46 - 22p_1 - 4p_2 = 0 \ldots\ldots\ldots\ldots\ldots(3).$$

From (1) and (2) we get $p_1 = -3$, $p_2 = 5$, and these values also satisfy (3). Hence we conclude that our *assumption* is correct, and that the scale of relation is $1 - 3x + 5x^2$.

We *might have found* that the value of p_1 and p_2 in (1) and (2) would not satisfy (3), and we must then have made another assumption.

For instance, we should have assumed the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

In that case we must have had given us another term in the series; for we should have required *three* equations to determine the values of p_1, p_2, p_3, and a *fourth* to verify our results. In the next example we shall assume the scale of relation to *consist of four* terms, as on trial it would be found $1 + p_1x + p_2x^2$ would not answer the conditions.

Ex. 2. Find the scale of relation of the series

$1 + 10x + 34x^2 + 67x^3 + 115x^4 + 247x^5 + 716x^6 + \&c.$

Assume the scale of relation to be $1 + p_1x + p_2x^2 + p_3x^3$.

Then
$$\begin{aligned} 67 + 34p_1 + 10p_2 + p_3 &= 0. \\ 115 + 67p_1 + 34p_2 + 10p_3 &= 0. \\ 247 + 115p_1 + 67p_2 + 34p_3 &= 0. \\ 716 + 247p_1 + 115p_2 + 67p_3 &= 0. \end{aligned}$$

From the first three equations we find $p_1 = -3, p_2 = 4, p_3 = -5$; and these values satisfy the *fourth* equation.

Hence the scale of relation is $1 - 3x + 4x^2 - 5x^3$.

91. ***To find the sum of a recurring series.***

Let $a_0 + a_1x + a_2x^2 + \ldots$. be the series, whose scale of relation is $1 + p_1x + p_2x^2 + \ldots + p_{r-1}x^{r-1}$.

We have then

$$a_n + p_1a_{n-1} + p_2a_{n-2} + \ldots + p_{r-1}a_{n-r+1} = 0 \ldots\ldots (1),$$

where n has any positive integral value.

Let $\Sigma = a_0 + a_1x + a_2x^2 + \ldots + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \ldots$

Multiplying each side of the equation by $1 + p_1x + p_2x^2 + \ldots p_{r-1}x^{r-1}$, we have

$$\begin{aligned} &(1 + p_1x + p_2x^2 + \ldots + p_{r-1}x^{r-1})\,\Sigma \\ &= a_0 + a_1x + a_2x^2 + \ldots + a_{r-2}x^{r-2} + a_{r-1}x^{r-1} + \ldots \\ &\quad + p_1a_0x + p_1a_1x^2 + \ldots + p_1a_{r-3}x^{r-2} + p_1a_{r-2}x^{r-1} + \ldots \\ &\quad + p_2a_0x^2 + \ldots + p_2a_{r-4}x^{r-2} + p_2a_{r-3}x^{r-1} + \ldots \\ &\quad + \ldots\ldots\ldots\ldots \\ &\quad + p_{r-2}a_0x^{r-2} + p_{r-2}a_1x^{r-1} + \ldots \\ &\quad + p_{r-1}a_0x^{r-1} + \ldots \end{aligned}$$

Collecting the coefficients of like powers of x, and remembering that the coefficient of x^{r-1}, viz.,

$$a_{r-1} + p_1a_{r-2} + p_2a_{r-3} + \ldots + p_{r-2}a_1 + p_{r-1}a_0,$$

and all succeeding coefficients are each equal to zero, as we *learn from (1) above*, we get

$$(1 + p_1x + p_2x^2 + \ldots + p_{r-1}x^{r-1})\,\Sigma$$

$$= a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \ldots + (a_{r-2} + p_1a_{r-3} + \ldots + p_{r-2}a_0)x^{r-2}$$

$$\therefore \Sigma = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2 + \ldots + (a_{r-2} + p_1a_{r-3} + \ldots + p_{r-2}a_0)x^{r-2}}{1 + p_1x + p_2x^2 + \ldots + p_{r-1}x^{r-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}.$$

Cor. When $r = 3$, or the scale of relation is $1 + p_1x + p_2x^2$, we have

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x}{1 + p_1x + p_2x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

And so, when $r = 4$, or the scale of relation is $1 + p_1x + p_2x^2 + p_3x^3$,

$$\Sigma = \frac{a_0 + (a_1 + p_1a_0)x + (a_2 + p_1a_1 + p_2a_0)x^2}{1 + p_1x + p_2x^2 + p_3x^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}.$$

Ex. 1. Sum the infinite series $2 + 2x - 4x^2 - 22x^3 - 46x^4 - \&c.$

By Art. 90, Ex. 1, the scale of relation is $1 - 3x + 5x^2$.

Hence from (3) above, remembering that a_0, a_1, &c., represent the coefficients of the given series, and that p_1, p_2 are the coefficients of the powers of x in the scale of relation, we have

$$\Sigma = \frac{2 + \{2 + (-3)2\}x}{1 - 3x + 5x^2} = \frac{2 - 4x}{1 - 3x + 5x^2}.$$

Ex. 2. Find the sum of the infinite series

$$1 + 10\,x + 34\,x^2 + 67\,x^3 + 115\,x^4 + 247\,x^5 + 716\,x^6 + \&c.$$

We found the scale of relation to be $1 - 3\,x + 4\,x^2 - 5\,x^3$.

Hence, by (3) above,

$$\Sigma = \frac{1 + \{10 - 3\,(1)\}\,x + \{34 - 3\,(10) + 4\,(1)\}\,x^2}{1 - 3\,x + 4\,x^2 - 5\,x^3}$$

$$= \frac{1 + 7\,x + 8\,x^2}{1 - 3\,x + 4\,x^2 - 5\,x^3}.$$

Since, by the process of long division, or otherwise, the expression found in (A) above will give the original series, we give to the value of Σ there found the name of the *generating function* of the series.

92. *When the scale of relation can be broken up into simple factors, it is easy to find the general term of the series and the sum to* n *terms.*

Let the scale of relation be capable of being broken up into the $(r - 1)$ factors, $(1 - b_1x)$, $(1 - b_2x)$, &c.

Then the generating function can be resolved into partial fractions having these factors for denominators.

Thus we have

$$\Sigma = \frac{P_1}{1 - b_1x} + \frac{P_2}{1 - b_2x} + \ldots + \frac{P_{r-1}}{1 - b_{r-1}x}$$

$$= P_1(1 + b_1x + b_1^2x^2 + \ldots + b_1^{n-1}x^{n-1} + \ldots)$$

$$+ P_2(1 + b_2x + b_2^2x^2 + \ldots + b_2^{n-1}x^{n-1} + \ldots)$$

$$+ \&c.$$

$$+ P_{r-1}(1 + b_{r-1}x + b_{r-1}^2x^2 + \ldots + b_{r-1}^{n-1}x^{n-1} + \ldots)$$

We thus find the nth term of the series to be

$$(P_1b_1^{n-1} + P_2b_2^{n-1} + \ldots + P_{r-1}b_{r-1}^{n-1})\,x^{n-1}.$$

Also, the sum to n terms,

$$= P_1 \cdot \frac{1 - (b_1x)^n}{1 - b_1x} + P_2 \cdot \frac{1 - (b_2x)^n}{1 - b_2x} + \ldots$$

$$+ P_{r-1} \cdot \frac{1 - (b_{r-1}x)^n}{1 - b_{r-1}x}.$$

Ex. Find the sum to n terms and to infinity of the series

$$8 - 29x + 257x^2 - 1691x^3 + 12053x^4 - \&c.$$

The scale of relation is easily found to be $1 + 5x - 14x^2$.

$$\therefore \Sigma = \frac{8 + \{-29 + 5(8)\}x}{1 + 5x - 14x^2} = \frac{8 + 11x}{(1 - 2x)(1 + 7x)}.$$

Resolving into partial fractions, we have

$$\Sigma = \frac{3}{1 - 2x} + \frac{5}{1 + 7x}$$

$$= 3\{1 + 2x + (2x)^2 + \ldots. + (2x)^{n-1} + \ldots.\}$$

$$+ 5\{1 - 7x + (7x)^2 + \ldots. + (-7x)^{n-1} + \ldots.\}.$$

Therefore sum to n terms, or,

$$S = 3 \cdot \frac{1 - (2x)^n}{1 - 2x} + 5 \cdot \frac{1 - (-7x)^n}{1 + 7x}.$$

When the given recurring term does not contain the powers of x, but is of the form $a_0 + a_1 + a_2 + \ldots.$, we may first find the scale of relation and the sum of the series $a_0 + a_1x + a_2x^2 + \ldots.$, and put $x = 1$ in the result.

Summation of Series.

93. Certain series may be summed by particular artifices.

In the series $u_1 + u_2 + \ldots. + u_n$, suppose u_n to be broken up into the difference of two terms, one of which is the same function of $(n - 1)$ as the other is of n.

I. Thus, suppose $u_n = U_n - U_{n-1}$.

Then, putting for n all values from 1 to n, we have

$$S = (U_1 - U_0) + (U_2 - U_1) + \ldots. + (U_n - U_{n-1})$$

$$= U_n - U_0.$$

II. Suppose $u_n = U_{n-1} - U_n$.

We have, similarly,

$$S = (U_0 - U_1) + (U_1 - U_2) + \ldots + (U_{n-1} - U_n)$$
$$= U_0 - U_n.$$

Ex. 1. Sum the series

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1).$$

We have

$$u_n = n(n+1) = \tfrac{1}{3} n(n+1)\{(n+2) - (n-1)\}$$
$$= \tfrac{1}{3} n(n+1)(n+2) - \tfrac{1}{3}(n-1)n(n+1)$$
$$= U_n - U_{n-1},$$

where $U_n = \tfrac{1}{3} n(n+1)(n+2)$,

and therefore $U_0 = \tfrac{1}{3}(0)(1)(2) = 0.$

Hence, S or $U_n - U_0 = \tfrac{1}{3} n(n+1)(n+2) - 0.$

$$\therefore S = \tfrac{1}{3} n(n+1)(n+2).$$

Ex. 2. Sum the series

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots \frac{1}{(2n-1)(2n+1)}.$$

Here $u_n = \dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2} \cdot \dfrac{(2n+1) - (2n-1)}{(2n-1)(2n+1)}$

$$= \frac{1}{2} \cdot \frac{1}{2n-1} - \frac{1}{2} \cdot \frac{1}{2n+1} = U_{n-1} - U_n,$$

where $U_n = \dfrac{1}{2} \cdot \dfrac{1}{2n+1}$, and $\therefore U_0 = \dfrac{1}{2} \cdot \dfrac{1}{2(0)+1} = \dfrac{1}{2}.$

Hence, by II. above

$$S = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2n+1} = \frac{n}{2n+1}.$$

94. *When the* nth *term of a series is of the form*

$$u_n = (an + b)(a \cdot \overline{n+1} + b) \ldots (a \cdot \overline{n+m-1} + b),$$

where a, b *are constants, and* m *the number of factors, then* $S = U_n - U_0$, *where*

$$U_n = \frac{(an + b)(a.\overline{n+1} + b) \ldots . (a.\overline{n+m} + b)}{(m+1)a};$$

that is, U_n *is found by taking one factor more, and dividing by as many times the common difference as there are then factors.*

$$\text{For } u_n = \frac{u_n}{(m+1)a}\left\{(a.\overline{n+m} + b) - (a.\overline{n-1} + b)\right\}$$

$$= \frac{u_n(a.\overline{n+m} + b)}{(m+1)a} - \frac{(a.\overline{n-1} + b)u_n}{(m+1)a}$$

$$= \frac{(an + b)(a.\overline{n+1} + b) \ldots . (a.\overline{n+m} + b)}{(m+1)a}$$

$$- \frac{(a.\overline{n-1} + b)(an + b) \ldots . (a.\overline{n+m-1} + b)}{(m+1)a}$$

$$= U_n - U_{n-1},$$

$$\text{where } U_n = \frac{(an+b)(a.\overline{n+1}+b) \ldots . (a.\overline{n+m}+b)}{(m+1)a} \ldots (1).$$

Hence, by Art. 93, $S = U_n - U_0$, where U_n has the value in (1).

Ex. 1. Sum the series

$$3.5.7 + 5.7.9 + \ldots . + (2n+1)(2n+3)(2n+5).$$

We have $u_n = (2n+1)(2n+3)(2n+5)$.

Here the common difference is 2, and with one more factor, the number of factors will be 4.

$$\text{Hence } U_n = \frac{(2n+1)(2n+3)(2n+5)(2n+7)}{4(2)}$$

$$= \frac{1}{8}(2n+1)(2n+3)(2n+5)(2n+7).$$

$$\text{Therefore } U_0 = \frac{1}{8}(1)(3)(5)(7) = \frac{105}{8}.$$

$$\therefore\ S = \frac{1}{8}\{(2n+1)(2n+3)(2n+5)(2n+7) - 105\}$$

$$= 2n^4 + 16n^3 + 43n^2 + 44n.$$

When u_n consists of the *sum of several terms* of the form in Art. 94, it is evident that U_n must also consist of the sum of an equal number of terms, each obtained according to the law there proved. The next example is an illustration.

Ex. 2. Sum the series $1^3 + 2^3 + 3^3 + \ldots + n^3$.

We have $u_n = n^3 = n\{(n-1)(n+1) + 1\}$

$$= (n-1)\,n(n+1) + n.$$

$$\text{Hence}\quad U_n = \frac{(n-1)\,n(n+1)(n+2)}{4\,(1)} + \frac{n(n+1)}{2\,(1)}$$

$$= \frac{1}{4}n^2(n+1)^2.$$

Therefore $U_0 = 0$.

Hence S or $U_n - U_0 = \frac{1}{4}n^2(n+1)^2 = \left\{\frac{1}{2}n(n+1)\right\}^2$.

95. *When the* n*th term of a series is of the form*

$$u_n = \frac{1}{(an+b)(a.\overline{n+1}+b)\ldots\ldots(a.\overline{n+m-1}+b)},$$

where a, b *are constants, and* m *the number of factors, then* $S = U_0 - U_n$, *where*

$$U_n = \frac{1}{(m-1)a.(a.\overline{n+1}+b)(a.\overline{n+2}.b)\ldots\ldots(a.\overline{n+m-1}.b)};$$

that is, where U_n *is found by omitting the first factor, and dividing by as many times the common difference as there are factors remaining.*

For u_n

$$= \frac{1}{(m-1)a} \cdot \frac{(a.\overline{n+m-1}+b) - (an+b)}{(an+b)(a.\overline{n+1}+b)\ldots\ldots(a.\overline{n+m-1}+b)}$$

$$= \frac{1}{(m-1)\,a}\left\{\frac{1}{(an+b)(a.n+1+b)\ldots.(a.n+m-2+b)} - \frac{1}{(a.n+1+b)\ldots.(a.n+m-1+b)}\right\}$$

$$= U_{n-1} - U_n,$$

where $U_n = \dfrac{1}{(m-1)\,a\,.\,(a\,.\,n+1+b)\ldots.(a\,.\,n+m-1+b)}$ ..... (1).

Hence, by Art. 93, $S = U_0 - U_n$, where U_n has the value in (1).

Ex. 1. Sum the series $\dfrac{1}{1\,.\,2\,.\,3} + \dfrac{1}{2\,.\,3\,.\,4} + \ldots. + \dfrac{1}{n(n+1)\,(n+2)}$.

Here $u_n = \dfrac{1}{n(n+1)\,(n+2)}$.

Therefore $U_n = \dfrac{1}{2(1)\,(n+1)\,(n+2)} = \dfrac{1}{2(n+1)\,(n+2)}$.

Hence also, $U_0 = \dfrac{1}{2(1)\,(2)} = \dfrac{1}{4}$.

Therefore $U_0 - U_n$, or $S = \dfrac{1}{4} - \dfrac{1}{2(n+1)\,(n+2)}$.

Hence also, since the limit of $\dfrac{1}{2(n+1)\,(n+2)}$ is 0, when $n = \infty$, we have $\Sigma = \dfrac{1}{4}$.

Ex. 2. Sum the series $\dfrac{1}{1\,.\,2\,.\,3} + \dfrac{3}{2\,.\,3\,.\,4} + \ldots. + \dfrac{2n-1}{n(n+1)\,(n+2)}$.

Here $u_n = \dfrac{2n-1}{n(n+1)(n+2)}$

$$= \frac{2}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}.$$

Therefore $U_n = \dfrac{2}{1(1)(n+2)} - \dfrac{1}{2(1).(n+1)(n+2)}$

$$= \frac{2}{n+2} - \frac{1}{2(n+1)(n+2)}$$

$$= \frac{4n+3}{2(n+1)(n+2)}.$$

Hence $U_0 = \dfrac{3}{2(1)(2)} = \dfrac{3}{4}$.

Therefore $U_0 - U_n$, or $S = \dfrac{3}{4} - \dfrac{4n+3}{2(n+1)(n+2)}$.

Hence also, since $\dfrac{4n+3}{2(n+1)(n+2)} = \dfrac{4+\frac{3}{n}}{2(n+1)(1+\frac{2}{n})}$

$= 0$, when $n = \infty$, we have $\Sigma = \dfrac{3}{4}$.

Figurate Numbers.

96. We give the name **Figurate Numbers** to those of the following series, where the nth term of any series is the sum of n terms of the preceding series :—

1st order, 1, 1, 1, 1, &c.
2nd „ 1, 2, 3, 4, &c.
3rd „ 1, 3, 6, 10, &c.
&c.

For the first order, we have $S_n = n$.

And for the second, $S_n = \frac{1}{2} n(n+1)$.

This is the nth term of the 3rd order.

Hence, to find the sum of n terms of the third order, we have

$$u_n = \tfrac{1}{2} n(n+1).$$

Therefore (Art. 94),

$$U_n = \tfrac{1}{2} \cdot \frac{n(n+1)(n+2)}{3(1)} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

And therefore $U_0 = 0$.

Hence $U_n - U_0$ or $S_n = \dfrac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$.

To find S_n for the rth order,

Assume that for the pth order, $S_n = \dfrac{n(n+1)\ldots(n+p-1)}{1 \cdot 2 \ldots p}$.

Then for the $(p+1)$th order, $u_n = \dfrac{n(n+1)\ldots(n+p-1)}{1 \cdot 2 \ldots p}$.

Therefore

$$U_n = \frac{n(n+1)\ldots(n+p)}{(p+1) \cdot 1 \cdot 2 \ldots p} = \frac{n(n+1)\ldots(n+\overline{p+1}-1)}{1 \cdot 2 \ldots \overline{p+1}}.$$

$$\text{and } U_0 = 0.$$

Therefore $U_n - U_0$ or $S_n = \dfrac{n(n+1)\ldots(n+\overline{p+1}-1)}{1 \cdot 2 \ldots \overline{p+1}}$.

Hence, by assuming S_n to be formed according to a certain law for the pth order, we have proved it to be formed according to the *same law* for the $(p+1)$th order. Now, we know it is true for the 2nd and 3rd orders. It is therefore true for the 4th, and therefore for the 5th, and so generally true.

Hence for the rth order,

$$S_n = \frac{n(n+1)\ldots(n+r-1)}{1 \cdot 2 \ldots r}.$$

Polygonal Numbers.

97. These numbers are called **linear, triangular, square, pentagonal,** and generally **polygonal,** from the fact that if a dot be *taken to represent* unity, they may all be represented *in the form of the* corresponding polygons. Thus—

Linear Numbers, . , &c., or 1, 2, 3, 4, &c.

Triangular „ . , &c., or 1, 3, 6, 10, &c.

Square „ . , &c., or 1, 4, 9, 16, &c.

Pentagonal „ . , &c., or 1, 5, 12, 22, &c.

The *first order* of polygonal numbers is the series in which each term is *unity.*

And hence, by observing the above diagrams, it is easy to see that the several orders may be arranged as follows :—

1st order,	1,	1,	1,	1,	&c.
2nd „	1,	1 + 1,	1 + 1 + 1,	1 + 1 + 1 + 1,	&c.
3rd „	1,	1 + 2,	1 + 2 + 3,	1 + 2 + 3 + 4,	&c.
4th „	1,	1 + 3,	1 + 3 + 5,	1 + 3 + 5 + 7,	&c.
5th „	1,	1 + 4,	1 + 4 + 7,	1 + 4 + 7 + 10,	&c.
&c.					

The law is evident. The nth number of each order after the first is the sum of n terms of an A. P., whose first term is unity; and the common difference in the rth order is $r - 2$.

Hence the nth term of the rth order

$$= 1 + (1 + \overline{r - 2}) + (1 + 2 \,.\, \overline{r - 2}) + \text{&c., to } n \text{ terms}$$

$$= \left\{2 + \overline{n - 1} \,.\, (r - 2)\right\} \frac{n}{2}$$

$$= n + \tfrac{1}{2}\, n\,(n - 1)\,(r - 2).$$

Hence, for the rth order, we have u_n

$$= n + \tfrac{1}{2}(r - 2) \,.\, (n - 1)n.$$

Hence, $\mathrm{U}_n = \dfrac{n(n + 1)}{2} + \tfrac{1}{2}(r - 2) \,.\, \dfrac{(n - 1)\, n(n + 1)}{3}$,

and $\therefore\ \mathrm{U}_0 = 0.$

Hence (Art. 93) for the rth order,

$$S_n = \frac{n}{6}(n + 1)\,\{(r - 2)\,(n - 1) + 3\}.$$

Cor. Let $r = 3, 4, 5$, successively, then,

For triangular numbers, $S_n = \frac{1}{6} n(n + 1)(n + 2)$.
„ square „ $S_n = \frac{1}{6} n(n + 1)(2n + 1)$.
„ pentagonal „ $S_n = \frac{1}{6} n(n + 1)(3n - 3 + 3)$
$= \frac{1}{2} n^2(n + 1)$.

98. *To find the number of cannon balls in a pyramidal heap.*

(1.) When the base is an *equilateral triangle.*

Let n be the number of balls in a side of the base.

Then, number of balls in lowest layer

$$= 1 + 2 + 3 + \ldots + n = \tfrac{1}{2} n(n + 1),$$

the nth triangular number.

Hence, by Art. 93, $S_n = \frac{1}{6} n(n + 1)(n + 2)$.

(2.) When the base is a *square.*

We have, similarly, $S_n = \frac{1}{6} n(n + 1)(2n + 1)$.

(3.) When the pile is *deficient*, the first m courses being removed.

Here, if S_n = the number in the *complete* pile,
and S_m = the number in the removed courses,
$S_n - S_m$ = the number in the incomplete pile.

99. *To find the number of balls when the base is rectangular, and not square.*

The pile will now terminate in a single row at the top.

Let l be the length of the top row.

Then, the nth row, reckoned from the top, will contain $l + n - 1$ balls, and its breadth will be n.

Then, number of balls in the lowest or nth layer

$$= (l + n - 1)\, n.$$

Hence $u_n = (l + n - 1)\, n = l \,.\, n + (n - 1)\, n$.

Therefore (Art. 94)

$$U_n = l \,.\, \frac{n(n + 1)}{2} + \frac{(n - 1)\, n(n + 1)}{3}, \text{ or, since } U_0 = 0,$$

$$S_n = l \cdot \frac{n(n+1)}{2} + \frac{(n-1)\,n(n+1)}{3}$$

$$= \tfrac{1}{6}\, n(n+1)\left\{3\,l + 2(n-1)\right\}$$

$$= \tfrac{1}{6}\, n(n+1)\left\{3\,(l+n-1) - n + 1\right\}$$

$$= \tfrac{1}{6}\, n(n+1)\,(3\,l_1 - n + 1),$$

when l_1 is the length, and n the breadth, of the lowest row.

Method of Differences.

100. Suppose we have to find the sum of the series

3, 6, 11, 18, 27. 38, &c.

Taking the differences of each two consecutive terms, and then the differences of each two of these differences, and so on, until the differences become a series of identical terms, we have;

Series, ...	3, 6, 11, 18, 27, 38, &c.
1st order of differences, .	3, 5, 7, 9, 11, &c.
2nd order of differences, .	2, 2, 2, 2, &c.

It will be seen that *the* n*th term of any of these series is the sum of its* 1st term, *and the first* (n − 1) *terms of the succeeding series*..... (A).

Thus, in the 1st order,

$$u_n = 3 + 2(n-1).$$

Therefore, by Art. 94, $U_n = 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)\,n}{2 \cdot 1}$.

$$= 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)n}{1 \cdot 2}; \text{ or, since } U_0 = 0$$

$$S_n = 3 \cdot \frac{n}{1} + 2 \cdot \frac{(n-1)\,n}{1 \cdot 2}.$$

$$\therefore S_{n-1} = 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)\,(n-1)}{1 \cdot 2}.$$

Hence, by (A) above, in the given series,

$$u_n = 3 + 3 \cdot \frac{n-1}{1} + 2 \cdot \frac{(n-2)\,(n-1)}{1 \cdot 2}.$$

$$\therefore\ U_n = 3 \cdot \frac{n}{1} + 3 \cdot \frac{(n-1)n}{1 \cdot 2} + 2 \cdot \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3};$$

or, since $U_0 = 0$,

$$S_n = 3n + 3 \cdot \frac{n(n-1)}{1 \cdot 2} + 2 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

$$= \frac{n}{6}(n^2 + 3n + 13).$$

We shall in the next article give the general theorem.

101. *If* u_1, u_2, u_3, *&c. be a series, and* d_1, d_2, d_3, *&c. the first terms of the successive order of differences, the differences in the* r*th order being identical, then we have*

$$S_n = nu_1 + \frac{n(n-1)}{1 \cdot 2} d_1 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d_2 + \ldots.$$

$$+ \frac{n(n-1)\ldots.(n-r)}{\lfloor r+1} d_r.$$

In the $(r-1)$th order, we have $u_n = d_{r-1} + (n-1)d_r$.

Therefore $U_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r$; and therefore, since

$$U_0 = 0,\ S_n = \frac{n}{1} d_{r-1} + \frac{(n-1)n}{1 \cdot 2} d_r.$$

Therefore $S_{n-1} = \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r$.

In the $(r-2)$th order, we therefore have

$$u_n = d_{r-2} + \frac{n-1}{1} d_{r-1} + \frac{(n-2)(n-1)}{1 \cdot 2} d_r.$$

Therefore $U_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r$;

and therefore, since $U_0 = 0$,

$$S_n = \frac{n}{1} d_{r-2} + \frac{(n-1)n}{1 \cdot 2} d_{r-1} + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3} d_r.$$

We may easily show therefore by induction that—

For the $(r - p)$th order

$$S_n = \frac{n}{1} d_{r-p} + \frac{(n-1)n}{1.2} d_{r-p+1} + \ldots.$$

$$+ \frac{(n-p)\,(n-p+1)\ldots.(n-1)n}{1.2\ldots.(p+1)} d_r.$$

Put $p = r$, then for the series *preceding* the 1st order of differences, that is, for the given series, since $d_0 = u_1$, we have

$$S_n = \frac{n}{1} u_1 + \frac{(n-1)n}{1.2} d_1 + \frac{(n-2)\,(n-1)\,n}{1.2.3} d_2 + \ldots.$$

$$+ \frac{(n-r)\ldots.(n-1)\,n}{1.2\ldots.(r+1)} d_r,$$

$$= nu_1 + \frac{n(n-1)}{1.2} d_1 + \frac{n(n-1)\,(n-2)}{1.2.3} d_2 + \ldots.$$

$$\div \frac{n(-1)\ldots.(n-r)}{\lfloor r+1} d_r.$$

Ex. XV.

Find the generating functions of the following six series:

1. $1 + 8x + 22x^2 + 50x^3 + 106x^4 + \ldots.$
2. $1 + 3x + x^2 - 5x^3 - 7x^4 - \ldots.$
3. $2 + 13x + 38x^2 + 61x^3 - 22x^4 - \ldots.$
4. $3 + 11x + 57x^2 + 309x^3 + 1683x^4 + \ldots.$
5. $9 - 11x - 52x^2 + 129x^3 + 235x^4 - \ldots.$
6. $6 + 16x + 25x^2 + 17x^3 - 33x^4 - 142x^5 - 277x^6 - \ldots.$

Sum to n terms, and to infinity, the following eight series:

7. $\frac{1}{3} - 13x + 69x^2 - 337x^3 + 170x^4 - \ldots.$
8. $6 - 33x + 249x^2 - 1707x^3 + 12021x^4 - \ldots.$
9. $1 - 2x + 7x^2 - 8x^3 + 37x^4 - 14x^5 + 211x^6 + \ldots.$
10. $1 - 7x + 22x^2 - 70x^3 + 211x^4 - 637x^5 + 1912x^6 - \ldots.$
11. $2 - 11x + 54x^2 - 271x^3 + 1354x^4 - \ldots.$

12. $1 + 4x + 10x^2 + 34x^3 - 26x^4 + \ldots$.

13. $3 + 14x + 36x^2 + 69x^3 + 113x^4 + 168x^5 + 234x^6 + \ldots$.

14. $4 + 29x + 131x^2 + 491x^3 + 1679x^4 + 5459x^5 + 17231x^6 + \ldots$.

15. Find the scale of relation of the recurring series

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots.$$

16. Find the generating function of the series

$$a + (a + b)x + (a + 2b)x^2 + (a + 3b)x^3 + \ldots.$$

17. If $a_0, a_1, a_2, \ldots$ form as A. P., and $b_0, b_1, b_2, \ldots$ form a G. P., show that the series

$$\frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots \text{ is recurring.}$$

18. If $1 + p_1x + p_2x^2$ is the scale of relation of the series $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, show that

$$a_n(a^2_n - a_{n-1}a_{n+1}) + a_{n-1}(a_{n-1}a_{n+2} - a_na_{n+1}) + a_{n-2}(a^2_{n+1} - a_na_{n+2}) = 0.$$

Sum to n terms the following ten series:

19. $1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \ldots$.

20. $(x + 1)(x + 2)(x + 3) + (x + 2)(x + 3)(x + 4) + \ldots$.

21. $\frac{1}{2 \cdot 4 \cdot 6} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{4 \cdot 6 \cdot 8} + \ldots$.

22. $\frac{4}{2 \cdot 3 \cdot 4} + \frac{7}{3 \cdot 4 \cdot 5} + \frac{10}{4 \cdot 5 \cdot 6} + \ldots$.

23. $\frac{1^3 + 1}{2} + \frac{2^3 + 1}{3} + \frac{3^3 + 1}{4} + \ldots$.

24. $\frac{1}{(a+b)(a+2b)(a+3b)} + \frac{1}{(a+2b)(a+3b)(a+4b)} + \ldots$.

25. $1^2 + 3^2 + 5^2 + \ldots$.

26. $1^3 + 3^3 + 5^3 + \ldots$.

27. $\dfrac{1}{1\,.\,2} + \dfrac{2}{1\,.\,2\,.\,3} + \dfrac{3}{1\,.\,2\,.\,3\,.\,4} + \ldots$.

28. $\dfrac{1}{\lfloor\underline{3}} + \dfrac{12}{\lfloor\underline{4}} + \ldots + \dfrac{3n^2 + 2n - 4}{\lfloor\underline{n+2}}$.

29. $\dfrac{1}{m} + \dfrac{m-n}{m(m-1)} + \dfrac{(m-n)(m-n-1)}{m(m-1)(m-2)} + \ldots$ to $m-n+1$ terms.

30. $\dfrac{1}{2\,.\,4} + \dfrac{1\,.\,3}{2\,.\,4\,.\,6}\cdot\dfrac{1}{4} + \dfrac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}\cdot\dfrac{1}{4^2} + \ldots$ to infinity.

31. Find the number of shot in a triangular pile, each side of whose base contains 10 balls.

32. There are 12 balls in the length, and 8 in the breadth of the base of a rectangular pile, and it terminates in a single row. Find the number of balls.

33. Find the number of balls in a rectangular pile when m, n represent respectively the number in the length and breadth of the base, and there are p layers.

34. If $x \div \{(1-x)^2 - cx\}$ be expanded in a series of ascending powers of x, show that the coefficient of x^r is

$$r\left\{1 + \frac{r^2-1}{2\,.\,3}c + \frac{(r^2-1)(r^2-4)}{2\,.\,3\,.\,4\,.\,5}c^2 + \frac{(r^2-1)(r^2-4)(r^2-9)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}c^3 + \&c.\right\}.$$

35. Show that

$$\frac{1}{m} - n\cdot\frac{1}{m+p} + \frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{m+2p} - \&c. + (-1)^n\cdot\frac{1}{m+np} = \frac{\lfloor\underline{np^n}}{m(m+p)\ldots(m+np)}.$$

36. If p_r be the coefficient of the $(r+1)$th term of $(1+x)^n$, where n is a positive integer, then

$$\frac{p_1}{p_0} + \frac{2p_2}{p_1} + \frac{3p_3}{p_2} + \ldots + \frac{np_n}{p_{n-1}} = \frac{1}{2}n(n+1),$$

and $(p_0 + p_1)(p_1 + p_2)(p_2 + p_3) + \ldots (p_{n-1} + p_n) = \dfrac{p_1 p_2 p_3 \ldots p_n}{1\,.\,2\,.\,3\ldots n}(n+1)^n$.

CHAPTER XII.

Scales of Notation.

102. An ordinary number as 6437 may be thus written:

$$6 \cdot 10^3 + 4 \cdot 10^2 + 3 \cdot 10 + 7.$$

We call 10 the radix of the decimal system of notation.

And generally, if r be the radix of any other scale, and $p_0, p_1, p_2, \ldots . p_{n-1}$, the n digits commencing with the one at the right hand, we may express a number N thus:

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_2r^2 + p_1r + p_0.$$

It is easy to see that in order to express all numbers in any given scale, it is necessary and sufficient to have, besides zero, one digit less than the radix of the scale.

Thus, in ordinary notation, we have 9 digits.

Hence also, for the duodenary scale we shall require characters to represent *ten* and *eleven*. It is usual to represent them by t and e respectively.

Ex. Express 1420 (senary scale), and $1te3$ (duodenary scale), as common numbers.

1420 (senary scale) $= 1 \cdot 6^3 + 4 \cdot 6^2 + 2 \cdot 6 + 0 = 372.$

$1te3$ (duodenary scale) $= 1 \cdot 12^3 + 10 \cdot 12^2 + 11 \cdot 12 + 3 = 3303.$

103. *To transform an integer from any scale of notation to any other scale.*

Let N be the number, and r the radix of the scale in which we are to express it.

We shall therefore have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_2r^2 + p_1r + p_0 \ldots . (1),$$

where p_0, p_1, p_2, &c., are the digits whose values are required.

From (1), dividing each side by r, we have

$$\text{Quotient} = p_{n-1}r^{n-2} + p_{n-2}r^{n-3} + \ldots . + p_2r + p_1 \ldots (2),$$

and remainder $= p_0$.

We obtain then the first or right hand digit by dividing the given number by the proposed radix, and taking the remainder.

Again, dividing each side of (2) by r, we get—

$$\text{Quotient} = p_{n-1}r^{n-3} + p_{n-2}r^{n-4} + \ldots + p_2 \ldots (3),$$

and Remainder $= p_1$.

Hence, *we obtain the second digit by dividing the first quotient by the radix, and taking the remainder.*

And so we may proceed until we have found all the digits. We have therefore the following rule:—

RULE.—*Divide the given number by the proposed radix, then the quotient so obtained again by the radix, and so on. Then the successive remainders are the respective digits required, commencing with the right hand one.*

Ex. 1. Transform the common number 3526 into the *ternary* and *septenary* scales.

3	3526	
3	1175	1
3	391	2
3	130	1
3	43	1
3	14	1
3	4	2
3	1	1
		1

7	3526	
7	503	5
7	71	6
7	10	1
7	1	3
		1

Hence 3526, in the common scale, is expressed by *11211121* in the ternary scale, and by 13165 in the *septenary scale.*

Ex. 2. Transpose 35*et* from the duodenary scale to the septenary scale.

First method.

Expressing 35*et* as a common number, we have

$$35et = 3 \cdot 12^3 + 5 \cdot 12^2 + 11 \cdot 12 + 10 = 6046.$$

Then, as in the last example,

7	6046	
7	863	5
7	123	2
7	17	4
7	2	3
		2

Therefore, the number required in the septenary scale is 23425.

Second method.

We may proceed to divide the given number, 35 *et*; at once by 7. Thus—

7	35 *et*	
7	5*e*	5
7	*t*3	2
7	15	4
7	2	3
		.. .2

The operation requires a little explanation. Dividing 35 by 7, we get 5, and 0 over; for 35 in the duodenary scale is $3 \times 12 + 5$ *or* 41 *in the* common scale.

Then dividing $6e$ by 7 we get e, and 6 over, for

$$6e = 6 \times 12 + 11 = 83.$$ And so on.

In the next examples, the student is recommended to work out, as above, the process for himself.

Ex. 3. The numbers 357 and 76 are in the duodenary scale; find their product.

$$\begin{array}{r} 357 \\ 76 \\ \hline 1896 \\ 2031 \\ \hline 21et6 \\ \hline \end{array}$$

Ex. 4. Find the quotient of 1420422 divided by 241, both numbers being in the senary scale.

$$\begin{array}{r} 241)1420422(3502 \\ 1203 \\ \hline 2134 \\ 2125 \\ \hline 522 \\ 522 \\ \hline \end{array}$$

Ex. 5. Find the area of a room which measures 27 ft. 9 in. by 10 ft. 10 in.

Expressing these in the duodenary scale, we have

$$\begin{array}{r} 23{\cdot}9 \\ 16{\cdot}t \\ \hline 1e16 \\ 11t6 \\ 239 \\ \hline \end{array}$$

$376{\cdot}76 = 376{\cdot}76$ sq ft. in the duodenary scale.

This expressed in the common scale

$= (3 \cdot 12^2 + 7 \cdot 12 + 6)$ sq. ft. $(7 \times 12 + 6)$ sq. in.

$= 522$ sq. ft. 90 sq. in.

104. *To transpose a given proper fraction from any scale of notation to any other scale.*

If we transform the numerator and denominator each into the new scale, we transpose the fraction into the new scale.

It may, however, be required to transform it into a fraction having the form of a decimal in ordinary notation. Such fractions are called **radix fractions.**

We proceed to investigate the method of doing this.

Let F be the given proper fraction,

And let q_1, q_2, q_3, &c., be the digits of the radix fraction, commencing from the left.

We then have

$$\mathrm{F} = \frac{q_1}{r} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \text{&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

where q_1, q_2, q_3, &c., are integers whose values are required.

Multiplying each side of (1) by r, we have

$$\mathrm{F}r = q_1 + \frac{q_2}{r} + \frac{q_3}{r_2} + \text{&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, the first digit, q_1, of the radix fraction is found by multiplying the given fraction by the proposed radix, and taking the integral portion.

The remaining fractional part is $\frac{q_2}{r} + \frac{q_3}{r_2} +$ &c., and it is therefore evident that the second digit q_2 of the radix fraction is found by multiplying this by the radix. And so on.

Hence we have the following rule for bringing a given fraction to a radix fraction.

RULE.—*Multiply the given fraction by the radix, and the integral portion of the product is the first digit. Multiply the fractional part by the radix, and the integral portion of the product is the second digit. And so on.*

Ex. 1. Transform ·8125 from the common scale to the scale whose radix is 8.

$$\begin{array}{r} \cdot 8125 \\ 8 \\ \hline 6\cdot 5000, \qquad \therefore\ q_1 = 6. \\ 8 \\ \hline 4\cdot 0 \quad , \qquad \therefore\ q_2 = 4. \end{array}$$

Hence the required radix fraction is ·64.

Proof.—·64, in the scale whose radix is 8, $= \frac{6}{8} + \frac{4}{8^2}$ $= \cdot 75 + \cdot 0625 = \cdot 8125.$

Ex. 2. Transform 713·44 in the scale eight to the scale six.

$$\begin{array}{r|l|l|l} 6 & 713 & \cdot 44 & \\ & & 6 & \\ \hline 6 & 114 \ldots . 3 = p_0, & 3\cdot 30, & \therefore\ q_1 = 3. \\ \hline 6 & 14 \ldots . 4 = p_1, & 6 & \\ \hline 6 & 2 \ldots . 0 = p_2, & 2\cdot 2\ , & \therefore\ q_2 = 2. \\ \hline & \ldots . 2 = p_3, & 6 & \\ & & 1.4\ , & \therefore\ q_3 = 1. \\ & & 6 & \\ & & 3\cdot 0\ , & \therefore\ q_4 = 3. \end{array}$$

Hence the number required is 2043·3213.

105. *A number whose radix is* r *is divisible by* r − 1, *when the sum of its digits is so divisible.*

Let the number N be such that

$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_2r^2 + p_1r + p_0 \ldots$ (1), where $p_0, p_1, p_2, \ldots . , p_{n-1}$, are the digits commencing at *the right hand.*

We have

$$N = p_{n-1}(r^{n-1} - 1) + p_{n-2}(r^{n-2} - 1) + \ldots . p_2(r^2 - 1) + p_1(r - 1) + (p_{n-1} + p_{n-2} + \ldots . + p_2 + p_1 + p_0).$$

Now each of the quantities $r^{n-1} - 1$, $r^{n-2} - 1$, ... , $r^2 - 1$, $r - 1$, is divisible by $r - 1$. Hence the whole of the right side of this equation is divisible by $r - 1$, if $(p_{n-1} + p_{n-2} + \ldots . + p_2 + p_1 + p_0)$ is so divisible; *i.e.*,

N is divisible by $r - 1$, when $(p_{n-1} + p_{n-2} + \ldots . + p_2 + p_1 + p_0)$, the sum of its digits, is so divisible.

COR. *Hence also, a number whose radix is* r, *will leave the same remainder when divided by* r − 1, *as the sum of its digits leaves when divided by* r − 1.

106. *A number whose radix is* r *is divisible by* r + 1, *when the difference between the sum of the digits in the odd places and the sum of the digits in the even places is so divisible.*

We have

$$N = p_{n-1}r^{n-1} + p_{n-2}r^{n-2} + \ldots . + p_3r^3 + p_2r^2 + p_1r + p_0 \ldots (1).$$

$$= p_{n-1}\{r^{n-1} - (-1)^{n-1}\} + p_{n-2}\{r^{n-2} - (-1)^{n-2}\} + \ldots . + p_3(r^3 + 1) + p_2(r^2 - 1) + p_1(r + 1) + \{p_0 - p_1 + p_2 - p_3 + \ldots . + (-1)^{n-1}p_{n-1}\}.$$

Now (Vol. I., Art. 30) each of the quantities $r + 1$, $r^2 - 1$, $r^3 + 1$, ... , $r^{n-1} - (-1)^{n-1}$, is divisible by $r + 1$.

Hence the whole of the right side of this equation is divisible by $r + 1$, if $\{p_0 - p_1 + p_2 - p_3 + \ldots . (-1)^{n-1}p_{n-1}\}$ is so divisible.

But $p_0 - p_1 - p_2 - p_3 + \ldots . + (-1)^{n-1}p_{n-1}$ is the difference between the sums of the digits in the odd and even places.

Hence N is divisible by $r + 1$, when the difference between the sums of the digits in the odd and even places is so *divisible.*

COR. Hence, also, *when a number, whose radix is* r, *is divided by* r + 1, *it will leave the same remainder as the difference between the sums of the digits in the odd and even places will leave when so divided.*

EX. XVI.

1. Express the common numbers 654 and 387 in the senary scale.

2. Transform 372 and 516 from the denary to the septenary scale, multiply the results together, and divide the product by 516 (septenary scale). Express the quotient in the denary scale.

3. Transform 4163 from the septenary to the scale whose radix is eight.

4. The number 86 expressed in another scale is 95. Find the radix.

5. The number 172 expressed in a different scale is 124. What is the radix.

6. If the number 516 be divided by 2, the quotient when expressed in another scale has the same digits as the original number. Find the scale.

7. Transform 79·875 from the denary scale to the scale whose radix is 8.

8. Express 4521 (senary scale) in the duodenary scale.

9. Find the area of a room measuring 52 ft. 9 in. by 30 ft. 7 in., expressing your result in the duodenary scale.

10. Simplify $\frac{313 \times 1034\frac{2}{3}}{45\frac{5}{10} \times 210}$, the number being in scale six. Give your answer in scale seven.

11. The number 52614 is in scale seven. Find its square root in that scale.

12. *Reduce* $\cdot\dot{3}\dot{2}$ (scale six) to a vulgar fraction in the *same scale.*

13. Show that 1331 is a cube number in any scale above three.

14. The difference between any number and the number composed of the same digits in reverse order is divisible by $r - 1$, where r is the radix of the scale.

15. Show that the difference between two numbers having the same digits is divisible by $r - 1$, whatever be the order of the digits in the second number.

16. Given a system of weights 1 lb., 3 lbs., 3^2 lbs., &c., (one of each kind); show how to weigh 1135 lbs.

17. If $\phi(n)$ represent the remainder after dividing a number n by $r - 1$, show that

$$\phi\{\phi(p) \cdot \phi(q)\} = \phi(pq).$$

18. If the digits of a number n be added together, the digits of the result added together, and so on, until the sum is represented by one digit; and we represent the final result by $F(n)$, show that

$$F(n) \cdot F(p) \cdot F(q) \ldots \text{ and } F(npq \ldots)$$

are either equal or differ by a multiple of $(r - 1)$.

19. If a number whose digits are p_0, p_1, p_2, is a perfect square in any scale, find the relation between p_0, p_1, p_2.

20. Show that if the sum of two numbers is a multiple of the radix, the sum of the last digits of their cubes is equal to the radix.

21. Every number having its digits repeated in threes, and being in any scale of the series—*four*, *seven*, *ten*, *thirteen*, &c.—is divisible by 3, and the quotient contains a factor, the sum of whose digits is equal to the radix.

22. The sum of the numbers (scale s) represented by the permutations of any three digits of the scale amongst each other, is always divisible by the number (s 1) of the scale whose radix is $(s + 1)$.

23. Every number less than 2^{n+1} can be expressed by the sum of certain terms of the series

$$1, 2, 2^2, \ldots, 2^n.$$

24. Every number less than 3^{n+1} can be expressed by terms of the series

$$1, 3, 3^2, \ldots\ldots, 3^n,$$

the coefficients of the terms being either $\pm$ 1.

CHAPTER XIII.

Properties of Numbers.

107. By the term **number** in the present chapter we mean a positive integer.

Owing to the extensive nature of the Theory of Numbers we shall be able to give in this work only the more elementary propositions.

A **Prime** number is one which has no divisor except itself and unity..

Two numbers are said to be **prime to each other** when they contain no other common divisor than *unity*.

108. *If a number* a *be prime to each of the numbers* b *and* c, *it is prime to their product* bc.

For suppose a, b, c to be each broken up into their elementary factors.

Then a contains no factor except unity, which b contains; and neither does it contain any factor except unity, which c contains.

Now, bc when broken up into elementary prime factors cannot contain any other such factors than what can be found in b and c.

Therefore a contains no factor common to bc other than unity.

Hence a is prime to bc.

Cor. 1. Hence if a be prime to any number of quantities b, c, d, &c., it is prime to their product bcd, &c.; and if a *be prime to any* quantity b, it is prime to each of the factors *of* b.

Cor. 2. If a be prime to b, it is also prime to $bbb\ldots$, that is, *to any power* of b.

Cor. 3. If a be a divisor of bc, and be prime to b, it must be a divisor of c.

Cor. 4. If $\frac{a}{b}$ is a fraction in its lowest terms, then $\frac{a^n}{b^n}$ is in its lowest terms.

109. *If* $\frac{\mathrm{a}}{\mathrm{b}}$ *be a fraction in its lowest terms, and* $\frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{d}}$; *then will* c *and* d *be equimultiples of* a *and* b.

For $\frac{a}{b}$ is the fraction $\frac{c}{d}$ reduced to its *lowest terms.*

Now in order to express $\frac{c}{d}$ in its lowest terms we have to divide numerator and denominator by the greatest common measure x, suppose.

We must therefore have $\frac{c}{x} = a$, or $c = ax$;

$$\text{and } \frac{d}{x} = b, \text{ or } d = bx.$$

110. *If* N *be an integer, and* $\sqrt[n]{\mathrm{N}}$ *a surd, we cannot express the latter by a fraction.*

For suppose $\sqrt[n]{\mathrm{N}} = \frac{a}{b}$, we then have $\mathrm{N} = \frac{a^n}{b^n}$, ... (1).

Now if $\frac{a}{b}$ be not in its lowest terms, we may so reduce it.

Then we shall have $\frac{a^n}{b^n}$ in its lowest terms, and also a fraction.

Hence in (1) we have, *an integer = a fraction*, which is absurd.

111. *Any number is prime if it cannot be divided by some number equal to, or less than, its square root.*

For every number N which is not a *prime* is composed of two factors, as a and b, so that we have

$$\mathrm{N} = ab.$$

Now if $a = b$, we have $a = b = \sqrt{N}$, so that N is divisible by $\sqrt{N}$.

Again, if $a > b$, we have $b < \sqrt{N}$;

And, if $a < b$, we have $a < \sqrt{N}$.

Hence, for every number not prime, there always exists a divisor, either equal to, or less than, its square root; and therefore, every number not having such a divisor is prime.

Ex. To determine whether 101 is a prime number.

We have $\sqrt{101} < 11$.

Now, as 101 is odd, we need only try the prime divisors 3, 5, 7; and we find it is not divisible by either of them.

Hence 101 is a prime number.

112. *No rational algebraical formula can contain prime numbers only.*

For, if possible, suppose that the expression

$$p + qx + rx^2 + sx^3 + \&c. \ldots .$$

expresses prime numbers only.

Suppose that when $x = m$, the expression equals P; and when $x = m + n\mathrm{P}$, the value of the expression is P_1.

Then we have

$$\mathrm{P} = p + qm + rm^2 + sm^3 + \&c. \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\begin{aligned}\text{and } \mathrm{P}_1 &= p + q(m + n\mathrm{P}) + r(m + n\mathrm{P})^2 + s(m + n\mathrm{P})^3 + \&c. \\ &= (p + qm + rm^2 + sm^3 + \&c.) \\ &\quad + (qn + 2rmn + 3sm^2n + \&c.)\,\mathrm{P} \\ &\quad + (rn^2 + 3smn^2 + \&c.)\,\mathrm{P}^2 + \&c. \\ &= \mathrm{P} + \text{terms containing P as a factor.}\end{aligned}$$

Hence P_1 is divisible by P, and therefore not prime.

And hence the algebraical expression

$$p + qx + rx^2 + sx^3 + \&c.$$

cannot represent prime numbers only.

Remark. *There* are several remarkable algebraical formulæ *which represent* a large number of primes,

Thus $x^2 + x + 41$, by making $x = 0, 1, 2, 3,$ &c. 39, will give a series of forty numbers, all primes.

$x^2 + x + 17$ gives similarly seventeen primes.

$2x^2 + 29$ gives twenty-nine primes.

$2^n + 1$ is prime for $n = 1, 2, 4, 8,$ or 16.

113. *The number of primes is infinite.*

For suppose p to be the greatest prime number. Then the product of all the prime numbers up to p is

$$2 . 3 . 5 . 7 . . . . p.$$

And this product is divisible by each of the prime numbers. Hence

$$(2 . 3 . 5 . 7 . . . . p) + 1$$

is divisible by none of the prime numbers $2, 3, 5, 7, ... p$.

Now, if this quantity is a prime, it is a greater prime than p; and if it be a composite number, it must have a prime factor greater than p, since none of the prime numbers 2, 3, 5, 7, ... p will divide it.

Hence, in any case, there exists a prime number greater than any supposed prime p.

114. *If* a *be prime to* b, *and each of the terms of the series*

$$b, 2b, 3b, ... (a - 1)b$$

be divided by a, *all the remainders will be different.*

For suppose any two terms of this series, mb and m_1b, to have the same remainder, where m and m_1 are of course each less than a.

Then their difference $(mb - m_1b)$ will be divisible by a.

But since $mb - m_1b = (m - m_1)\,b$, it is composed of two factors, one of which is prime to a, and the other of which is less than a. It therefore cannot be divisible by a.

Hence no terms of the given series can leave the same remainder when divided by a.

Cor. 1. Since all these remainders are less than a, and they are all different, and since there are $(a - 1)$ terms in the given series, it follows that the series

$$1, 2, 3, \ldots . a - 1$$

represents the remainders, not regarding their order.

Cor. 2. If a be prime to b, the terms of the A. P.

$$b + c, 2b + c, 3b + c, \ldots . (a - 1)b + c,$$

when divided by a, will give for remainders the series

$$1, 2, 3, \ldots . (a - 1),$$

no regard being had to their order.

Fermat's Theorem.

115. *If* n *be a prime number, and* N *be a number prime to* n, *then* $N^{n-1} - 1$ *is divisible by* n.

We shall give two demonstrations of this important theorem.

First proof.

By Art. 114, Cor. 1, the terms of the series

$$N, 2N, 3N, \ldots , (n - 1)N,$$

when divided by n, give for remainders (disregarding their order) the series

$$1, 2, 3, \ldots , (n - 1).$$

Hence, assuming $q_1, q_2, q_3, \ldots , q_{n-1}$ as the quotients corresponding to these remainders, we must have

$$N \,.\, 2N \,.\, 3N \ldots . (n - 1)N$$

$$= (1 + q_1 n)(2 + q_2 n)(3 + q_3 n) \ldots . (\overline{n-1} + q_{n-1} n)\text{ ; or,}$$

$$N^{n-1} \,.\, 1 \,.\, 2 \,.\, 3 \ldots . (n-1) = 1 \,.\, 2 \,.\, 3 \ldots . (n-1) + \text{terms containing } n \text{ as a factor}$$

$$= 1 \,.\, 2 \,.\, 3 \ldots . (n - 1) + Mn, \text{ suppose.}$$

$$\therefore (N^{n-1} - 1) \,.\, 1 \,.\, 2 \,.\, 3 \ldots . (n - 1) = Mn.$$

Now n, being a prime number, is prime to each of the factors 1, 2, 3, ... , $(n-1)$. And hence

$$\frac{\mathrm{M}}{1 \cdot 2 \cdot 3 \ldots . (n-1)} \text{ is an integer,}$$

since $\mathrm{N}^{n-1} - 1$ and n are integers.

Hence we have $\mathrm{N}^{n-1} - 1 = \mathrm{P}n$, when P is integral.

$\therefore$ $\mathrm{N}^{n-1} - 1$ is divisible by n.

Second proof.

We have $\mathrm{N}^n = \{(\mathrm{N}-1) + 1\}^n =$

$$(\mathrm{N}-1)^n + n(\mathrm{N}-1)^{n-1} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \ldots r}(\mathrm{N}-1)^{n-r}$$
$$+ \ldots . + n(\mathrm{N}-1) + 1.$$

Now every term of the second side of this identity, excepting the first and last, is divisible by n.

For the general coefficient $\frac{n(n-1)\ldots.(n-r+1)}{1 \cdot 2 \ldots r}$ is an integer, and is so divisible, since r is $< n$, and n is prime.

Hence we have

$$\mathrm{N}^n = (\mathrm{N}-1)^n + 1 + \mathrm{P}_0 n, \text{ where } \mathrm{P}_0 \text{ is an integer.}$$
$$\therefore \quad \mathrm{N}^n - \mathrm{N} = (\mathrm{N}-1)^n - (\mathrm{N}-1) + \mathrm{P}_0 n;$$

or, putting successively, $\mathrm{N}-1$, $\mathrm{N}-2$, ... , 2, for N, we have

$$(\mathrm{N}-1)^n - (\mathrm{N}-1) = (\mathrm{N}-2)^n - (\mathrm{N}-2) + \mathrm{P}_1 n,$$
$$(\mathrm{N}-2)^n - (\mathrm{N}-2) = (\mathrm{N}-3)^n - (\mathrm{N}-3) + \mathrm{P}_2 n,$$
$$\&c. \quad = \quad \&c.$$
$$2^n - 2 \quad = \quad 1^n - 1 + \mathrm{P}_{n-2}$$

where P_1, P_2, &c., are integers.

Hence adding these equations, we have—

$$\mathrm{N}^n - \mathrm{N} = 1^n - 1 + (\mathrm{P}_0 + \mathrm{P}_1 + \mathrm{P}_2 + \ldots . \mathrm{P}_{n-2})n$$
$$= \mathrm{P}n, \text{ suppose, where P is an integer.}$$

Or, $\mathrm{N}\,(\mathrm{N}^{n-1} - 1) = \mathrm{P}n$.

Now n is prime to N, and therefore $\frac{\mathrm{P}}{\mathrm{N}}$ is integral.

Hence $N^{n-1} - 1$ is divisible by n.

Extension of Fermat's Theorem.

116. *If* n *be prime to* N, *and* ϕ(n) *represent how many numbers there are less than* n, *and prime to it, then*

$$N^{\phi(n)} - 1 \text{ is divisible by } n.$$

Let $a, b, c, d, \ldots . m, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$

be the $\varphi(n)$ numbers which are less than n and prime to it. Then if the series be multiplied by N we have the second series

$$aN, bN, cN, \ldots . , mN, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now it may be shown, as in Art. 114, that if the terms of the series be severally divided by n, we shall have for remainders, neglecting their order, the series (1).

Hence, as in Art. 115, assuming q_a, q_b, q_c, &c., q_n as the quotients corresponding to these remainders, we must have

$$aN \,.\, bN \,.\, cN \ldots . \, mN$$
$$= (a + q_1 n)\ (b + q_2 n)\ (c + q_3 n) \ldots . \ (m + q_m n)\ ;\ \text{or,}$$
$$abc \ldots . \, m \,.\, N^{\phi(n)}$$
$$= abc \ldots . \, m + \text{terms containing } n \text{ as a factor}$$
$$= abc \ldots . \, m + Mn, \text{ suppose.}$$
$$\therefore \ \left\{N^{\phi(n)} - 1\right\} abc \ldots . \, m = Mn.$$

Now each of the quantities $a, b, c, \ldots . , m$ is prime to n and less than n. Hence n is prime to the product $abc \ldots . \, m$.

Therefore $N^{\phi(n)} - 1$ is divisible by n.

Cor. Fermat's theorem follows at once from this.

For suppose n a prime number, then the numbers less than n and prime to it are

$$1, 2, 3, \ldots . , (n-1).$$

Hence, here $\phi\ (n) = n - 1$.

Hence, if n be a prime number, and n be prime to N,

$$N^{n-1} - 1 \text{ is divisible by } n.$$

Wilson's Theorem.

117. *If* n *be a prime number, then*

$$\lfloor \underline{n-1} + 1$$

is divisible by n.

Let a represent one of the numbers of the series

$$1, 2, 3, \ldots, (n-1) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

a is therefore prime to n.

Multiplying each term of this series by a, we have the second series

$$a, 2a, 3a, \ldots (n-1)a \ldots\ldots\ldots\ldots\ldots (2).$$

Now (Art. 114, Cor. 1) if the terms of this series be divided by n, we have for remainders, not regarding their order, the series in (1).

Hence some one of the numbers in series (2) give *unity* for a remainder when divided by n.

If we represent this number by ma, we have

$$ma - 1 \text{ divisible by } n = b_0 n, \text{ suppose} \ldots\ldots\ldots\ldots (3).$$

Now m cannot be equal to a, except

$$a = 1, \text{ or } a = n - 1 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

For suppose we have $m = a$, we then have from (3) $a^2 - 1 = b_0 n$ or $(a+1)(a-1) = b_0 n$.

Now n is a prime number;

Hence either $a + 1$ or $a - 1$ is divisible by n.

But a is less than n, and hence when n divides $a + 1$, we must have $a = n - 1$.

And so, when n divides $a - 1$, we must have $a = 1$.

Hence, leaving out the numbers 1 and $n - 1$, we may say that it is always possible that *any one* of the numbers

$$2, 3, 4, \ldots, (n-2),$$

may be multiplied by *some other* number of the same series, *so as to form a product* of the form $b_0 n + 1$.

Suppose then we multiply these numbers so as to form such products as $b_1n + 1$, $b_2n + 1$, &c. By multiplying together these products we have

$$2 \,.\, 3 \,.\, 4 \,.\,.\,.\,. (n - 2) = (b_1n + 1)\,(b_2n + 1).....$$
$$= \mathrm{M}n + 1,$$

where $\mathrm{M}n$ represents the terms having n as a factor.

Hence, multiplying each side by $n - 1$, we have

$$1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. (n - 1) = (\mathrm{M}n + 1)\,(n - 1)$$
$$= \mathrm{M}(n - 1)\,n + n - 1\,; \text{ or}$$
$$1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. (n - 1) + 1 = \{\mathrm{M}\,(n - 1) + 1\}n.$$

Hence $\lfloor \underline{n - 1} + 1$ is divisible by n.

Cor. Hence $\left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n - 1}{2}\right)^2 \pm 1$ *is divisible by* n, *when* n *is prime, and* > 2, *the upper or lower sign being taken as* n *is of the form* $4\,\mathrm{m} + 1$ *or* $4\,\mathrm{m} - 1$.

Since $\frac{n - 1}{2}$ is integral, we have

$$1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. (n - 1)$$

$$= \left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n-1}{2}\right) . \left(\frac{n+1}{2} . \frac{n+3}{6} .\,.\,.\,. \overline{n-3} . \overline{n-2} . \overline{n-1}\right)$$

$$= 1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n - 1}{2} . \left\{\left(n - \frac{n - 1}{2}\right) .\,.\,.\,. \overline{n-3} . n - \overline{2} . \overline{n-1}\right\}$$

$$= 1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n - 1}{2} . \left\{\mathrm{M}n + (-1)^{\frac{n-1}{2}}\, 1 \,.\, 2 \,.\, 3 \,.\,.\,. \frac{n - 1}{2}\right\}$$

where $\mathrm{M}n$ represents terms containing n as a factor;

$$= \left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n-1}{2}\right) \mathrm{M}n + (-1)^{\frac{n-1}{2}} \left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n-1}{2}\right)^2;$$

or, adding *unity* to each side, $\lfloor \underline{n - 1} + 1$

$$= \left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n-1}{2}\right) \mathrm{M}n + (-1)^{\frac{n-1}{2}} . \left(1 \,.\, 2 \,.\, 3 \,.\,.\,.\,. \frac{n-1}{2}\right)^2 + 1.$$

Now we have just shown that $\lfloor \underline{n - 1} + 1$ is divisible *by* n.

Hence $(-1)^{\frac{n-1}{2}}\left(1 . 2 . 3 \ldots . \frac{n-1}{2}\right)^2 + 1$ is divisible by n.

Now $(-1)^{\frac{n-1}{2}} = \pm 1$, according as n is of the form of $4n + 1$ or $4n - 1$.

Hence the truth of the proposition.

118. *If* a, b, c, ... , l *are a series of numbers prime to each other, and* ϕ(n) *represents how many numbers there are which are prime to* n *and less than it, then*

$$\phi(abc \ldots . l) = \phi(a) . \phi(b) . \phi(c) \ldots . \phi(l).$$

First, suppose $N = ab$.

We may arrange the numbers from 1 to ab in b columns thus :

$$\begin{array}{rrrr}
1, & 2, & 3, \ldots & k, \ldots a, \\
a+1, & a+2, & a+3, \ldots & a+k, \ldots 2a, \\
2a+1, & 2a+2, & 2a+3, \ldots & 2a+k, \ldots 3a, \\
\vdots & \vdots & \vdots & \vdots \quad \vdots \\
(b-1)a+1, & (b-1)a+2, & (b-1)a+3 \ldots . & (b-1)a+k, \ldots ab.
\end{array}$$

If we consider any column as the one commencing with k, we see that all the numbers in it are prime to a, or all the numbers contain some factor common with a, according as k is prime to a or not.

Hence the number of columns which contain all the numbers of the column prime to a is equal to the number of quantities in the series 1, 2, 3, . . . k, . . . a, which are prime to a. And no other column contains any number prime to a.

Hence $\phi(a)$ is the number of columns containing numbers prime to a out of all the numbers from 1 to ab.....(1).

Again, by Art. 114, Cor. 1, if the numbers in any column, as that commencing with k, be divided by b, the remainders, when k is prime to b, will form the series

$$1, 2, 3, \ldots . , (b-1).$$

Hence in every column there are as many numbers prime to b as there are of such numbers in the series $1, 2, 3, \ldots . (b-1)$.

Therefore $\phi(b)$ is the number of quantities in each column which are prime to b.....(2).

Hence from (1) and (2) we conclude that—

The number of quantities in all the columns which are prime to a and prime to b, and therefore prime to ab, is $\phi(a) \cdot \phi(b)$.

$$\text{Therefore } \phi(ab) = \phi(a) \cdot \phi(b).$$

If $N = abc \ldots . l$, we have

$$\begin{aligned} \phi(abc \ldots . l) &= \phi(a) \cdot \phi(bc \ldots . l) \\ &= \phi(a) \cdot \phi(b) \cdot \phi(c \ldots l) = \&c. \\ &= \phi(a) \cdot \phi(b) \cdot \phi(c) \ldots . \phi(l). \quad Q.E.D. \end{aligned}$$

NOTE. It will be seen that we consider here *unity* as prime to a, b, c, &c.

119. *To find the number of positive integers which are less than* N *and prime to it.*

Suppose $N = a^p b^q c^r \ldots . l^s$, where $a, b, c, \ldots , l$ are prime factors.

Now the number of positive integers which are less than a^p and prime to it is found by subtracting from a^p the number of quantities in the series

$$a, 2a, 3a, \ldots , a^{p-1}a \text{ or } a^p.$$

The number of these is evidently a^{p-1}.

Hence we must have $\phi(a^p) = a^p - a^{p-1}$,

$$\text{or } \phi(a^p) = a^p\left(1 - \frac{1}{a}\right); \text{ and so}$$

$$\phi(b^q) = b^q\left(1 - \frac{1}{b}\right),$$

$$\phi(c^r) = c^r\left(1 - \frac{1}{c}\right),$$

$$\&c., = \quad \&c.$$

$$\phi(l^s) = l^s\left(1 - \frac{1}{l}\right).$$

Hence, multiplying, we have

$$\phi(a^p) \cdot \phi(b^q) \cdot \phi(c^r) \ldots \ldots \phi(l^s)$$

$$= a^p b^q c^r \ldots \ldots l^s \cdot \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots \left(1 - \frac{1}{l}\right).$$

$$= \mathrm{N} \cdot \left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots \left(1 - \frac{1}{l}\right).$$

But by last Art.,

$$\phi(\mathrm{N}) = \phi a^p) \cdot \phi(b^q) \cdot \phi(c^r) \ldots \ldots \phi(l^s).$$

Hence $\phi(\mathrm{N}) = \mathrm{N}\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) \ldots \left(1 - \frac{1}{l}\right)$.

120. *To find the number of divisors in a given integer.*

Suppose $\mathrm{N} = a^p b^q c^r \ldots \ldots$, where $a, b, c \ldots \ldots$ are primes.

Now N is divisible by every power of a not greater than a^p.

Hence, including *unity* as one of its divisors, N is divisible by every term of the series

$$1, a, a^2, \ldots, a^p.$$

And so it is divisible by every term of each of the series

$$1, b, b^2, \ldots, b^q,$$

$$1, c, c^2, \ldots, c^r,$$

&c.

N is therefore divisible by every combination of the terms of these series.

Hence N is divisible by every term of the product

$$(1 + a + a^2 + \ldots \ldots a^p)(1 + b + b^2 + \ldots \ldots + b^q)$$

$$(1 + c + c^2 + \ldots \ldots + c^r) \ldots \ldots,$$

and by no other quantity.

Now the *number of terms* of this product is found by putting $a = b = c = \&c. = 1$.

Therefore, number of divisors of N, including itself and unity

$$= (p + 1)(q + 1)(r + 1) \ldots \ldots$$

Cor. 1. Since every divisor has a corresponding divisor for a quotient, it follows that the number of ways in which N may be broken into two factors is half the number of divisors.

(i.) When N is *not a perfect square,*

One at least of the indices is odd, and therefore

$$(p + 1)(q + 1)(r + 1) \ldots .$$

is even.

Hence, number of ways in which N may be broken up into two factors

$$= \tfrac{1}{2}(p + 1)(q + 1)(r + 1) \ldots .$$

(ii.) When N *is a perfect square,*

All the indices, $p, q, r, \ldots .$ are even, and therefore

$$(p + 1)(q + 1)(r + 1) \ldots .$$

is odd.

But $\sqrt{N}$ is a divisor, and hence, since $N = \sqrt{N} \cdot \sqrt{N}$, there is *one way* of resolving N into two factors, by means of *one divisor.*

Hence, number of ways in which N may be resolved into two factors

$$= \tfrac{1}{2}\{(p + 1)(q + 1)(r +) \ldots . + 1\}.$$

It must be remembered that (N, 1) is one of the pairs of factors.

Cor. 2. *The sum of the divisors of any number* N, including among them itself and unity,

$$= \text{the product } (1 + a + a^2 + \ldots . + a^p)(1 + b + b^2 + \ldots . + b^q)$$
$$(1 + c + c^2 + \ldots . + c^r) \ldots .$$
$$= \frac{a^{p+1} - 1}{a - 1} \cdot \frac{b^{q+1} - 1}{b - 1} \cdot \frac{c^{r+1} - 1}{c - 1} \ldots .$$

Cor. 3. If n be the number of prime factors, a, b, c, &c. (not including N or unity), in N,

The number of ways in which N may be resolved into *two factors prime to each other* $= 2^{n-1}$.

For we must have a^p in one factor of any pair; and so we must have b^q, c^r, &c.; and we cannot have any lower powers of a, b, c, &c., in any of the factors which are prime to each other.

Hence the *number of ways* in which we may resolve N into two factors prime to each other is the same as that of the number $abc \ldots .$;

We have therefore, by Cor. 1, making there $p = q = r = \&c. = 1$,

Number of ways in which N may be resolved into two factors prime to each other,

$$= \tfrac{1}{2}(1 + 1)(1 + 1)(1 + 1) \ldots . \text{ to } n \text{ factors} = 2^{n-1}.$$

Ex. XVII.

1. If any number divide each of two other numbers, it will divide their sum and difference.

2. If two numbers be prime to each other, then will their sum and difference be prime to each of them.

3. If a square number be multiplied by a square, the product will be a square.

4. All numbers belong to one of the forms $3n$, $3n \pm 1$.

5. Every square number is of the form $5n$ or $5n \pm 1$.

6. Every cube number is of the form $7n$ or $7n \pm 1$.

7. A number which is a perfect square will divide by 5 without a remainder, or will leave 1 or 4 for a remainder.

8. Show that $n(n^2 + 5)$ is divisible by 6, when n is an integer.

9. Show that $n(n^2 - 1)(n^2 - 4)$ is divisible by 120, when n is an integer.

10. All even square numbers are of the form $16n$ or $4(4n + 1)$, and all odd squares of the form $8n + 1$.

11. Every number of the form $4n + 1$ is the sum of two *squares.*

12. If the difference between a and b is divisible by p, a is said to be congruous to b, for the modulus p. Hence show that if $a \equiv b \pmod{p}$, and $c \equiv d \pmod{p}$, then $a + c \equiv b + d \pmod{p}$.

13. If $a \equiv b \pmod{p}$, then $a^n \equiv b^n \pmod{p}$, and $ma \equiv mb \pmod{p}$.

14. If $ma \equiv mb \pmod{p}$, what is the condition that $a \equiv b \pmod{p}$?

15. If the congruence $\phi(x) \equiv 0 \pmod{p}$ is satisfied by $x = a$, show that every number of the form $a + mp$ is also a solution, and that among these numbers there always exists one, and only one, between the limits 0 and $p - 1$ inclusive.

16. If $ax + b \equiv 0 \pmod{p}$, how many solutions are possible in the three cases: (i.) when a is prime to p; (ii.) when a and p have a common measure, which will not divide b; (iii.) when a, b, p have a common measure?

17. If p be a prime number, show that

$$(x + 1)(x + 2)(x + 3) \ldots (x + p - 1) \equiv x^{p-1} - 1 \pmod{p},$$

and hence deduce Wilson's theorem.

18. If $2^n - 1$ be a prime number, then will $2^{n-1}(2^n - 1)$ be a perfect number—that is, a number which is equal to the sum of all its aliquot parts, or its divisors.

19. If $3 \cdot 2^n - 1$, $6 \cdot 2^n - 1$, $18 \cdot 2^{2n} - 1$ are prime numbers, show that

$2^{n+1}(18 \cdot 2^{2n} - 1)$, and $2^{n+1}(3 \cdot 2^n - 1)(6 \cdot 2^n - 1)$,

are amicable numbers—that is, numbers each of which is equal to the sum of the divisors of the other.

20. Any power of the sum of two squares may be separated into the sum of two squares.

21. If a, b, and n are positive integers, and b less than $2a - 1$, show that the integral part of $(a + \sqrt{a^2 - b})^n$ is an odd number.

22. *If n be a prime number, and m be odd, show that $m^n + 4$ cannot be a perfect square.*

23. If ϕ (N) is the number of integers not greater than N and prime to it, and 1, d_1, d_2, &c., are all the divisors of N, then

$$\phi(1) + \phi(d_1) + \phi(d_2) + \ldots = \mathrm{N}.$$

24. If p be the G. C. M. of m and n, then $x^p - 1$ is the G. C. M. of $x^m - 1$ and $x^n - 1$.

25. If p be a prime number, and a one of the numbers $1, 2, 3, \ldots \overline{p-1}$, then $a + kp$ contains square numbers or not according as $a^{\frac{p-1}{2}} - 1$, or $a^{\frac{p-1}{2}} + 1$, is divisible by p.

26. With the notation of Ex. 23, show that, when N is an odd number, $\varphi(2\,\mathrm{N}) = \varphi(\mathrm{N})$.

27. Find the sum of the positive integers which are less than a given number (N) and prime to it.

28. Show that the sum of the squares of these integers

$$= \frac{\mathrm{N}^3}{3}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots + \frac{\mathrm{N}}{6}(1-a)(1-b)(1-c)\ldots;$$

and that the sum of their cubes

$$= \frac{\mathrm{N}^2}{4}\left\{\mathrm{N}^2\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots + (1-a)(1-b)(1-c)\ldots\right\},$$

where a, b, c, ... , are the prime factors of N.

29. If p be a prime number, then the congruence

$$a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m \equiv 0 \pmod{p}$$

cannot admit of more than m solutions less than p.

30. The congruence $x^p \equiv 1 \pmod{p}$ has always $\varphi(p-1)$ roots, where $\varphi(n)$ represents the number of integers less than n and prime to it, and where p is a prime number.

CHAPTER XIV.

THEORY OF INDICES. IMAGINARY QUANTITIES.

Indices.

121. In the first volume of this work we have not altogether omitted this subject; we shall here complete the theory as far as space will permit.

I. *Positive integral indices.*

It is convenient to conceive of a^m, where m is a positive integer, as the result of *multiplying* unity by m factors each equal to a.

Thus $a^m = 1\,(a \,.\, a \,.\, a \,.\,.\,.\,.$ to m factors).

II. *Negative integral indices.*

Here, remembering that *negative* is the *reverse* of *positive*, we shall be quite consistent if we allow a^{-m} to represent the result of *dividing* unity by a, and the quotients successively by a for m operations.

Thus, $a^{-m} = 1 \div (a \,.\, a \,.\, a \,.\,.\,.\,.$ to m factors $= \frac{1}{a^m}$.

We get from this $a^m \times a^{-m} = 1$.

Now, by *assuming* that the formula $a^m \times a^n = a^{m+n}$, which we know holds for positive integral indices, to also hold when

$n = -m$, we have $a^m \times a^{-m} = a^{m-m} = a^0$;

and we conclude that $a^0 = 1$.

We have no need, however, to *assume* the principle at all.

For, referring to our definitions of a^m and a^{-m}, it at once follows, since a^m represents the result of *multiplying* unity by a, and the successive products by a for m operations, and that a^{-m} represents the result of *dividing* unity by a, and the successive quotients by a for m operations, that

a^0 represents unity *neither multiplied nor divided at all by* a;

that is, a^0 is consistently interpreted to mean unity.

Hence, whether m be a positive or negative integer, or zero, a^m admits of the same interpretation, provided we remember the algebraical meaning of *positive* and *negative*.

Again, the law that $a^m \times a^n = a^{m+n}$ holds good if either m or n, or both, be negative integers.

For, suppose $n = -p$, when p is a positive integer, we have

$$a^m \times a^n = a^m \times a^{-p} = a^m \times \frac{1}{a^p} = \frac{a^m}{a^p}.$$

If $m > p$, then $\frac{a^m}{a^p} = a^{m-p} = a^{m+n}$.

If $m < p$, then $\frac{a^m}{a^p} = \frac{1}{a^{p-m}} = a^{-(p-n)} = a^{m-p} = a^{m+n}$.

Hence $a^m \times a^n = a^{m+n}$, when n is negative.

And it may be similarly shown when both are negative.

Fractional Indices.

122. If m and n are any integers, it follows at once from the above that

$$(a^m)^n = a^{mn} = a^{m \times n}.$$

Hence, a^{mn} or $a^{m \times n}$ is the correct symbol for the nth *power* of a^m. And therefore $a^{m \div n}$ or $a^{\frac{m}{n}}$ will consistently represent the nth *root* of a^m. And this is true whether m and n be either one or both positive or negative.

(i.) We shall now show that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$, where p, q, r, s are integers, and thus show that the law $a^m \times a^n = a^{m+n}$ holds when m and n are fractional.

We have $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \sqrt[qs]{(a^p)^s} = \sqrt[qs]{a^{ps}}$ .....(1).

and $a^{\frac{r}{s}} = \sqrt[s]{a^r} = \sqrt[qs]{(a^r)^q} = \sqrt[qs]{a^{qr}}$.....(2).

$$\therefore \quad p^{\frac{p}{q}} \times a^{\frac{r}{s}} = \sqrt[qs]{a^{ps} \times a^{qr}} = \sqrt[qs]{a^{ps+qr}}$$

$$= a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

(ii.) We shall now show that $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}$, where p, q, r, s are integers, and thus that the law $(a^m)^n = a^{mn}$ holds when m and n are fractional.

We have $$\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = \sqrt[s]{\left(a^{\frac{p}{q}}\right)^r},$$

or by what has just been proved,

$$= \sqrt[s]{a^{\frac{pr}{q}}}$$

$$= \sqrt[s]{\sqrt[q]{a^{pr}}} = \sqrt[qs]{a^{pr}} = a^{\frac{pr}{qs}}.$$

Hence it has been universally established that the laws

(i.) $a^m a^n = a^{m+n}$, and

(ii.) $(a^m)^n = a^{mn}$,

are universally true, whether m, n be integral or fractional, positive or negative.

Imaginary Quantities.

123. An expression of the form $a + b\sqrt{-1}$ is called an **imaginary quantity**, a and b being real quantities. If we allow that $b\sqrt{-1}$ vanishes when $b = 0$, then the whole expression vanishes when $a = 0$, and $b = 0$.

Conversely, when $a + b\sqrt{-1} = 0$, *we must have* a = 0, b = 0.

For, transposing, $a = -b\sqrt{-1}$; or, squaring,

$$a^2 = b^2(-1) = -b^2; \text{ or,}$$

$$a^2 + b^2 = 0.$$

And the only values of a and b which satisfy this equality are

$$a = 0, b = 0.$$

124. *If* a + b $\sqrt{-1}$ = c + d $\sqrt{-1}$, *then* a = c, *and* b = d.

For, transposing,

$$(a - c) + (b - d)\sqrt{-1} = 0.$$

Hence, by the last Art., $a - c = 0$, or $a = c$.
and $b - d = 0$, or $b = d$.

125. DEF.—Two imaginary expressions are **conjugate** when they differ only in the sign of the coefficient of $\sqrt{-1}$.

Thus $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$ are conjugate expressions.

It follows, therefore, that

(i.) *The sum of two conjugate imaginary expressions is* REAL.

For $(a + b\sqrt{-1}) + (a - b\sqrt{-1}) = 2a$.

(ii.) *The difference of two conjugate imaginary expressions is* IMAGINARY.

The square root of the product of an imaginary expression by its conjugate is called its **modulus.**

Thus, since $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 + b^2$,
we call $\sqrt{a^2 + b^2}$ the modulus of $a + b\sqrt{-1}$, or of $a - b\sqrt{-1}$.

126. *If* M (p) *and* M (q) *represent respectively the moduli of* p *and* q, *then*

$$\mathrm{M}(p) \,.\, \mathrm{M}(q) = \mathrm{M}(pq).$$

Suppose $p = a + b\sqrt{-1}$,

and $q = c + d\sqrt{-1}$.

Then $\mathrm{M}(p) = \sqrt{a^2 + b^2}$, and $\mathrm{M}(q) = \sqrt{c^2 + d^2}$.

Therefore $\mathrm{M}(p) \,.\, \mathrm{M}(q) = \sqrt{(a^2 + b^2)(c^2 + d^2)}$..... (1).

But $pq = (a + b\sqrt{-1})(c + d\sqrt{-1})$

$= (ac - bd) + (ad + bc)\sqrt{-1}$.

$\therefore \mathrm{M}(pq) = \sqrt{(ac - bd)^2 + (ad + bc)^2}$

$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$..... (2).

Hence from (1),

$$\mathrm{M}(p) \,.\, \mathrm{M}(q) = \mathrm{M}(pq).$$

127. *The addition, subtraction, multiplication, or division of quantities of the form* $\mathrm{A} + \mathrm{B}\sqrt{-1}$ *gives a result of the same form.*

Let $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$ be the quantities.

(i.) *Addition* and *subtraction.*

$$(a + b\sqrt{-1}) \pm (c + d\sqrt{-1}) = (a \pm c) + (b \pm d)\sqrt{-1}.$$

(ii.) *Multiplication.*

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1}.$$

(iii.) *Division.*

$$\frac{a + b\sqrt{-1}}{c + d\sqrt{-1}} = \frac{(a + b\sqrt{-1})(c - d\sqrt{-1})}{(c + d\sqrt{-1})(c - d\sqrt{-1})}$$

$$= \frac{(ac + bd) + (bc - ad)\sqrt{-1}}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\sqrt{-1}.$$

128. *To find the powers of* $\sqrt{-1}$.

Every number must be of the form $4m$, $4m+1$, $4m+2$, or $4m + 3$.

Now $(\sqrt{-1})^{4m} = \{(\sqrt{-1}^2)\}^{2m} = (-1)^{2m} = 1.$

$(\sqrt{-1})^{4m+1} = (\sqrt{-1})^{4m} . (\sqrt{-1}) = 1 . \sqrt{-1} = \sqrt{-1}.$

$(\sqrt{-1})^{4m+2} = (\sqrt{-1})^{4m} . (\sqrt{-1})^2 = 1(-1) = -1.$

$(\sqrt{-1})^{4m+3} = (\sqrt{-1})^{4m} . (\sqrt{-1})^2 (\sqrt{-1})$

$= 1(-1)\sqrt{-1} = -\sqrt{-1}.$

129. *To find the cube roots of unity.*

These will evidently be the three values of x which satisfy the equation $x^3 = 1$, or $x^3 - 1 = 0$.

We have

$$(x - 1)(x^2 + x + 1) = 0.$$

Therefore $x - 1 = 0$, and $x^2 + x + 1 = 0$, will both furnish solutions.

From the former, $x = 1$;

And from the latter, $x = \frac{1}{2}(-1 \pm \sqrt{-3})$.

Hence the cube roots of unity are 1, $\frac{1}{2}(-1 \pm \sqrt{-3})$.
We may verify this result by *cubing* these quantities.
In each case the cube is *unity*.

COR. Similarly, the cube roots of -1 are

$$-1, \frac{1}{2}(1 \pm \sqrt{-3}).$$

And so, the fourth roots of 1 are $\pm 1, \pm \sqrt{-1}$;

and the fourth roots of -1 are $\pm \frac{1 \pm \sqrt{-1}}{\sqrt{2}}$.

Miscellaneous Examples.

1. Show that abc is $> (a + b - c)(a + c - b)(b + c - a)$, except when $a = b = c$.

2. If the number n be divided into any two parts, the difference of the squares of the parts is n times the difference of the parts.

3. Find to five places of decimals the value of x from the equation

$$\frac{2}{3 + x} + \frac{1}{5 - x} = 4.$$

4. Show that

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)$$
$$= (adf + bcf + bed - ace)^2 + (bce + aed + acf - bdf)^2.$$

5. Prove that

$$25\{(y - z)^7 + (z - x)^7 + (x - y)^7\}$$
$$\times \{(y - z)^3 + (z - x)^3 + (x - y)^3\}$$
$$= 21\{(y - z)^5 + (z - x)^5 + (x - y)^5\}^2.$$

6. Clear of surds the equations

$$\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 0, \text{ and}$$
$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{u} = 0.$$

7. Find x and y from the equations

$$x + \sqrt{\frac{x}{y}} = \frac{42}{y}, \text{ and}$$
$$2x^2y + 3x\sqrt{y} = 324.$$

8. Prove that

$$1 - n^2 + \frac{n^2(n^2 - 1^2)}{1^2 \cdot 2^3} + \frac{n^2(n^2 - 1^2)(n^2 - 2^2)}{1^3 \cdot 2^2 \cdot 3^3} + \ldots = 0,$$

when n is a positive integer.

9. Show that the sum of the squares of the coefficients of

$$(1 - x + x^2)^n$$

$$= \frac{2n}{(\lfloor n)^2}\left\{1 + \frac{n^2}{\lfloor 2} + \frac{n^2(n-1)^2}{\lfloor 4} + \ldots\right\}.$$

10. For what values of x is $x + \frac{3}{x} >$ or < 4; and find the least value of $\frac{(1 + x)(2 + x)}{3 + x}$.

11. If α, β be the roots of the quadratic $ax^2 + bx + c = 0$, show that the equation whose roots are $\frac{\alpha^4}{\beta}$ and $\frac{\beta^4}{\alpha}$ is

$$a^4cx^2 - b(b^4 - 5ab^2c + 5a^2c^2)x + ac^4 = 0.$$

12. What is the interpretation of the expression $x^{\sqrt{5}}$?

13. Show that unless $a = b = c = \ldots$, then

$$\frac{bcd\ldots.}{(b-a)(c-a)(d-a)\ \ldots} + \frac{acd\ldots.}{(a-b)(c-b)(d-b)\ldots.} + \&c. = 1.$$

14. Sum the series $1^p + 2^p + 3^p + \ldots + n^p$.

15. Show that

$$\tfrac{1}{5}\{(x - y)^5 + (y - z)^5 + (z - x)^5\}$$

$$= \tfrac{1}{6}\{(x-y)^3 + (y-z)^3 + (z-x)^3\}\{(x-y)^2 + (y-z)^2 + (z-x)^2\}.$$

16. Show that

$$\frac{a^{n+2}}{(a - b)(c - a)} + \frac{b^{n+2}}{(a - b)(b - c)} + \frac{c^{n+2}}{(b - c)(c - a)}$$

is always equal to the sum of the homogeneous products of *n dimensions of a, b, c.*

17. The integral part of $(1 + \sqrt{3})^{2m+1}$ is divisible by 2^{m+1}.

18. If $E^n = y + \sqrt{1 + y^2}$, then $y = \dfrac{E^n - E^{-n}}{2}$.

19. Resolve into factors the expression

$$1 + a_0 + (1 + a_0)\,a_1 + (1 + a_0)\,(1 + a_1)\,a_2 \ldots\ldots + \{(1 + a_0)\,(1 + a_1) \ldots\ldots (1 + a_{n-1})\,a_n\}.$$

20. Show that

$$\sqrt{\frac{b}{2} + \sqrt{ab - a^2}} + \sqrt{\frac{b}{2} - \sqrt{ab - a^2}} = \sqrt{2a}.$$

21. Show that $\left(\dfrac{-1 + \sqrt{-3}}{2}\right)^n + \left(\dfrac{-1 - \sqrt{-3}}{2}\right)^n = 2$, if n be a multiple of 3, and is equal to -1 if n be any other integer.

22. Show how to sum the series $a_m x^m + a_{m+n}x^{m+n} + a_{m+2n}x^{m+2n}$ + &c., when the sum of $a_0 + a_1x + a_2x^2 + \ldots$ is known.

23. If $(a + b\sqrt{-1})^{\frac{1}{n}} = c + d\sqrt{-1}$, then

$$(a - b\sqrt{-1})^{\frac{1}{n}} = c - d\sqrt{-1}.$$

24. If a_1, a_2, a_3, &c., a_n be unequal, and $m < n - 1$, then

$$\frac{a_1{}^m}{(a_1 - a_2)\,(a_1 - a_3) \ldots\ldots (a_1 - a_n)} + \frac{a_2{}^m}{(a_2 - a_1)\,(a_2 - a_3) \ldots\ldots (a_2 - a_n)} + \ldots\ldots + \frac{a_n{}^m}{(a_n - a_1)\,(a_n - a_2) \ldots\ldots (a_n - a_{n-1})} = 0.$$

25. If $\dfrac{ad - bc}{a - b - c + d} = \dfrac{ac - bd}{a - b - d + c}$, then each of them

$$= \tfrac{1}{4}(a + b + c + d).$$

26. Resolve into partial fractions

$$\frac{x^2 + mx + n}{(x + a)\,(x + b)\,(x + c)}.$$

27. Find the number of homogeneous products of n quantities a, b, c, &c., of r dimensions. Hence find the number of terms in the expansion of $(a + b + c + \&c.)^n$, n being integral.

28. Find a factor which will rationalize $a^{\frac{1}{2}} - b^{\frac{3}{5}}$, and extract the square root of $37 - 20\sqrt{3}$.

29. If $ax^3 = by^3 = cz^3$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$, show that $ax^2 + by^2 + cz^2 = (a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 d^2$.

30. If a debt a at compound interest is discharged in n years by annual payments of $\frac{a}{m}$, show that

$$(1 + r)^n (1 - mr) = 1.$$

31. If $x = h\sqrt{s^2 + \cdot 0075\, a^2}$, find the value of x when $h = 28{\cdot}4$ metres, $s = 3{\cdot}2$ metres, $a = 5{\cdot}2$ kilometres. Of what *dimension* is x?

32. Show that $\frac{p + q + r + s}{a + b + c + d}$ lies between the greatest and least of the fractions $\frac{p}{a}, \frac{q}{b}, \frac{r}{c}, \frac{s}{d}$, but differs from the average value of those quantities.

33. If $u_r = \frac{x \,.\, \overline{x+1} \,.\, \ldots \,.\, \overline{x+r-1}}{\lfloor r}$, show that

$$u_1 + u_2 + \ldots + u_n = \frac{\overline{x+1} \,.\, \overline{x+2} \,.\, \ldots \,.\, \overline{x+n}}{\lfloor n} - \underline{1}.$$

34. Show that the minimum value of

$$\frac{(a + x)(b + x)}{x} \text{ is } (\sqrt{a} + \sqrt{b})^2.$$

35. If $10^x = 2$, find x as a continued fraction.

36. Calculate T from the following formula:

$$T = \frac{2A}{mS\sqrt{2g}} \sqrt{H},$$

when $A = 3503{\cdot}6$ sq. ft., $S = 13{\cdot}532$ sq. ft., $m = {\cdot}548$, $g = 32{\cdot}1998$ ft., $H = 6{\cdot}3945$ ft.

37. Reduce to its simplest form

$$\frac{(yz - ax)^2 - (ac - y^2)(ab - z^2)}{(bc - x^2)(yz - ax) - (xz - by)(xy - cz)}$$

38. If p and q are each less than 1, show that

$$\frac{\log(1 - p)}{\log(1 - q)} \text{ is } > \frac{p - pq}{q}, \text{ and } < \frac{p}{q - pq}.$$

39. If $x = \dfrac{2ab + b^2}{a^2 + ab + b^2}$ and $y = \dfrac{a^2 - b^2}{a^2 + ab + b^2}$, show that $x^3 + y = y^3 + x$.

40. Prove that

$$\log_e x = \frac{1}{n} \cdot {}^{\infty}S_m \left(\frac{x^{mn} - 1}{m(x^n + 1)^m} \right),$$

where ${}^rS_m f(m)$ denotes the sum of r terms of the series whose mth term is $f(m)$.

41. Show that the integral part of $(5 + \sqrt{24})^n$ is odd if n be a positive integer.

42. If $\dfrac{A}{x} = \dfrac{B}{y} = \dfrac{C}{z}$, and $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, then

$$\frac{A^2}{a^2} + \frac{B^2}{b^2} + \frac{C^2}{c^2} = \frac{A^2 + B^2 + C^2}{x^2 + y^2 + z^2}.$$

43. Write down the coefficient of t^{2n} in the expression

$$1 - \tfrac{1}{2}(e^t - 1) + \tfrac{1}{3}(e^t - 1)^2 - \&c.$$
$$+ \frac{1}{2n + 1}(e^t - 1)^{2n} - \&c.$$

44. On a sum of money borrowed, interest is paid at the rate of 5 per cent. After a time £600 of the loan is paid off, and the interest on the remainder reduced to 4 per cent., and the yearly interest is now lessened one-third. What was the sum borrowed?

45. If $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \&c.$, find x in terms of y by the method of indeterminate coefficients.

46. Show how the solution of the equation $ax + by = c$ in *positive integers* may be obtained by the properties of

continued and converging fractions. Solve $17x + 19y = 250$, in positive integers.

47. In the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$, find the value of $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma +$ &c., where α, β, γ are the roots of the equation.

48. If
$$\frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^3}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^3}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{L},$$
$$\frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^4}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^4}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{M},$$
$$\frac{\alpha^5}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^5}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^5}{(\gamma-\alpha)(\gamma-\beta)} = \mathrm{N},$$
show that $\mathrm{LN} - \mathrm{M}^2 = \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 - \alpha\beta\gamma(\alpha + \beta + \gamma)$.

49. Sum to n terms, and to infinity
$$\frac{x}{(1+x)(1+ax)} + \frac{ax}{(1+ax)(1+a^2x)} + \frac{a^2x}{(1+a^2x)(1+a^3x)} + \&c.$$

50. Show that the result of eliminating x, y, z from the equations
$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$
is the same as the result of elimination from
$$a_1x + a_2y + a_3z = 0,$$
$$b_1x + b_2y + b_3z = 0,$$
$$c_1x + c_2y + c_3x = 0.$$

SECTION II.

PLANE TRIGONOMETRY.

CHAPTER I.

FUNCTIONS OF THE SUM AND DIFFERENCE OF ANGLES.

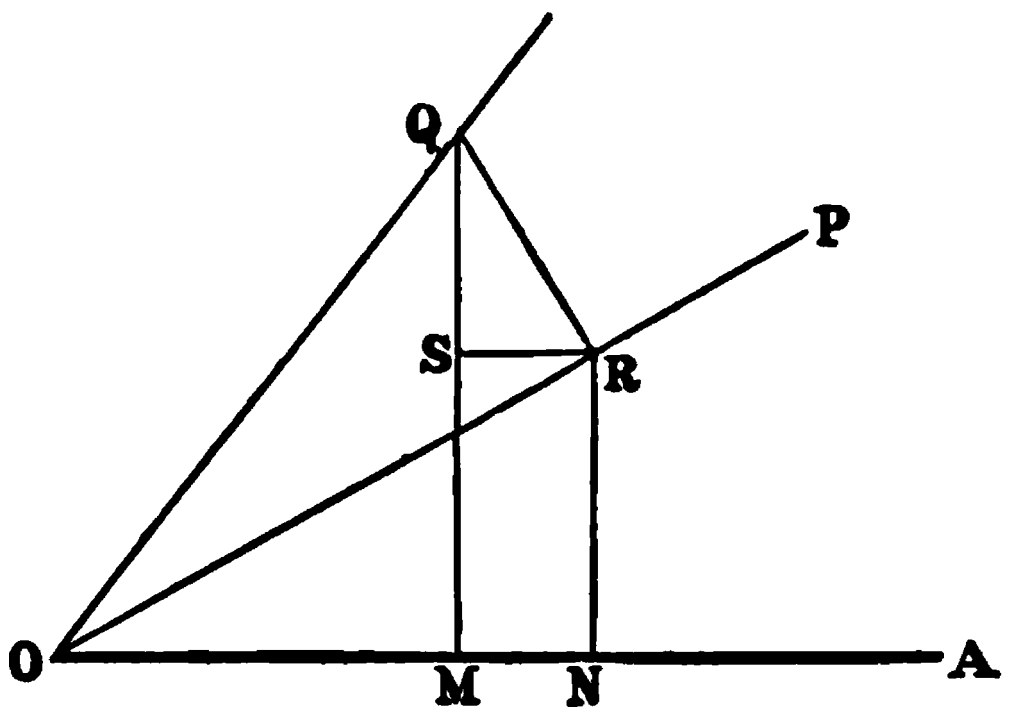
MULTIPLES AND SUB-MULTIPLES OF ANGLES.

1. *To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles.*

We shall first take the case where the sum of the angles in question is less than a right angle.

Let $\angle$ AOP = A, $\angle$ POQ = B.

Then $\angle$ AOQ = A + B; and considering OA as the initial line, OQ is *the bounding line of the compound angle* (A + B).



In the *bounding line* OQ take any point Q, and from Q draw QR perpendicular to OP, and meeting it in R; also draw QM, RN perpendiculars to OA, and RS perpendicular to QM, and therefore also parallel to OA.

Then $\angle SQR = 90° - \angle SRQ = \angle SRO = \angle AOP = A$.

Now, $\sin (A + B)$

$$= \sin AOQ = \frac{QM}{OQ} = \frac{SM + QS}{OQ} = \frac{RN + QS}{OQ}$$

$$= \frac{RN}{OQ} + \frac{QS}{OQ}$$

$$= \frac{RN}{OR} \cdot \frac{OR}{OQ} + \frac{QS}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP \cdot \cos POQ + \cos SQR \cdot \sin POQ$$

$$= \sin A \cos B + \cos A \sin B.$$

Again, $\cos (A + B)$

$$= \cos AOQ = \frac{OM}{OQ} = \frac{ON - MN}{OQ}$$

$$= \frac{ON - SR}{OQ} = \frac{ON}{OQ} - \frac{SR}{OQ}$$

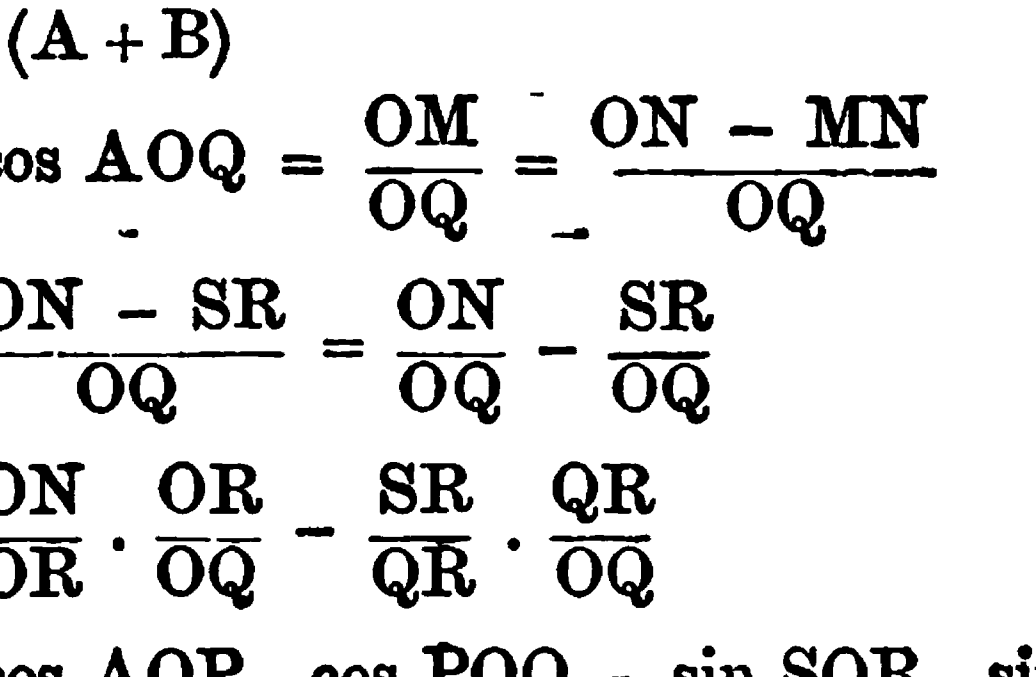
$$= \frac{ON}{OR} \cdot \frac{OR}{OQ} - \frac{SR}{QR} \cdot \frac{QR}{OQ}$$

$$= \cos AOP \cdot \cos POQ - \sin SQR \cdot \sin POQ$$

$$= \cos A \cos B - \sin A \sin B.$$

2. *To express the sine and cosine of the difference of two angles in terms of the sines and cosines of the angles.*

Let $\angle AOP = A$, $\angle POQ = B$.



Then $\angle AOQ = A - B$, and OQ is *the bounding line of the compound angle* $(A - B)$.

In the *bounding line* OQ take any point Q, and from Q draw QR perpendicular to OP, meeting it in R; also draw QM, RN perpendiculars to OA, and RS perpendicular to MQ produced, and therefore also parallel to OA.

Then $\angle RQS = 90° - \angle QRS = \angle PRS = \angle AOP = A.$

Now, $\sin(A - B) = \sin \overline{AOQ} = \frac{QM}{OQ} = \frac{SM - QS}{OQ} = \frac{RN - QS}{OQ}$

$$= \frac{RN}{OQ} - \frac{QS}{OQ}$$

$$= \frac{RN}{OR} \cdot \frac{OR}{OQ} - \frac{QS}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP \cdot \cos POQ - \cos RQS \cdot \sin POQ$$

$$= \sin A \cos B - \cos A \sin B.$$

And $\cos (A - B)$

$$= \cos \overline{AOQ} = \frac{OM}{OQ} = \frac{ON + NM}{OQ}$$

$$= \frac{ON + RS}{OQ} = \frac{ON}{OQ} + \frac{RS}{OQ}$$

$$= \frac{ON}{OR} \cdot \frac{OR}{OQ} + \frac{RS}{QR} \cdot \frac{QR}{OQ}$$

$$= \cos AOP \cdot \cos POQ + \sin RQS \cdot \sin POQ$$

$$= \cos A \cos B + \sin A \sin B.$$

It is easy to see that the two formulæ just obtained may be obtained from those in the last article by putting $-B$ for B.

We will collect for reference the results of this and the last article:

$$\text{Sin } (A + B) = \sin A \cos B + \cos A \sin B \ldots\ldots\ldots (1).$$
$$\text{Sin } (A - B) = \sin A \cos B - \cos A \sin B \ldots\ldots\ldots (2).$$
$$\text{Cos } (A + B) = \cos A \cos B - \sin A \sin B \ldots\ldots\ldots (3).$$
$$\textit{Cos } (A - B) = \cos A \cos B + \sin A \sin B \ldots\ldots\ldots (4).$$

3. By means of these formulæ we may easily obtain other useful formulæ.

We have,

$$(1) + (2),\ \sin(A + B) + \sin(A - B) = 2\sin A\cos B \ldots\ldots (5).$$
$$(1) - (2),\ \sin(A + B) - \sin(A - B) = 2\cos A\sin B \ldots\ldots (6).$$
$$(3) + (4),\ \cos(A + B) + \cos(A - B) = 2\cos A\cos B \ldots\ldots (7).$$
$$(4) - (3),\ \cos(A - B) - \cos(A + B) = 2\sin A\sin B \ldots\ldots (8).$$

(The student will be careful to remember that first side of (8) has its terms in an order the reverse of the other three.)

Put $A + B = C$, and $A - B = D$;

and $\therefore A = \frac{C + D}{2}$, $B = \frac{C - D}{2}$,

then we have, from (5), (6), (7), (8),

$$\sin C + \sin D = 2\sin\frac{C + D}{2}\cos\frac{C - D}{2} \ldots\ldots\ldots (9),$$

$$\sin C - \sin D = 2\cos\frac{C + D}{2}\sin\frac{C - D}{2} \ldots\ldots\ldots (10),$$

$$\cos C + \cos D = 2\cos\frac{C + D}{2}\cos\frac{C - D}{2} \ldots\ldots\ldots (11),$$

$$\cos D - \cos C = 2\sin\frac{C + D}{2}\sin\frac{C - D}{2} \ldots\ldots\ldots (12).$$

Again, we have,

$$\begin{aligned}
&\text{Sin}(A + B).\sin(A - B)\\
&= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)\\
&= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B\\
&= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B\\
&= \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A \ldots\ldots\ldots\ldots\ldots\ldots (13).
\end{aligned}$$

And $\cos(A + B).\cos(A - B)$

$$\begin{aligned}
&= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)\\
&= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B\\
&= \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A \ldots\ldots\ldots\ldots\ldots\ldots (14).
\end{aligned}$$

The formulæ obtained in this article are extremely useful, and are often referred to in trigonometrical transformations.

Thus (i.),

$$\sin 7\theta + \sin 5\theta = \sin(6\theta + \theta) + \sin(6\theta - \theta); \text{ or, by (5)},$$
$$= 2\sin 6\theta \,.\, \cos\theta.$$

Or, at once by (9),

$$\text{Sin } 7\theta + \sin 5\theta = 2\sin\frac{7\theta + 5\theta}{2}\cos\frac{7\theta - 5\theta}{2}$$
$$= 2\sin 6\theta \,.\, \cos\theta.$$

Again (ii.),

$$\cos\theta - \cos 3\theta = \cos(2\theta - \theta) - \cos(2\theta + \theta); \text{ or, by (8)},$$
$$= 2\sin 2\theta \,.\, \sin\theta.$$

Or, at once by (12),

$$\cos\theta - \cos 3\theta = 2\sin\frac{3\theta + \theta}{2}\sin\frac{3\theta - \theta}{2} = 2\sin 2\theta \,.\, \sin\theta.$$

And so (iii.)

$$\sin(\alpha - \beta) + \sin(\beta - \gamma) + (\sin\gamma - \delta) + \sin(\delta - \alpha)$$
$$= 2\sin\frac{\overline{\alpha - \beta} + \overline{\beta - \gamma}}{2}\cos\frac{\overline{\alpha - \beta} - \overline{\beta - \gamma}}{2}$$
$$+ 2\sin\frac{\overline{\gamma - \delta} + \overline{\delta - \alpha}}{2}\cos\frac{\overline{\gamma - \delta} - \overline{\delta - \alpha}}{2}$$
$$= 2\sin\frac{\alpha - \gamma}{2}\cos\frac{\alpha - 2\beta + \gamma}{2} + 2\sin\frac{\gamma - \alpha}{2}\cos\frac{\alpha + \gamma - 2\delta}{2}$$
$$= 2\sin\frac{\alpha - \gamma}{2}\left\{\cos\frac{\alpha - 2\beta + \gamma}{2} - \cos\frac{\alpha + \gamma - 2\varepsilon}{2}\right\}$$
$$= 2\sin\frac{\alpha - \gamma}{2} \times$$
$$\left\{\cos\left(\frac{\alpha - \beta + \gamma - \delta}{2} - \frac{\beta - \delta}{2}\right) - \cos\left(\frac{\alpha - \beta + \gamma - \delta}{2} + \frac{\beta - \delta}{2}\right)\right\};$$

or, by (8),

$$= 2\sin\frac{\alpha - \gamma}{2} \,.\, 2\sin\frac{\alpha - \beta + \gamma - \delta}{2}\sin\frac{\beta - \delta}{2}$$
$$= 4\sin\frac{\alpha - \gamma}{2} \,.\, \sin\frac{\beta - \delta}{2} \,.\, \sin\frac{\alpha - \beta + \gamma - \delta}{2}.$$

4. It will be remembered that formula (1) of Art. 2 was obtained in Vol. I. of this work by a less direct method, and we there found formulæ for $\cos\frac{A}{2}$, $\tan\frac{A}{2}$, &c.

We shall now obtain these formulæ, with others, from the results of the last Art.

We have $\sin(A + B) = \sin A \cos B + \cos A \sin B$,

and $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Put $B = A$, then since $A + B = 2A$, we at once have

$$\sin 2A = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A \quad \ldots\ldots(1),$$
$$\text{and } \cos 2A = \cos A \cos A - \sin A \,.\, \sin A$$
$$= \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A \quad \ldots\ldots(2).$$

And so, if we put $\frac{A}{2}$ for A, and therefore A for 2 A, we get

$$\sin A = 2 \sin\frac{A}{2}\cos\frac{A}{2} \quad \ldots\ldots(3),$$

$$\text{and } \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$

$$= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2} \quad \ldots\ldots(4).$$

5. *To find* $\tan(A \pm B)$, $\cot(A \pm B)$.

$$\tan(A \pm B) = \frac{\sin(A \pm B)}{\cos(A \pm B)} = \frac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B \mp \sin A \sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B}}{1 \cdot \mp \dfrac{\sin A}{\cos A} \cdot \dfrac{\sin B}{\cos B}}$$

$$= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad \ldots\ldots(1).$$

And so—

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A} \quad \ldots\ldots(2).$$

Cor. 1. Put B = A, and take the upper sign, then,

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad \ldots\ldots(3),$$

$$\text{and } \cot 2A = \frac{\cot^2 A - 1}{2 \cot A} \quad \ldots\ldots(4).$$

Again, put $\frac{A}{2}$ for A, and therefore A for 2 A, then,

$$\tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} \quad \ldots\ldots(5),$$

$$\text{and } \cot A = \frac{\cot^2 \frac{A}{2} - 1}{2 \cot \frac{A}{2}} \quad \ldots\ldots(6).$$

Cor. 2. Put A = 45°, then since tan 45° = 1, we have,

$$\tan(45° \pm B) = \frac{1 \pm \tan B}{1 \mp \tan B}.$$

6. *To expand* sin 3 A, cos 3 A, tan 3 A, cot 3 A.

$$\begin{aligned}
\sin 3A &= \sin(2A + A) = \sin 2A \cos A + \cos 2A \sin A \\
&= 2 \sin A \cos A \,.\, \cos A + (1 - 2 \sin^2 A) \sin A \\
&= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A \\
&= 3 \sin A - 4 \sin^3 A \quad \ldots\ldots(1).
\end{aligned}$$

$$\begin{aligned}
\cos 3A &= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A \\
&= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \,.\, \sin A \\
&= 2 \cos^3 A - \cos A - 2 (1 - \cos^2 A) \cos A \\
&= 4 \cos^3 A - 3 \cos A \quad \ldots\ldots(2).
\end{aligned}$$

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3 \sin A - 4 \sin^3 A}{4 \cos^3 A - 3 \cos A};$$

or, dividing numerator and denominator by $\cos^3 A$,

$$\tan 3A = \frac{3.\frac{\sin A}{\cos A}.\frac{1}{\cos^2 A} - 4.\frac{\sin^3 A}{\cos^3 A}}{4 - 3.\frac{1}{\cos^2 A}}$$

$$= \frac{3\tan A(1 + \tan^2 A) - 4\tan^3 A}{4 - 3(1 + \tan^2 A)}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$\cot 3A = \frac{1}{\tan 3A}$; or, by (3),

$$= \frac{1 - 3\tan^2 A}{3\tan A - \tan^3 A};$$

or, dividing numerator and denominator by $\tan^3 A$,

$$= \frac{\frac{1}{\tan^3 A} - 3.\frac{1}{\tan A}}{3.\frac{1}{\tan^2 A} - 1}$$

$$= \frac{\cot^3 A - 3\cot A}{3\cot^2 A - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Cor. Put $\frac{A}{3}$ for A, and therefore A for 3 A, we have—

$$\sin A = 3\sin\frac{A}{3} - 4\sin^3\frac{A}{3},$$

$$\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3},$$

$$\tan A = \frac{3\tan\frac{A}{3} - \tan^3\frac{A}{3}}{1 - 3\tan^2\frac{A}{3}},$$

$$\cot A = \frac{\cot^3\frac{A}{3} - 3\cot\frac{A}{3}}{3\cot^2\frac{A}{3} - 1}.$$

7. *To express the sine and cosine of a multiple angle in terms of sines and cosines of inferior multiples.*

$6-n$

We have, by Art. 2,

$$\text{Sin}\,(n+1)\,\text{A} + \sin(n-1)\,\text{A} = 2\sin n\,\text{A}\cos\text{A}$$

$$\therefore \sin(n+1)\,\text{A} = 2\sin n\,\text{A}\cos\text{A} - \sin(n-1)\,\text{A} \quad \ldots\ldots(1).$$

And so

$$\cos(n+1)\,\text{A} = 2\cos n\,\text{A}\cos\text{A} - \cos(n-1)\,\text{A} \quad \ldots\ldots(2).$$

8. *To find the sine, cosine, and tangent, of* (A + B + C).

N (i.) $\text{Sin}\,(\text{A}+\text{B}+\text{C}) = \sin(\text{A}+\text{B})\cos\text{C} + \cos(\text{A}+\text{B})\sin\text{C}$

$= (\sin\text{A}\cos\text{B} + \cos\text{A}\sin\text{B})\cos\text{C} + (\cos\text{A}\cos\text{B} - \sin\text{A}\sin\text{B})\sin\text{C}$

$= \sin\text{A}.\cos\text{B}\cos\text{C} + \sin\text{B}.\cos\text{A}\cos\text{C} + \sin\text{C}.\cos\text{A}\cos\text{B} - \sin\text{A}\sin\text{B}\sin\text{C}.$

(ii.) $\text{Cos}(\text{A}+\text{B}+\text{C}) = \cos(\text{A}+\text{B})\cos\text{C} - \sin(\text{A}+\text{B})\sin\text{C}$

$= (\cos\text{A}\cos\text{B} - \sin\text{A}\sin\text{B})\cos\text{C} - (\sin\text{A}\cos\text{B} + \cos\text{A}\sin\text{B})\sin\text{C}$

$= \cos\text{A}\cos\text{B}\cos\text{C} - \cos\text{A}.\sin\text{B}\sin\text{C} - \cos\text{B}.\sin\text{A}\sin\text{C} - \cos\text{C}.\sin\text{A}\sin\text{B}.$

(iii.) $\text{Tan}(\text{A}+\text{B}+\text{C}) = \dfrac{\sin(\text{A}+\text{B}+\text{C})}{\cos(\text{A}+\text{B}+\text{C})}$

$$= \frac{\sin\text{A}\cos\text{B}\cos\text{C} + \sin\text{B}\cos\text{A}\cos\text{C} + \sin\text{C}\cos\text{A}\cos\text{B} - \sin\text{A}\sin\text{B}\sin\text{C}}{\cos\text{A}\cos\text{B}\cos\text{C} - \cos\text{A}\sin\text{B}\sin\text{C} - \cos\text{B}\sin\text{A}\sin\text{C} - \cos\text{C}\sin\text{A}\sin\text{B}}.$$

Dividing numerator and denominator by cos A cos B cos C, we have,

$$\tan(A+B+C) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C}}{1 - \frac{\sin B}{\cos B}\cdot\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin C}{\cos C} - \frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A \cdot \tan B \tan C}{1 - \tan B \tan C - \tan A \tan C - \tan A \tan B}.$$

Cor. 1. If $A + B + C = 90°$, then, since $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90° = \infty$, we have, from (i.),

$1 = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$, or

$\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B = 1 + \sin A \sin B \sin C$.....(1).

Also, from (ii.), we similarly have,

$\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B = \cos A \cos B \cos C$.....(2).

And from this, dividing each side by cos A cos B cos C, we have,

$\tan B \tan C + \tan A \tan C + \tan A \tan B = 1$.....(3).

Cor. 2. So when $A + B + C = 180°$, the student will easily get the following formulæ:

$\sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B = \sin A \sin B \sin C$.....(4).

$\cos A \sin B \sin C + \cos B \sin A \sin C + \cos C \sin A \sin B = \cos A \cos B \cos C + 1$.....(5).

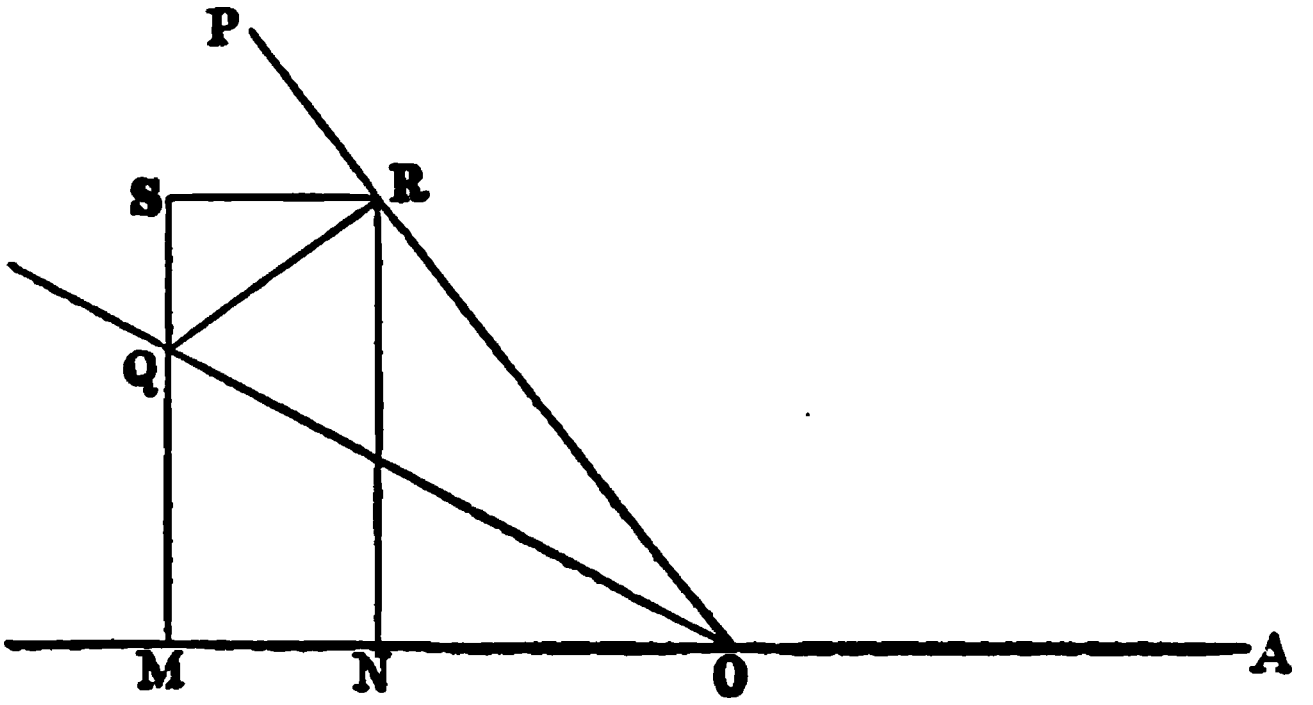
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.....(6).

9. ***To extend the formulæ of Arts. 1 and 2 to angles greater than a right angle.***

We shall take the case only when A is greater than a right angle, (A + B) being less than two right angles. Other cases will form good exercises for the student, and will not offer much difficulty.

Let $\angle AOP = A$, $\angle POQ = B$;

$\therefore \angle AOQ = A + B.$



Take any point Q in OQ the bounding line of the angle (A + B), and draw QR perpendicular to OP. From R and Q draw RN and QM perpendiculars to AO *produced*, and from R draw RS perpendicular to MQ *produced*.

Then $\angle SQR = 90° - \angle SRQ = \angle PRS$

$= \angle POM = 180° - \angle AOP = 180° - A.$

Now, $\sin (A + B) = \sin AOQ$

$$= \frac{QM}{OQ} = \frac{SM - SQ}{OQ} = \frac{RN - SQ}{OQ}$$

$$= \frac{RN}{OQ} - \frac{SQ}{OQ} = \frac{RN}{OR} \cdot \frac{OR}{OQ} - \frac{SQ}{QR} \cdot \frac{QR}{OQ}$$

$$= \sin AOP . \cos POQ - \cos SQR . \sin POQ$$

$$= \sin A \cos B - \cos (180° - A) \sin B.$$

Hence, since $\cos (180° - A) = - \cos A$, we easily get—

$$\sin (A + B) = \sin A \cos B + \cos A \sin B.$$

And in the same way we may obtain expressions for *cos (A + B), and the* sine and cosine of (A − B).

10. *To find the trigonometrical ratios of* 18°.

Put $\theta = 18°$, then $5\theta = 90°$,

and $\therefore\ 2\theta = 90° - 3\theta$

$\therefore \sin 2\theta = \sin(90° - 3\theta) = \cos 3\theta$, or

$2 \sin\theta \,.\, \cos\theta = 4\cos^3\theta - 3\cos\theta$;

or, dividing each side by $\cos\theta$,

$$2\sin\theta = 4\cos^2\theta - 3$$
$$= 4(1 - \sin^2\theta) - 3\text{; or}$$
$$4\sin^2\theta + 2\sin\theta = 1, \text{ or}$$
$$\sin^2\theta + \tfrac{1}{2}\sin\theta = \tfrac{1}{4}.$$

Solving this quadratic for $\sin\theta$, we have—

$$\sin\theta = \frac{\pm\sqrt{5} - 1}{4}.$$

Now, as $\theta = 18°$, the *sine* must be positive, and we must therefore take the upper sign;

Hence we have $\sin 18° = \dfrac{\sqrt{5} - 1}{4}$.

From this result, remembering that cos 18° is positive, we get—

$$\cos 18° = \sqrt{1 - \left(\frac{\sqrt{5} - 1}{4}\right)} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

And hence we may easily obtain the values of tan 18°, cot 18°, &c.

11. *To find the trigonometrical ratios of* 36°.

We have,

$$\cos 36° = \cos 2(18°) = 1 - 2\sin^2 18°$$
$$= 1 - 2\left(\frac{\sqrt{5} - 1}{4}\right)^2 = 1 - \frac{3 - \sqrt{5}}{4} = \frac{\sqrt{5} + 1}{4}.$$

And hence,

$$\sin 36° = \sqrt{1 - \left(\frac{\sqrt{5} + 1}{4}\right)^2} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

Cor. Since $\sin 18° = \cos 72°$, $\sin 36° = \cos 54°$, &c., we may arrange the results of the last two articles, thus:

$$\sin 18° = \frac{\sqrt{5} - 1}{4} = \cos 72°,$$

$$\cos 18° = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \sin 72°,$$

$$\sin 36° = \frac{\sqrt{10 - 2\sqrt{5}}}{4} = \cos 54°,$$

$$\cos 36° = \frac{\sqrt{5} + 1}{4} = \sin 54°.$$

Inverse Functions.

12. When the *sine*, *cosine*, *tangent*, &c., of an unknown angle are given, it is sometimes convenient to express the angle by an inverse function.

Thus, by $\sin^{-1} s$ is meant the angle whose *sine* is s;
by $\cos^{-1} c$ „ „ *cosine* is c; and
by $\tan^{-1} t$ „ „ *tangent* is t.

13. *To show that*

$$\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}.$$

Let $\tan^{-1} t_1 = A$, and $\therefore t_1 = \tan A$.
Also, let $\tan^{-1} t_2 = B$, and $\therefore t_2 = \tan B$. } .....(1).

Now $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$.

Hence, by Art. 12,

$$A + B = \tan^{-1} \frac{\tan A + \tan B}{1 - \tan A . \tan B}.$$

Or, substituting from (1),

$$\tan^{-1} t_1 + \tan^{-1} t_2 = \tan^{-1} \frac{t_1 + t_2}{1 - t_1 t_2}.$$

Cor. Hence also,

$$\tan^{-1} t_1 - \tan^{-1} t_2 = \tan^{-1} \frac{t_1 - t_2}{1 + t_1 t_2},$$

$$\text{and } 2 \tan^{-1} t_1 = \tan^{-1} \frac{2t}{1 - t^2}.$$

14. *To show that*

$$\sin^{-1} s_1 + \sin^{-1} s_2 = \sin^{-1}\{s_1 \sqrt{1 - s_2^2} + s_2 \sqrt{1 - s_1^2}\}.$$

Let $\sin^{-1} s_1 = A$, and $\therefore s_1 = \sin A$.
Also let $\sin^{-1} s_2 = B$, and $\therefore s_2 = \sin B$. } .....(1).

Then we have,

$\cos A = \sqrt{1 - s_1^2}$, and $\cos B = \sqrt{1 - s_2^2}$.....(2).

Now,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B;$$

hence, by Art. 12,

$$A + B = \sin^{-1}\{\sin A \cos B + \cos A \sin B\}.$$

Or, substituting from (1) and (2),

$$\sin^{-1} s_1 + \sin^{-1} s_2 = \sin^{-1}\{s_1 \sqrt{1 - s_2^2} + s_2 \sqrt{1 - s_1^2}\}.$$

Cor. So we get,

$$\cos^{-1} s_1 + \cos^{-1} s_2 = \cos^{-1}\{s_1 s_2 - \sqrt{1 - s_1^2} \cdot \sqrt{1 - s_2^2}\}.$$

15. We shall now give a few examples of questions solved by the aid of the preceding formulæ.

Ex. 1. Show that

$$\frac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}.$$

[As the required expression is to contain sin 3 A in the numerator, and sin 5 A in the denominator, we shall not alter the terms containing these quantities in the given expression. We observe that the first and last terms of the numerator may be thrown into the form—

$$\sin (3A - 2A) + \sin (3A + 2A),$$

where we have the sum of the *sines* of the difference and sum of two *angles*. We therefore apply the formulæ of Art. 2. And a similar *plan may be* pursued in the denominator. Thus :]

$$\frac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A}$$

$$= \frac{\sin(3A - 2A) + \sin(3A + 2A) + 2\sin 3A}{\sin(5A - 2A) + \sin(5A + 2A) + 2\sin 5A};$$

or, by Art. 3,

$$= \frac{2\sin 3A\cos 2A + 2\sin 3A}{2\sin 5A\cos 2A + 2\sin 5A}$$

$$= \frac{2\sin 3A(\cos 2A + 1)}{2\sin 5A(\cos 2A + 1)} = \frac{\sin 3A}{\sin 5A}.$$

Ex. 2. Prove that

$$\frac{\cos A - \cos 5A}{\sin 5A - \sin A} = \tan 3A.$$

[Our result is to be $\tan 3A$ or $\frac{\sin 3A}{\cos 3A}$; we must then try to transform the denominator so as to contain the factor $\cos 3A$. Throwing it into the form of the difference of the *sines* of the sum and difference of two angles, we shall easily get the required result. And a similar method will apply to the numerator. Thus :]

$$\frac{\cos A - \cos 5A}{\sin 5A - \sin A} = \frac{\cos(3A - 2A) - \cos(3A + 2A)}{\sin(3A + 2A) - \sin(3A - 2A)}$$

$$= \frac{2\sin 3A\sin 2A}{2\cos 3A\sin 2A} = \tan 3A.$$

Ex. 3. Show that

$$4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = 45°.$$

By Art. 13, Cor.,

$$\tan^{-1}1 - \tan^{-1}\tfrac{1}{5} = \tan^{-1}\frac{1 - \frac{1}{5}}{1 + 1\left(\frac{1}{5}\right)} = \tan^{-1}\tfrac{2}{3},$$

$$\tan^{-1}\tfrac{2}{3} - \tan^{-1}\tfrac{1}{5} = \tan^{-1}\frac{\frac{2}{3} - \frac{1}{5}}{1 + \frac{2}{3}\left(\frac{1}{5}\right)} = \tan^{-1}\tfrac{7}{17},$$

$$\tan^{-1}\tfrac{7}{17} - \tan^{-1}\tfrac{1}{5} = \tan^{-1}\frac{\frac{7}{17} - \frac{1}{5}}{1 + \frac{7}{17}\left(\frac{1}{5}\right)} = \tan^{-1}\tfrac{9}{46},$$

$$\tan^{-1}\tfrac{9}{46} - \tan^{-1}\tfrac{1}{5} = \tan^{-1}\frac{\frac{9}{46} - \frac{1}{5}}{1 + \frac{9}{46}\left(\frac{1}{5}\right)} = \tan^{-1}\left(-\tfrac{1}{239}\right).$$

Adding together these equations, and reducing, we have,

$$\tan^{-1} 1 - 4 \tan^{-1} \tfrac{1}{5} = \tan^{-1}(-\tfrac{1}{2}\tfrac{1}{3}\tfrac{9}) = -\tan^{-1}\tfrac{1}{2}\tfrac{1}{3}\tfrac{9}.$$

Or, transposing,

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1}\tfrac{1}{2}\tfrac{1}{3}\tfrac{9} = \tan^{-1} 1.$$

Now $\tan^{-1} 1$ is the angle whose tangent is unity, and therefore $\tan^{-1} 1 = 45°$.

Hence we have

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1}\tfrac{1}{2}\tfrac{1}{3}\tfrac{9} = 45°.$$

Ex. 4. Eliminate θ from the equations,

$$(a + b) \tan(\theta - \phi) = (a - b) \tan(\theta + \phi) \ldots\ldots\ldots(1),$$
$$a \cos 2\phi + b \cos 2\theta = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1),

$$\frac{a + b}{a - b} = \frac{\tan(\theta + \phi)}{\tan(\theta - \phi)} = \frac{\sin(\theta + \phi)\cos(\theta - \phi)}{\cos(\theta + \phi)\sin(\theta - \phi)}.$$

Hence,

$$\frac{a}{b} = \frac{\sin(\theta + \phi)\cos(\theta - \phi) + \cos(\theta + \phi)\sin(\theta - \phi)}{\sin(\theta + \phi)\cos(\theta - \phi) - \cos(\theta + \phi)\sin(\theta - \phi)}$$

$$= \frac{\sin\{(\theta + \phi) + (\theta - \phi)\}}{\sin\{(\theta + \phi) - (\theta - \phi)\}} = \frac{\sin 2\theta}{\sin 2\phi}$$

$$\therefore b \sin 2\theta = a \sin 2\phi \ldots \quad \ldots\ldots\ldots\ldots(3).$$

We have also from (2),

$$b \cos 2\theta = c - a \cos 2\phi \ldots \ldots\ldots\ldots(4).$$

Squaring (3) and (4), and adding, we have,

$$b^2(\sin^2 2\theta + \cos^2 2\theta) = a^2 \sin^2 2\phi + (c - a \cos 2\phi)^2, \text{ or}$$
$$b^2 = a^2(\sin^2 2\phi + \cos^2 2\phi) + c^2 - 2ac \cos 2\phi.$$
$$\therefore b^2 = a^2 + c^2 - 2ac \cos 2\phi, \text{ or}$$
$$2ac \cos 2\phi = a^2 + c^2 - b^2, \text{ the relation required.}$$

Ex. I.

1. *If* $\sin \alpha = \frac{1}{\sqrt{2}}$, and $\sin \beta = \frac{1}{2}$, then $\sin(\alpha - \beta) = \frac{\sqrt{3} - 1}{2\sqrt{2}}$.

2. If $\sin \alpha = \frac{1}{3}$, and $\sin \beta = \frac{1}{5}$, then $\cos(\alpha + \beta)$
$$= \frac{8\sqrt{3} - 1}{15}.$$

3. Show that $\cos 2x = \frac{\sqrt{3}}{2}$, and $\sin 2x = \frac{1}{2}$, when
$$\sin x = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

4. When $\cos x = \frac{1}{2}$, and $\cos y = \frac{\sqrt{3} + 1}{2\sqrt{2}}$, then $\cos(x - y)$
$$= \frac{1}{\sqrt{2}}.$$

5. If $\sin \alpha = \frac{1}{8}(\sqrt{6} - \sqrt{2})$, show that
$$\tan(45° + \alpha) + \tan(45° - \alpha) = \sqrt{3} - \frac{1}{\sqrt{3}}.$$

6. Given $\sin 72° = \frac{\sqrt{5 + \sqrt{5}}}{2\sqrt{2}}$, show that
$$\tan 36° = \sqrt{5 - 2\sqrt{5}}.$$

7. Given $\sin 18° = \frac{\sqrt{5} - 1}{4}$, show that
$$\sin 9° = \sqrt{\frac{4 - \sqrt{10 + 2\sqrt{5}}}{8}}, \quad \cos 9° = \sqrt{\frac{4 + \sqrt{10 + 2\sqrt{5}}}{8}}.$$

8. Find the trigonometrical ratios of $7°\frac{1}{2}$, $22°\frac{1}{2}$, $37°\frac{1}{2}$.

9. If $\alpha = 7°\frac{1}{2}$, show that $\frac{\cos 3\alpha}{\cos 7\alpha} = \frac{\sin 9\alpha}{\sin 5\alpha}$.

10. If $\sin \alpha = \frac{4}{5}$, $\sin \beta = \frac{5}{13}$, $\sin \gamma = \frac{16}{65}$, show that
$$\sin(\alpha + \beta + \gamma) = 1.$$

11. If $\tan A = \frac{1}{x - 1}$, and $\cot B = x$, then
$$\tan(A - B) = \frac{1}{x^2 - x + 1}.$$

Adding together these equations, and reducing, we have,

$$\tan^{-1} 1 - 4 \tan^{-1} \tfrac{1}{5} = \tan^{-1}(-\tfrac{1}{239}) = -\tan^{-1} \tfrac{1}{239}.$$

Or, transposing,

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = \tan^{-1} 1.$$

Now $\tan^{-1} 1$ is the angle whose tangent is unity, and therefore $\tan^{-1} 1 = 45°$.

Hence we have

$$4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239} = 45°.$$

Ex. 4. Eliminate θ from the equations,

$$(a + b) \tan (\theta - \phi) = (a - b) \tan (\theta + \phi) \ldots\ldots\ldots(1),$$
$$a \cos 2\phi + b \cos 2\theta = c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1),

$$\frac{a + b}{a - b} = \frac{\tan (\theta + \phi)}{\tan (\theta - \phi)} = \frac{\sin (\theta + \phi) \cos (\theta - \phi)}{\cos (\theta + \phi) \sin (\theta - \phi)}.$$

Hence,

$$\frac{a}{b} = \frac{\sin (\theta + \phi) \cos (\theta - \phi) + \cos (\theta + \phi) \sin (\theta - \phi)}{\sin (\theta + \phi) \cos (\theta - \phi) - \cos (\theta + \phi) \sin (\theta - \phi)}$$

$$= \frac{\sin\{(\theta + \phi) + (\theta - \phi)\}}{\sin\{(\theta + \phi) - (\theta - \phi)\}} = \frac{\sin 2\theta}{\sin 2\phi}$$

$$\therefore b \sin 2\theta = a \sin 2\phi \ldots \quad \ldots\ldots\ldots\ldots(3).$$

We have also from (2),

$$b \cos 2\theta = c - a \cos 2\phi \ldots \ldots\ldots\ldots(4).$$

Squaring (3) and (4), and adding, we have,

$$b^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2 \sin^2 2\phi + (c - a \cos 2\phi)^2, \text{ or}$$
$$b^2 = a^2 (\sin^2 2\phi + \cos^2 2\phi) + c^2 - 2ac \cos 2\phi.$$
$$\therefore b^2 = a^2 + c^2 - 2ac \cos 2\phi, \text{ or}$$

$2ac \cos 2\phi = a^2 + c^2 - b^2$, the relation required.

Ex. I.

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10. If $\sin \alpha = \frac{4}{5}$, $\sin \beta = \frac{5}{13}$, $\sin \gamma = \frac{16}{65}$, show that
$$\sin (\alpha + \beta + \gamma) = 1.$$

11. If $\tan A = \frac{1}{x - 1}$, and $\cot B = x$, then
$$\tan (A - B) = \frac{1}{x^2 - x + 1}.$$

42. $$\operatorname{Sin}^2 A \cdot \sin\frac{B+C}{2}\sin\frac{B-C}{2} + \sin^2 B \sin\frac{C+A}{2}\sin\frac{C-A}{2} + \sin^2 C \sin\frac{A+B}{2}\sin\frac{A-B}{2} = (\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A).$$

43. $$\operatorname{Sin} A \cdot \sin\frac{B+C}{2}\sin\frac{B-C}{2} + \sin B \cdot \sin\frac{C+A}{2}\sin\frac{C-A}{2} + \sin C \cdot \sin\frac{A+B}{2}\sin\frac{A-B}{2} = 0.$$

44. $$\operatorname{Cos}(2A+B) = \cos(A+2B)\sec(A-B) - \sin(2A+B)\tan(A-B).$$

45. $$\operatorname{Sin} 2(\alpha-\beta) + \sin 2(\beta-\gamma) + \sin 2(\gamma-\alpha) = 4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\alpha-\gamma).$$

46. $$\operatorname{Cos} 2(\alpha-\beta) + \cos 2(\beta-\gamma) + \cos 2(\gamma-\alpha) = 4\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha) - 1.$$

47. $$\operatorname{Tan}(\alpha+\beta+\gamma) = \frac{\sin 3\alpha \sin(\beta-\gamma) + \sin 3\beta \sin(\gamma-\alpha) + \sin 3\gamma \sin(\alpha-\beta)}{\cos 3\alpha \sin(\beta-\gamma) + \cos 3\beta \sin(\gamma-\alpha) + \cos 3\gamma \sin(\alpha-\beta)}.$$

48. $$\frac{1}{2\sin\alpha -}\ \frac{1}{2\sin\alpha -}\ \frac{1}{2\sin\alpha -}\ \ldots\ \frac{1}{2\sin\alpha + x} = \frac{\sin n\alpha + x\cos\overline{n-1}\cdot\alpha}{x\sin n\alpha - \cos\overline{n+1}\cdot\alpha}, \ n \text{ even};$$

$$\text{and} = \frac{\cos n\alpha - x\sin\overline{n-1}\cdot\alpha}{x\cos n\alpha + \sin\overline{n+1}\cdot\alpha}, \ n \text{ odd},$$

where n is the number of quotients of the continued fraction.

If A, B, C are the angles of a triangle, show that :

49. $$\operatorname{Cos} A + \cos B + \cos C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1.$$

50. $\text{Sin A} + \sin \text{B} + \sin \text{C}$

$$= 4 \cos \frac{\text{A}}{2} \cos \frac{\text{B}}{2} \cos \frac{\text{C}}{2}.$$

51. $\text{Cot} \frac{\text{A}}{2} + \cot \frac{\text{B}}{2} + \cot \frac{\text{C}}{2}$

$$= \cot \frac{\text{A}}{2} \cot \frac{\text{B}}{2} \cot \frac{\text{C}}{2}.$$

52. $\text{Sin}^2 \frac{\text{A}}{2} + \sin^2 \frac{\text{B}}{2} + \sin^2 \frac{\text{C}}{2}$

$$= 1 - 2 \sin \frac{\text{A}}{2} \sin \frac{\text{B}}{2} \sin \frac{\text{C}}{2}.$$

53. $\text{Cos}^2 \frac{\text{A}}{2} + \cos^2 \frac{\text{B}}{2} + \cos^2 \frac{\text{C}}{2}$

$$= 2 + 2 \sin \frac{\text{A}}{2} \sin \frac{\text{B}}{2} \sin \frac{\text{C}}{2}.$$

54. $\text{Sin } 2\text{A} + \sin 2\text{B} + \sin 2\text{C}$

$$= 4 \sin \text{A} \,.\, \sin \text{B} \,.\, \sin \text{C}.$$

55. $\text{Cos } 2\text{A} + \cos 2\text{B} + \cos 2\text{C}$

$$= -4 \cos \text{A} \cos \text{B} \cos \text{C} - 1.$$

56. $\text{Sin} \frac{\text{A} + \text{B}}{2} + \sin \frac{\text{A} + \text{C}}{2} + \sin \frac{\text{B} + \text{C}}{2}$

$$= 4 \cos \frac{\pi - \text{A}}{4} \cos \frac{\pi - \text{B}}{4} \cos \frac{\pi - \text{C}}{4}.$$

57. $\text{Sin} \frac{\text{A}}{2} + \sin \frac{\text{B}}{2} + \sin \frac{\text{C}}{2}$

$$+ 4 \sin \frac{\text{A} + \text{B}}{4} \sin \frac{\text{A} + \text{C}}{4} \sin \frac{\text{B} + \text{C}}{4} = 1.$$

58. $\text{Cot A cot B} + \cot \text{A cot C} + \cot \text{B cot C} = 1.$

59. $\text{Tan} \frac{\text{A}}{2} \tan \frac{\text{B}}{2} + \tan \frac{\text{A}}{2} \tan \frac{\text{C}}{2}$

$$+ \tan \frac{\text{B}}{2} \tan \frac{\text{C}}{2} = 1.$$

60. $\text{Cos}^2 \frac{C}{2}$

$$= \frac{(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C)}{4 \sin B \sin C}.$$

61. $\text{Tan}^2 \frac{A}{2}$

$$= \frac{(\sin A + \sin C - \sin B)(\sin A + \sin B - \sin C)}{(\sin A + \sin B + \sin C)(\sin B + \sin C - \sin A)}$$

62. $\text{Cot } mA \cot mB + \cot mA \cot mB$

$$+ \cot mB \cot mC = 1.$$

If $A + B + C = 90°$, show that:

63. $\text{Cos } 2A + \cos 2B + \cos 2C$

$$= 1 + 4 \sin A \sin B \sin C.$$

64. $\text{Tan } A + \tan B + \tan C$

$$= \tan A \tan B \tan C + \sec A \sec B \sec C.$$

65. $\frac{\text{Cos } B + \sin C - \sin A}{\text{Cos } A + \sin C - \sin B} = \frac{1 + \tan \frac{1}{2} B}{1 + \tan \frac{1}{2} A}.$

66. $(\text{Tan } A + \tan B + \tan C)(\cot A + \cot B + \cot C)$

$$= 1 + \text{cosec } A \text{ cosec } B \text{ cosec } C.$$

If α, β, γ be in A. P., show that:

67. $\text{Sin } \alpha - \sin \gamma = 2 \sin (\alpha - \beta) \cos B.$

68. $\frac{\text{Tan } \beta}{\text{Tan } (\beta - \gamma)} = \frac{\sin \alpha + \sin \gamma}{\sin \alpha - \sin \gamma} = \frac{\tan \frac{1}{2} (\alpha + \gamma)}{\tan \frac{1}{2} (\alpha - \gamma)}.$

69. $\frac{1}{\text{Tan } \alpha + \tan \gamma} + \frac{1}{2} \tan \beta$

$$= \frac{1}{\cot \alpha + \cot \gamma} + \frac{1}{2} \cot \beta.$$

70. If $\alpha + \beta + \gamma = 180°$, and the sines and cosines of α, β, γ be in A. P., prove that:

(1) $\text{Sin } \frac{1}{2} (\alpha - \beta) \sin \frac{1}{2} \gamma = \sin \frac{1}{2} \alpha \sin \frac{1}{2} (\beta - \gamma).$

(2) $\text{Cos } \beta = 4 \cos \frac{1}{2} \alpha \sin \frac{1}{2} \beta \cos \frac{1}{2} \gamma + 1.$

Prove that

71. $\text{Tan}^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45°.$

72. $\text{Tan}^{-1}\frac{4}{3} + \tan^{-1} 7 = 135°.$

73. $\text{Sin}^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}\frac{3}{\sqrt{10}} = 45°.$

74. $\text{Tan}^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{3}{4}\frac{3}{1} + \tan^{-1}\frac{1}{5} = 45°.$

75. $2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right) = \cos^{-1}\frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}.$

76. $\text{Cos}^{-1}\frac{\tan\theta + \tan\alpha}{1 + \tan\theta\tan\alpha}$

$$= 2\tan^{-1}\left\{\sqrt{\tan(45° - \theta)\tan(45° - \alpha)}\right\}.$$

77. If $1 + \cos A + \cos B + \cos C = 0$, where $A + B + C = 0$, show that either A, B, or C is an odd multiple of 180°.

78. If $A + B + C = (3 \pm 1)\frac{\pi}{2}$,

$$1 \pm 2\cos A\cos B\cos C = \cos^2 A + \cos^2 B + \cos^2 C;$$

but, if $A + B + C = (2 \pm 1)\frac{\pi}{2}$,

$$1 \pm 2\sin A\sin B\sin C = \sin^2 A + \sin^2 B + \sin^2 C.$$

79. Show that, unless $A = \frac{2x + 1}{r}\pi$,

$$\frac{\sin(n-r)A + 2\sin nA + \sin(n+r)A}{\sin(m-r) + 2\sin mA + \sin(m+r)A} = \frac{\sin nA}{\sin mA},$$

and explain the reason for the exception.

80. Given $x = a\cos(\phi + \alpha)$, $y = b\cos(\phi + \beta)$, $z = c\cos(\phi + \gamma)$, show that

$$\frac{x}{a}\sin(\beta - \gamma) + \frac{y}{b}\sin(\gamma - \alpha) + \frac{z}{c}\sin(\alpha - \beta) = 0.$$

81. If the tangents of the angles of a triangle are in geometric progression, so that $\tan A = r \tan B = r^2 \tan C$, show that

$$\tan C = \left(\frac{1 + r + r^2}{r^3}\right)^{\frac{1}{2}}, \text{ and that } \sin 2C = r \sin 2A.$$

82. Eliminate θ from the equations,

$$a \cos(\theta + \alpha) = x, \text{ and } b \cos(\theta + \beta) = y.$$

83. Prove that in any triangle,

$$(\cos \tfrac{1}{2} A + \cos \tfrac{1}{2} B + \cos \tfrac{1}{2} C)(\cos \tfrac{1}{2} B + \cos \tfrac{1}{2} C - \cos \tfrac{1}{2} A)$$
$$\times (\cos \tfrac{1}{2} C + \cos \tfrac{1}{2} A - \cos \tfrac{1}{2} B)(\cos \tfrac{1}{2} A + \cos \tfrac{1}{2} B - \cos \tfrac{1}{2} C)$$
$$= 4 \cos^2 \tfrac{1}{2} A \cos^2 \tfrac{1}{2} B \cos^2 \tfrac{1}{2} C.$$

84. Transform the equation,

$$\alpha^2 \cos^4 \tfrac{1}{2} A + \beta^2 \cos^4 \tfrac{1}{2} B + \gamma^2 \cos^4 \tfrac{1}{2} C$$
$$- 2\alpha\beta \cos^2 \tfrac{1}{2} A \cos^2 \tfrac{1}{2} B - 2\beta\gamma \cos^2 \tfrac{1}{2} B \cos^2 \tfrac{1}{2} C$$
$$- 2\gamma\alpha \cos^2 \tfrac{1}{2} C \cos^2 \tfrac{1}{2} A = 0,$$

into the form

$$(l\alpha + m\beta + n\gamma)(\alpha \sin A + \beta \sin B + \gamma \sin C)$$
$$= p(\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C),$$

determining the values of l, m, n, p.

85. If $1, \frac{p_1}{q_1}, \frac{p_2}{q_2}$, &c., be successive convergents to $\sqrt{2}$, then will

$$2 \tan^{-1} \frac{q_{n-1}}{q_n} + (-1)^n \tan^{-1} \frac{1}{q_{2n}} = \frac{\pi}{4}.$$

86. Given $\cos \theta = \frac{\sin \beta}{\sin \alpha}$, $\cos \phi = \frac{\sin \gamma}{\sin \alpha}$, $\cos(\theta + \phi) = \sin \beta \sin \gamma$, then

$$\tan^2 \alpha = \tan^2 \beta + \tan^2 \gamma.$$

87. Given that $\tan 3\theta = (2 \pm \sqrt{3}) \tan \theta$, show that $\tan^2 3\theta = 1$.

88. If $\tan \frac{1}{2} \alpha = \tan^3 \frac{1}{2} \beta$, and $\tan \beta = 2 \tan \phi$: then will $2\phi = \alpha + \beta$.

89. If $a\cos(x+\alpha) = b\cos(x+\beta) = c\cos(x+\gamma)$; then

$$\cos(\alpha-\beta) = \frac{c^2-a^2-b^2}{2ab}, \text{ and } \tan\gamma = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}.$$

90. If $\sin(\beta+x) + \sin(2\alpha-\beta+x)$
$= (\cos^2\beta - \cos^2\alpha)\sin(\alpha+2x) - \sin\alpha(\sin^2\beta - \sin^2\alpha)$;
then will

$$\sec x = \sin(\alpha+\beta)\tan(\alpha-\beta).$$

91. If $\sin\frac{\theta}{2} = m\sin\frac{\phi}{2}$, and $\theta+\phi=\alpha$, then

$$\tan\tfrac{1}{2}(\theta-\phi) = \frac{m-1}{m+1}\tan\tfrac{1}{2}\alpha.$$

92. If $\tan\theta = m\tan\phi$, and $\theta+\phi=\alpha$, then

$$\sin\alpha = \frac{m+1}{m-1}\sin(\theta-\phi).$$

93. If $\{1+m\cos(x+z)\}\sin y = (1+m\cos z)\sin(x+y)$, then $\tan\frac{1}{2}x = \cot y + m\cos(y-z)\operatorname{cosec} y$.

94. If $2\tan(x-y-z)+\tan z = \tan(x-z)$, then

$$\tan z = \frac{\sin y - \sin(x-y)\cos x}{\sin(x-y)\sin x}.$$

95. If $\sin\theta = \cos^2\alpha - \sin^2\alpha\sqrt{1-c^2\cos^2\theta}$, then

$$1-\tan\frac{\theta}{2} = \pm\left(1+\tan\frac{\theta}{2}\right)\tan\alpha\sqrt{1-c^2\sin^2\alpha}.$$

96. If $\tan\alpha = \tan^3\frac{1}{2}\theta$, and $\cos 2\theta = \frac{1}{3}(2m^2-5)$, then will $\cos^{\frac{2}{3}}\alpha + \sin^{\frac{2}{3}}\alpha = \left(\frac{2}{m}\right)^{\frac{2}{3}}$.

97. If $x\cos\phi = (1-x\sin\phi)\tan\alpha$,
and $x-\sin\phi = \cos\phi\,.\tan\beta$,
then will $\alpha-\beta=\phi$.

98. If $m - \operatorname{cosec}\theta = \sin\theta$, $n = \sec\theta - \cos\theta$, then will

$$m^{\frac{2}{3}} n^{\frac{2}{3}} \left(m^{\frac{2}{3}} + n^{\frac{2}{3}}\right) = 1.$$

99. If $\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$,
where $\alpha + \beta + \gamma = 180°$,
then will $\cot\theta = \cot\alpha + \cot\beta + \cot\gamma$,
and $\cot^2\theta = \cot^2\alpha + \cot^2\beta + \cot^2\gamma + 2$.

100. If $\cos(\phi - \theta + \alpha)\cos(\theta - \alpha) = \cos(\phi - \theta - \alpha)\cos(\theta + \alpha)$,
show that each of these quantities

$$= \tfrac{1}{2}\{\cos\phi + \cos 2\alpha \,.\, \cos(2\theta - \phi)\}.$$

CHAPTER II.

CIRCULAR MEASURE. GENERAL VALUES OF ANGLES.

Circular Measure.

16. We have explained how angles are measured by degrees and by grades. There is another method of measuring angles, and one much used in the higher branches of Mathematical Analysis. The fundamental proposition we have to establish is as follows:—*If the vertex of an angle be taken as centre, and an arc be drawn cutting the lines forming the angle, then the angle is measured by the ratio which the length of the arc intercepted between the lines bears to the radius.*

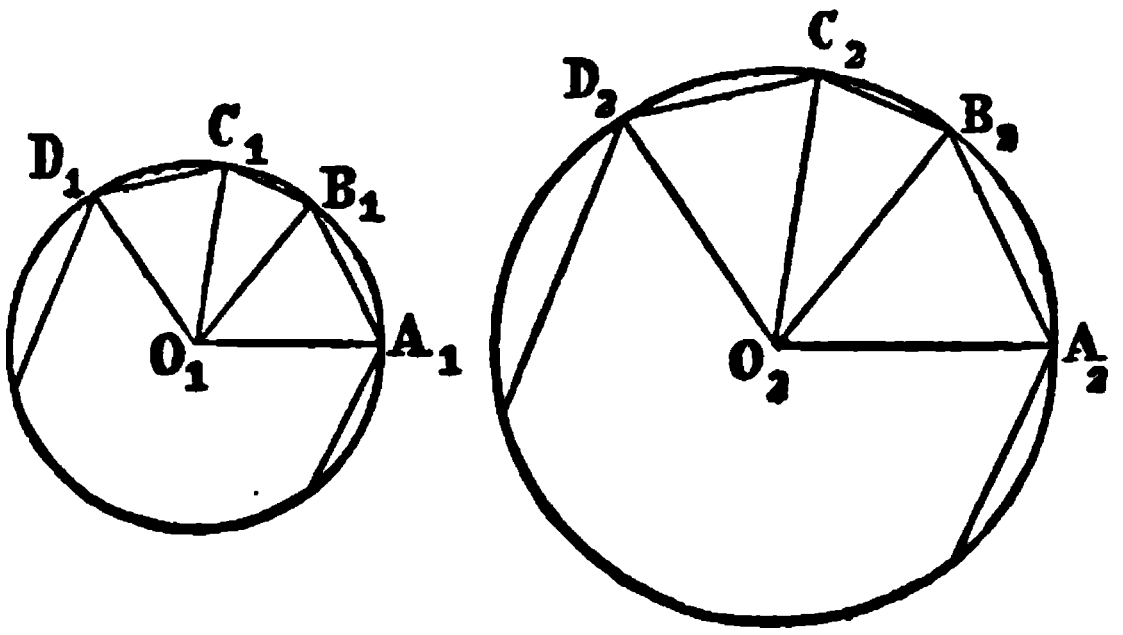
We shall first prove the following preliminary propositions.

17. *The circumference of a circle bears to its diameter a constant ratio.*

Let two circles be drawn with centres O_1 and O_2.

And inscribe in the former a polygon having any number of sides A_1B_1, B_1C_1, C_1D_1, &c.

Also, inscribe in the latter a similar polygon $A_2B_2C_2D_2$, &c. Join O_1A_1, O_1B_1, &c.



Then evidently, $$\frac{A_1B_1}{A_1O_1} = \frac{A_2B_2}{A_2O_2},$$

And so, $$\frac{B_1C_1}{A_1O_1} = \frac{B_2C_2}{A_2O_2}, \text{ \&c.}$$

Hence, by addition,

$$\frac{A_1B_1 + B_1C_1 + \text{\&c.}}{A_1O_1} = \frac{A_2B_2 + B_2C_2 + \text{\&c}}{A_2O_2}.$$

Now, suppose the sides of the polygon to diminish indefinitely, and their number to increase indefinitely.

Then, the sum of the sides of the polygon in each circle will be the circumference of the circles. Hence, we have,

$$\frac{\text{circumference of } A_1B_1C_1D_1\text{\&c.}}{\text{rad. } A_1O_1} = \frac{\text{circumference of } A_2B_2C_2D_2 \text{ \&c.}}{\text{rad. } A_2O_2}$$

Hence the circumference of a circle bears to its diameter a constant ratio.

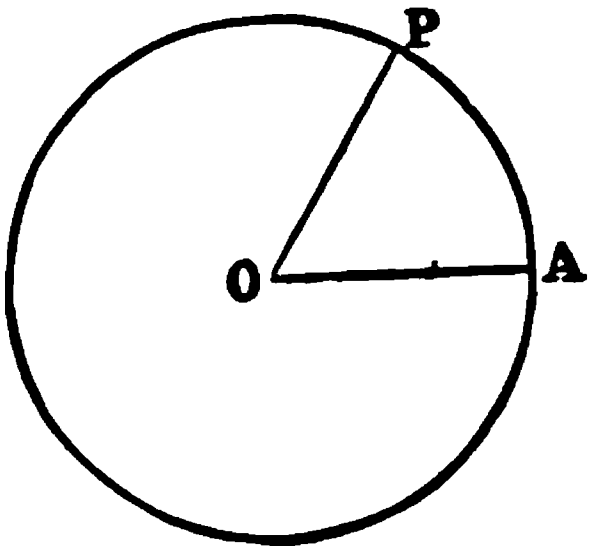
It is usual to represent this ratio by the symbol π.

Hence the circumference of a circle $= 2r\pi$.

The value of π will be computed farther on. Its value is 3·14159 . . . . , converging fractions to which are

$$\frac{1}{3}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \text{ \&c.}$$

18. *To find the angle whose arc is equal in length to the radius of the circle.*



Let a circle be drawn with centre O and radius OA, and let the angle AOP be such that its arc AP is equal in length to the radius.

Then we have

$$\text{Arc AP} = r \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Now (Euc. VI. 33) angles at the centre are as the arcs on which they stand; and hence

$$\frac{\angle \text{AOP}}{4 \text{ right angles}} = \frac{\text{arc AP}}{\text{circumference of circle}}, \text{ or}$$

$$\frac{\angle \text{AOP}}{360^\circ} = \frac{r}{2\pi r} = \frac{1}{2\pi}$$

$$\therefore \angle \text{AOP} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}.$$

Now, by the last article, π is a constant quantity; and hence the angle corresponding to the arc which is equal in length to the radius has the **constant value** $\frac{180^\circ}{\pi}$, whatever be the radius of the circle.

If we take this angle as a unit or standard of reference, then any other angle may be expressed in terms of this unit.

Suppose a given angle A to be θ times this standard angle.

Then θ is called the **circular measure** of the angle A.

Cor. 1. Hence if θ be the circular measure of an angle A, we have

$$\text{A}^\circ = \frac{180^\circ}{\pi}\,\theta = 57^\circ{\cdot}29577 \times \theta.$$

Sometimes it is convenient to express the angle at once in seconds; we then have, since $57^\circ{\cdot}29577 = 206265''$ nearly,

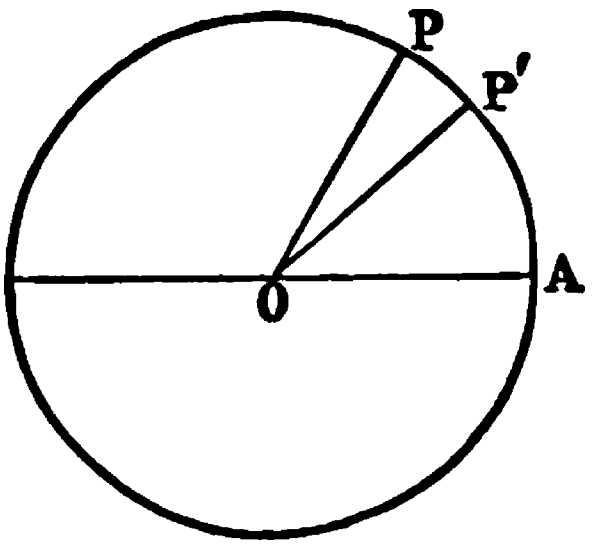
$$\therefore \text{angle} = 206265'' \times \theta.$$

COR. 2. Number of grades in angle A

$$= \frac{200}{\pi}\theta = 63{\cdot}66197\ldots. \times \theta.$$

19. *The circular measure of an angle is the ratio of the length of the arc to the radius; or,*

$$\textit{circular measure} = \frac{\textit{arc}}{\textit{radius}}.$$



Let AP be an arc which is equal in length to the radius;

And let AOP′ be an angle whose circular measure is required.

Then, by Art. 18,

Circular measure of $\angle$ AOP′ $= \frac{\angle \text{AOP}'}{\angle \text{AOP}}$; or (Euc. VI. 33),

$$= \frac{\text{arc AP}'}{\text{arc AP}};$$

But arc AP = radius, and hence

$$\text{Circular measure of } \angle \text{ AOP}' = \frac{\text{arc}}{\text{radius}}.$$

COR. If a represent the length of the arc, we have

$$\theta = \frac{a}{r}, \text{ or } a = r\theta.$$

It is sometimes convenient to represent an angle by its circular measure; thus, since

$$\text{circular measure of } 180^\circ = \frac{r\pi}{r} = \pi,$$

it is convenient to use π instead of 180°.

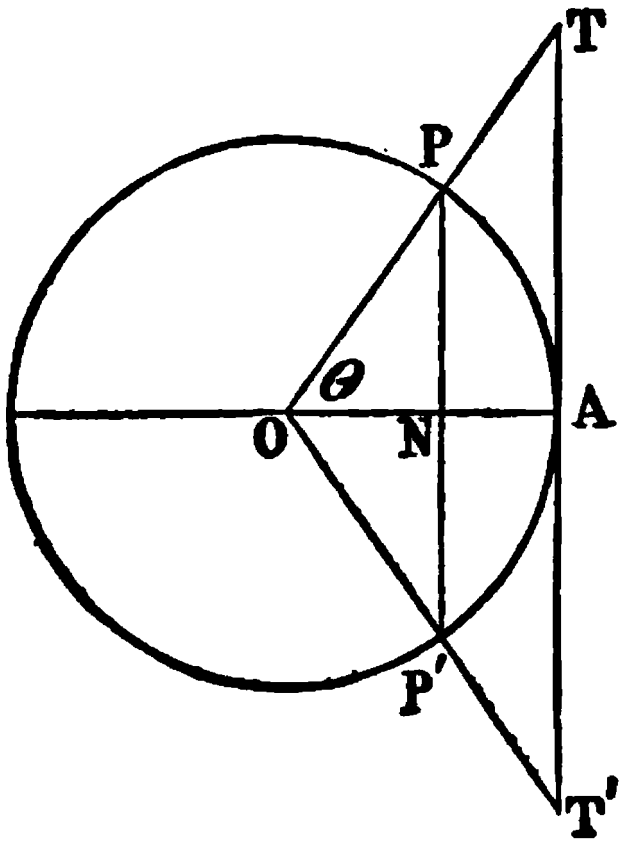
We may then write

$$\sin \text{A} = \sin (\pi - \text{A})$$
$$\cos \text{A} = -\cos (\pi - \text{A}), \text{ \&c.}$$

In the same way we allow the Greek letters θ, ϕ, ψ, &c., to represent the circular measure, and the angles themselves.

The student will have no difficulty in this respect, when he considers that the *unit* of circular measure is $\frac{180°}{\pi}$.

20. *When* $\theta = 0$, *the limit of each of the ratios* $\frac{\sin\theta}{\theta}$, *and* $\frac{\tan\theta}{\theta}$ *is unity.*



Let OA be an initial line, and AOP any angle, the arc corresponding to which is AP.

Make an angle AOP′ equal to AOP.

Join PNP′; it will cut OA at right angles in N.

Draw the line TT′ touching the circle at A, and meeting OP and OP′ produced in T and T′.

Then it is easily seen that PN = NP′, and TA = AT′ in magnitude.

And further,

PNP′, arc PAP′, TAT′ are in ascending order of magnitude.

But $\quad PNP' = 2\,PN = 2r\sin\theta,$

$\quad arc\,PAP' = 2\,arc\,PA = 2r\theta,$

and $\quad TAT' = 2\,TA = 2r\tan\theta.$

Hence,

$2r\sin\theta,\ 2r\theta,\ 2r\tan\theta$ are in ascending order of magnitude.

$\therefore\ \sin\theta,\ \theta,\ \tan\theta$ „ „ ..... (1).

Hence also,

$\frac{\sin\theta}{\sin\theta}, \frac{\sin\theta}{\theta}, \frac{\sin\theta}{\tan\theta}$, are in descending order;

or, $\quad 1, \frac{\sin\theta}{\theta}, \cos\theta$, in descending order.

But (Vol. I., page 345) the limit of $\cos\theta$ is 1, when $\theta = 0$.

Hence the limit of $\frac{\sin\theta}{\theta} = 1$, when $\theta = 0$,

and $\frac{\tan\theta}{\theta} = \frac{1}{\cos\theta} \cdot \frac{\sin\theta}{\theta} = 1 \times 1 = 1$, when $\theta = 0$.

Cor. Hence, when θ is small, we have,

$$\left.\begin{aligned} \sin\theta &= \theta \\ \text{and} \quad \tan\theta &= \theta \end{aligned}\right\} \text{very nearly.}$$

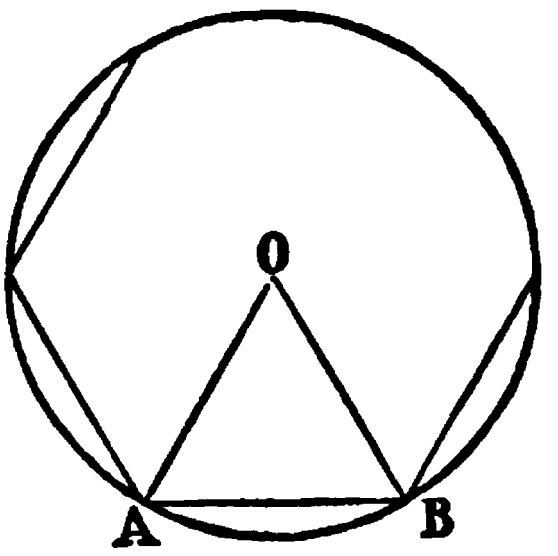
And further, when θ is very small,

$$\frac{\sin n\theta}{\sin\theta} = \frac{\sin n\theta}{n\theta} \cdot \frac{n\theta}{\sin\theta} = n \cdot \frac{\sin n\theta}{n\theta} \div \frac{\sin\theta}{\theta} = n.$$

21. *To find the areas of a regular polygon and a circle.*

Let AB be the side of a regular polygon of n sides;

And let O be the centre of the circumscribing circle, whose radius is r, suppose.



Then, since the angles about a point are together equal to 2π, we have,

$$\angle \text{AOB} = \frac{2\pi}{n}.$$

$$\therefore \text{ area of } \Delta \text{ AOB} = \tfrac{1}{2} \text{AO} \cdot \text{OB} \sin\frac{2\pi}{n} = \frac{r^2}{2} \sin\frac{2\pi}{n}.$$

And since the polygon consists of n such triangles, we therefore have

$$\textit{area of regular polygon of } \text{n } \textit{sides} = \frac{nr^2}{2} \sin\frac{2\pi}{n} \ldots\ldots (1).$$

To find the area of a circle whose radius is r, we may conceive the circle to be *the limit of the regular* polygon of n sides, when n is indefinitely increased.

We then have, *when* n *is infinite,*
area of circle whose radius is r

$$= \frac{nr^2}{2} \cdot \frac{2\pi}{n} \cdot \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}; \text{ or, since } \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1, \text{ when } n = \infty,$$

$$= \frac{nr^2}{2} \cdot \frac{2\pi}{n} \cdot 1,$$

$$= r^2\pi \quad \ldots\ldots\ldots\

23. *To find a general expression for all angles having the same cosine; that is, to show that, when* n *is integral,*

$$\cos\theta = \cos(2n\pi \pm \theta).$$

We have $\cos\theta = \cos(\pm\theta)$;

And we may increase or diminish the angle $(\pm\theta)$ by any multiple of 2π, without altering the value of the cosine.

Hence, adding $2n\pi$, where n is any integer, we have

$$\cos\theta = \cos(2n\pi \pm \theta).$$

24. *To show that* $\tan\theta = \tan(n\pi + \theta)$, *where* n *is any integer.*

We have $\tan\theta = \tan(\pi + \theta)$;

Hence it follows that the tangent of an angle is not altered by *adding* π to the angle;

And since we may write the above equation thus:

$$\tan(\pi + \theta) = \tan\theta,$$

it follows that the tangent is unaltered, if we *subtract* π from the angle.

Hence we conclude that *any multiple of* π may be added or subtracted without changing the value of the tangent.

We therefore have

$$\tan\theta = \tan(n\pi + \theta),$$

where n is any integer.

COR. We may easily gather from the results of the last three articles, that

$$\cot\theta = \cot(n\pi + \theta),$$

$$\sec\theta = \sec(2n\pi \pm \theta),$$

$$\operatorname{cosec}\theta = \operatorname{cosec}\{n\pi + (-1)^n\theta\}.$$

It would be a useful exercise for the student to illustrate these *results geometrically.*

Ex. II.

1. Find the circular measure of 1°, 1^r, $1'$, $1''$.

2. The circular measure of the sum of two angles is $\frac{5}{12}\pi$, and the greater angle exceeds the less by 17°. Find the angles.

3. Find the length of any arc of 45° in a circle whose radius is 100 feet.

4. Show that the radius of a circle is 35 feet, when the length of any arc of 60° is $36\frac{2}{3}$.

5. The semi-diameter of a disc whose distance is d, is found to be $10'$; show that the area of the disc is $\frac{d^2\pi^3}{(1080)^2}$ nearly.

6. Show that $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$; and hence, prove that

$$\sin 10'' = \cdot 000048481368\ldots.$$

7. Prove that at a distance of 3 miles, a length of 5 feet will subtend an angle of $1''$ nearly.

8. If h be the height of a mountain, r the radius of the earth, and d the distance which a person can see from the mountain top; prove that $d = \sqrt{2rh}$ nearly.

Solve the following equations:

9. $\text{Sin}\,\theta + \cos\theta = \sqrt{2}$.

10. $\text{Sin}\,\theta + \sin 3\theta = \sin 2\theta$.

11. $\text{Sin}\,5\theta - \sin 3\theta = \sin 4\theta$.

12. $\text{Sin}\,m\theta = \cos n\theta$.

13. $\text{Cos}\,\theta - \cos 2\theta = 1$.

14. $\text{Tan}\,5\theta \pm \tan 2\theta = 0$.

15. When the tangent of half an angle is found from the *tangent of the* whole angle, it has two values. Why is this?

16. Show, if α be one value of θ which satisfies the equation $\sin\theta = \sin\alpha$, that the three values of $\sin\frac{\theta}{3}$ will be

$$\sin\frac{\alpha}{3},\ \sin\left(\frac{\pi}{3}-\frac{\alpha}{3}\right),\ -\sin\left(\frac{\pi}{3}+\frac{\alpha}{3}\right).$$

17. Having given that $2\sin\frac{\theta}{2} = \sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$, where either sign of the radicals may be taken, explain the meaning of the four values of $\sin\frac{\theta}{2}$ thus obtained, by means of the formula

$$\sin\theta = \sin\left\{n\pi + (-1)^n\theta\right\}.$$

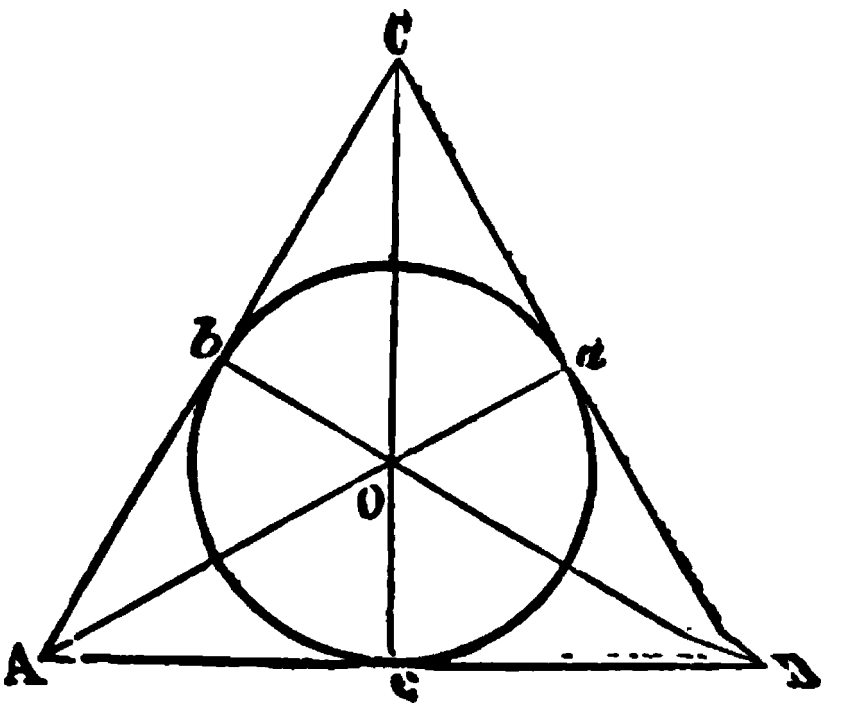
18. A sector, whose angle is θ, is cut from a circle, and formed into a cone. Show that the vertical angle of the cone is $\sin^{-1}\frac{\theta}{2\pi}$.

CHAPTER III.

INSCRIBED, ESCRIBED, AND CIRCUMSCRIBED CIRCLES OF A TRIANGLE. QUADRILATERALS.

The Inscribed Circle.

25. Let O be the centre of the inscribed circle of the triangle ABC, and let the circle touch the sides in a, b, c.



Then Oa, Ob, Oc are perpendicular to the sides, and

$$Oa = Ob = Oc = r.$$

Let S represent the area of the Δ *ABC*.

Then,

$$S = \Delta BOC + \Delta AOC + \Delta AOB$$

$$= \tfrac{1}{2} ar + \tfrac{1}{2} br + \tfrac{1}{2} cr = \tfrac{1}{2} (a + b + c) r = sr.$$

$$\therefore r = \frac{S}{s} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}} \ldots\ldots\ldots\ldots (1).$$

Since $S = \tfrac{1}{2} bc \sin A$, and $s = \tfrac{1}{2} (a + b + c)$.

We may evidently also write,

$$r = \frac{bc}{a + b + c} \sin A = \frac{ac}{a + b + c} \sin B$$

$$= \frac{ab}{a + b + c} \sin C \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Cor. 1. Since the angles A, B, C are bisected by the lines AO, BO, CO respectively, we have

$$\frac{r}{AO} = \sin \tfrac{1}{2} A \text{ or } AO = \frac{r}{\sin \frac{1}{2} A} = \frac{2r}{\sin A} \cos \tfrac{1}{2} A;$$

or, by (2),

$$AO = \frac{2bc}{a + b + c} \cos \tfrac{1}{2} A; \text{ and so}$$

$$BO = \frac{2ac}{a + b + c} \cos \tfrac{1}{2} B$$

$$CO = \frac{2ab}{a + b + c} \cos \tfrac{1}{2} C.$$

Cor. 2. Since $Ab = Ac$, $Bc = Ba$, $Ca = Cb$, we have

$$2\,Ab + 2\,Bc + 2\,Ca = a + b + c = 2s, \text{ or}$$

$$Ab + Bc + Ca = s, \text{ or } Ab + Bc + Cb = s;$$

$$\text{or } b + Bc = s; \ \therefore Bc = Ba = s - b.$$

And so,

$$Ab = Ac = s - a, \text{ and } Ca = Cb = s - c.$$

The Escribed Circles.

26. An escribed circle is one which touches one side of a triangle, and two others produced. Every triangle has therefore the escribed circles.

We shall denote those which touch the sides a, b, c respectively by r_a, r_b, r_c.

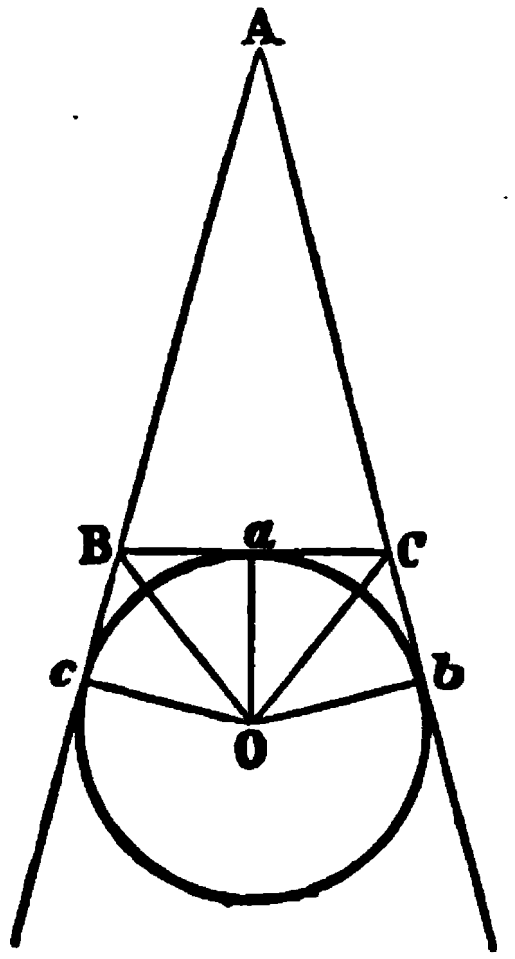
Let the sides AB, AC of the Δ ABC be produced. Bisect the exterior angles at B and C by lines meeting in O.

Then it may be easily shown that O is the centre of a circle touching the side BC and the sides AB, AC produced.

Let it touch these lines respectively in a, b, c.

Then we have Oa, Ob, Oc perpendiculars to BC, AC, AB respectively, and

$$Oa = Ob = Oc = r_a.$$



$$\begin{aligned}\text{Now } S &= \Delta ABC \\ &= \Delta AOC + \Delta AOB - \Delta BOC \\ &= \tfrac{1}{2} r_a b + \tfrac{1}{2} r_a c - \tfrac{1}{2} r_a a = \tfrac{1}{2}(b + c - a) r_a \\ &= (s - a) r_a.\end{aligned}$$

$$\therefore r_a = \frac{S}{s - a} \quad \ldots\ldots(1),$$

$$\text{and so } r_b = \frac{S}{s - b} \quad \ldots\ldots(2),$$

$$r_c = \frac{S}{s - c} \quad \ldots\ldots(3).$$

Cor. Hence,

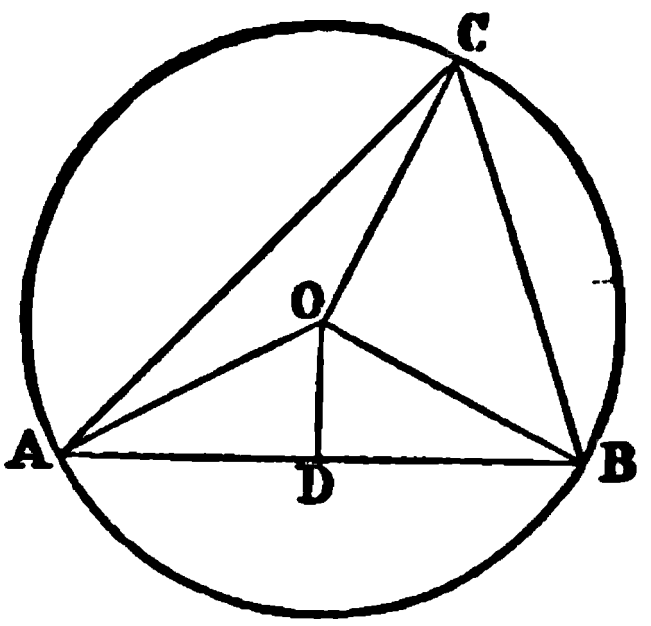
$$\text{(i.) } \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{(s - a) + (s - b) + (s - c)}{S} = \frac{s}{S} = \frac{1}{r}.$$

$$\text{(ii.) } r r_a r_b r_c = \frac{S}{s} \cdot \frac{S}{s - a} \cdot \frac{S}{s - b} \cdot \frac{S}{s - c} = \frac{S^4}{S^2} = S^2.$$

$$\therefore S = \sqrt{r r_a r_b r_c}.$$

The Circumscribed Circle.

27. Let O be the centre of the circumscribed circle.



Then AO = BO = CO = R, suppose.

Now (Euc. III. 20),

$$\angle AOB = 2 \angle ACB = 2C.$$

Draw OD perpendicular to AB, then evidently

$$\angle AOD = \tfrac{1}{2} AOB = C.$$

And $AD = \frac{1}{2} AB = \frac{1}{2} c$.

Now $AO = \dfrac{AD}{\sin AOD} = \dfrac{\frac{1}{2} c}{\sin C}$, or

$$R = \frac{C}{2 \sin B}.$$ Hence we may write

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

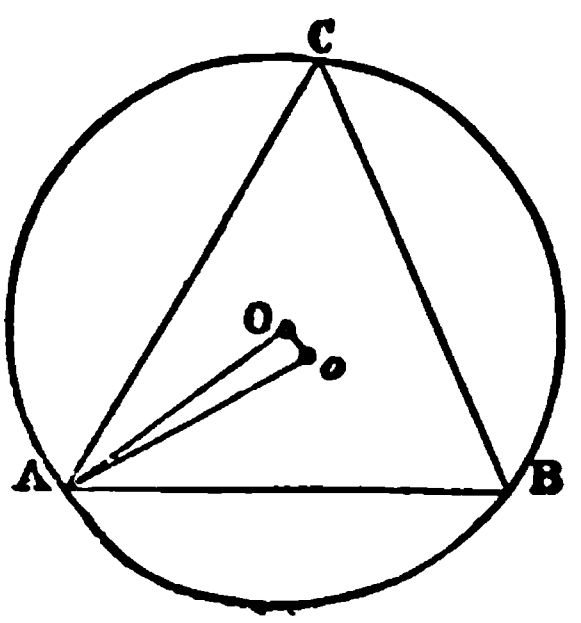
And since $\dfrac{a}{2 \sin A} = \dfrac{abc}{4 \cdot \frac{1}{2} bc \sin A} = \dfrac{abc}{4S}$, we also have

$$R = \frac{abc}{4S} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \quad \ldots\ldots\ldots (2).$$

Cor. Since (Art. 24) $r = \dfrac{S}{s}$, we have

$$Rr = \frac{abc}{4S} \cdot \frac{S}{s} = \frac{abc}{4s} = \frac{abc}{2(a+b+c)}.$$

28. *The distance between the centres of the circumscribed and inscribed circles is* $\sqrt{R^2 - 2Rr}$.



Let O be the centre of the circumscribed circle, and o the centre of the inscribed circle.

Join AO, Ao, Oo, then

$Oo^2 =$

$AO^2 + Ao^2 - 2\,AO \cdot Ao \cdot \cos OAo \ldots (1).$

Now AO = R; and, by Art. 25, Cor.,

$$Ao = \frac{r}{\sin \frac{1}{2} A}.$$

Also, $\angle OAo = \angle OAB - \angle oAB = (90° - C) - \frac{1}{2} A$
$= \frac{1}{2}(B - C)$.

Hence, from (1),

$$Oo^2 = R^2 + \frac{r^2}{\sin^2 \frac{1}{2} A} - \frac{2Rr}{\sin \frac{1}{2} A} \cos \frac{1}{2}(B - C) \ldots\ldots (2).$$

But $\frac{r}{\sin \frac{1}{2} A} = \frac{S}{s} \cdot \sqrt{\frac{bc}{(s-b)(s-c)}} = \sqrt{bc \cdot \frac{s-a}{s}}$,

and $\cos \frac{1}{2}(B - C)$
$= \cos \frac{1}{2} B \cdot \cos \frac{1}{2} C + \sin \frac{1}{2} B \cdot \sin \frac{1}{2} C$

$$= \sqrt{\frac{s(s-b)}{ac}} \cdot \sqrt{\frac{s(s-c)}{ab}} + \sqrt{\frac{(s-a)(s-c)}{ac}} \cdot \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \left(\frac{s}{a} + \frac{s-a}{a}\right) \sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{2s-a}{a} \sin \frac{1}{2} A;$$

$$\therefore \frac{\cos \frac{1}{2}(B - C)}{\sin \frac{1}{2} A} = \frac{2s - a}{a}.$$

Also (Art. 27, Cor.) $Rr = \frac{abc}{4s}$.

Hence

$$Oo^2 = R^2 + bc \cdot \frac{s-a}{s} - 2 \cdot \frac{abc}{4s} \cdot \frac{2s-a}{a}$$

$$= R^2 - \frac{abc}{2s}, \text{ or, by Art. 27, Cor.,}$$

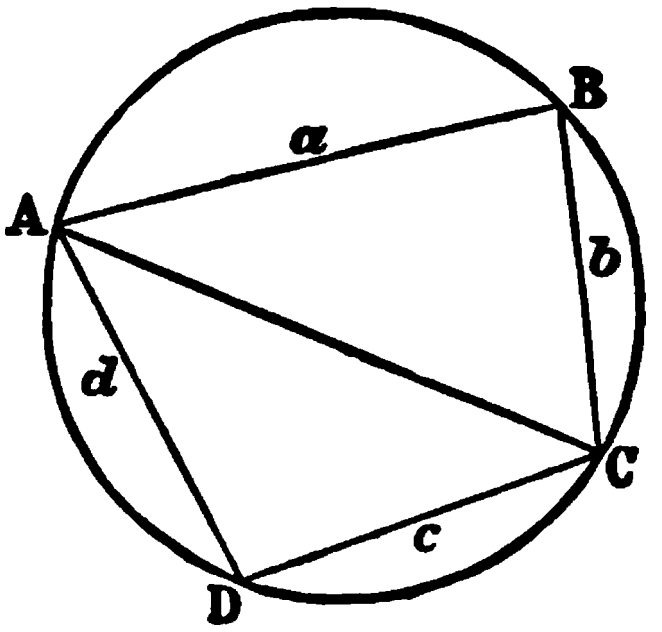
$$Oo^2 = R^2 - 2Rr.$$

COR. Similarly may we show that the distances of the centres of the escribed circles from the centre of the circumscribed circle are respectively

$$\sqrt{R^2 + 2Rr_a}, \sqrt{R^2 + 2Rr_b}, \sqrt{R^2 + 2Rr_c}.$$

Quadrilaterals.

29. *To find the area of a quadrilateral which can be inscribed in a circle.*



Let ABCD be the quadrilateral;

Then (Euc. III., 22),

$$A + C = 180°,$$

$$\text{and } B + D = 180°.$$

Let us represent the sides AB, BC, CD, DA respectively by a, b, c, d.

Join AC.

Then area ABCD $= \Delta ABC + \Delta ADC$

$$= \tfrac{1}{2} ab \sin B + \tfrac{1}{2} cd \sin D.$$

But $\sin D = \sin (180° - B) = \sin B.$

Hence area ABCD $= \tfrac{1}{2} (ab + cd) \sin B$ ..... (1).

Now, from the triangle ABC,

$$a^2 + b^2 - AC^2 = 2 ab \cos B \text{..... (2).}$$

And from the triangle ADC,

$$c^2 + d^2 - AC^2 = 2 cd \cos D$$

$$= - 2 cd \cos B \text{..... (3).}$$

(2) − (3) then

$$a^2 + b^2 - c^2 - d^2 = 2 (ab + cd) \cos B;$$

$$\therefore \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2 (ab + cd)}.$$

Hence $\sin^2 B$

$= 1 - \dfrac{(a^2 + b^2 - c^2 - d^2)^2}{4 (ab + cd)^2}$, or, arranging,

$$= \frac{(b + c + d - a) (a + c + d - b) (a + b + d - c) (a + b + c - d)}{4 (ab + cd)^2}.$$

Let $2s = a + b + c + d$, we then easily get

$$\sin^2 B = \frac{4(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2}, \text{ or}$$

$$\sin B = \frac{2}{ab+cd}\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Hence, substituting in (1), we have

$$\text{Area ABCD} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

COR. 1. Let $d = 0$, that is, let the quadrilateral become the Δ ABC.

Then $S = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.

COR. 2. We may find the length of the diagonal AC thus:

From (2)

$$AC^2 = a^2 + b^2 - 2ab\cos B$$

$$= a^2 + b^2 - 2ab \cdot \frac{a^2+b^2-c^2-d^2}{2(ab+cd)};$$

or, simplifying,

$$= \frac{(ac+bd)(ad+bc)}{ab+cd}.$$

$$\therefore AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}.$$

And so,

$$BD = \sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

Hence, $AC \cdot BD = ac + bd$.

COR. 3. We also easily find

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-a)(s-d)}{ad+bc}} = \cos \tfrac{1}{2}C,$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{ad+bc}} = \sin \tfrac{1}{2}C,$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}} = \cot \tfrac{1}{2}C.$$

Cor. 4. If R be the radius of the circumscribing circle, then, since the circle circumscribes the Δ ABC,

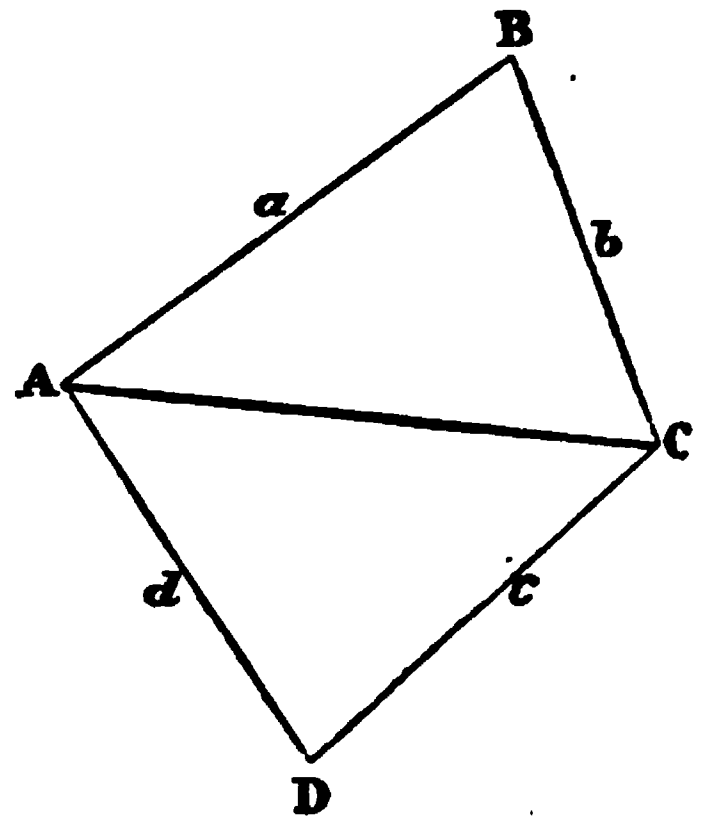
$$R = \frac{AC}{2 \sin B}; \text{ or, by substitution and reduction,}$$

$$R = \tfrac{1}{4} \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(s-a)(s-b)(s-c)(s-d)}}.$$

30. *To express the area of any quadrilateral in terms of its sides, and two of its opposite angles.*

We have,

$$S = \Delta ABC + \Delta ADC$$
$$= \tfrac{1}{2}(ab \sin B + cd \sin D) \ldots\ldots\ldots\ldots\ldots\ldots (1).$$



Now $\cos B = \dfrac{a^2 + b^2 - AC^2}{2ab}$, and $\cos D = \dfrac{c^2 + d^2 - AC^2}{2cd}$.

From these we easily get,

$$2ab \cos B - 2cd \cos D = a^2 + b^2 - c^2 - d^2; \text{ or,}$$

$$2ab\left(2\cos^2\frac{B}{2} - 1\right) - 2cd\left(1 - 2\sin^2\frac{D}{2}\right) = a^2 + b^2 - c^2 - d^2;$$

or, arranging, &c.,

$$(a + b)^2 - (c - d)^2 = 4ab\cos^2\frac{B}{2} + 4cd\sin^2\frac{D}{2} \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and so

$$(c + d)^2 - (a - b)^2 = 4\,ab \sin^2 \frac{B}{2} + 4\,cd \cos^2 \frac{D}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(2) × (3), then, putting $2s = a + b + c + d$, we get

$$(s-a)(s-b)(s-c)(s-d) = a^2b^2 \sin^2 \frac{B}{2} \cos^2 \frac{B}{2} + c^2d^2 \sin^2 \frac{D}{2} \cos^2 \frac{D}{2} + abcd \left(\sin^2 \frac{B}{2} \sin^2 \frac{D}{2} + \cos^2 \frac{B}{2} \cos^2 \frac{D}{2} \right)$$

$$= \left(ab \sin \frac{B}{2} \cos \frac{B}{2} + cd \sin \frac{D}{2} \cos \frac{D}{2} \right)^2 + abcd \left(\cos \frac{B}{2} \cos \frac{D}{2} - \sin \frac{B}{2} \sin \frac{D}{2} \right)^2$$

$$= \tfrac{1}{4} (ab \sin B + cd \sin D)^2 + abcd \cos^2 \tfrac{1}{2} (B + D).$$

Hence, substituting from (1),

$$(s - a)(s - b)(s - c)(s - d) = S^2 + abcd \cos^2 \tfrac{1}{2} (B + D).$$

$$\therefore S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \tfrac{1}{2} (B + D). \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

And since $A + B + C + D = 2\pi$, we have $\frac{1}{2} (A + C) = \pi - \frac{1}{2} (B + D)$.

$$\therefore \cos \tfrac{1}{2} (A + C) = - \cos \tfrac{1}{2} (B + D), \text{ or}$$

$$\cos^2 \tfrac{1}{2} (A + C) = \cos^2 \tfrac{1}{2} (B + D).$$

Hence from (4), if 2ω = the sum of either two opposite angles, we have

$$S^2 = (s - a)(s - b)(s - c)(s - d) - abcd \cos^2 \omega.$$

Cor. When the quadrilateral is inscribed in a circle we have

$$2\omega = 180°, \text{ or } \omega = 90°.$$

Hence,

$$S^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2 90°,$$

or $S = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, as in Art. 29.

31. *To find the area of a quadrilateral described about a circle.*

It is easy to show that $a + c = b + d$.

$$\therefore\ s = \tfrac{1}{2}(a + c + a + c) = a + c \text{ or } b + d.$$

$$\therefore\ s - a = c,\ \ s - b = d,\ \ s - c = a,\ \ s - d = b.$$

Hence $(s-a)(s-b)(s-c)(s-d) = abcd$.

Then, by Art. 30,

$$S^2 = abcd - abcd\cos^2\omega = abcd\sin^2\omega.$$

$$\therefore\ S = \sqrt{abcd}\,.\sin\omega,$$

where 2ω is the sum of two opposite angles of the quadrilateral.

CHAPTER IV.

SUBSIDIARY ANGLES.

32. When a trigonometrical formula is not adapted to logarithmic computation, it is usual to introduce what is called a **subsidiary angle** into the expression in order to put it into the form of a product of factors.

Since the value of the *sine* and *cosine* of an angle ranges from zero to unity, it follows that if a be not greater than b, it is perfectly legitimate to assume either

$$\sin\phi = \frac{a}{b}, \text{ or } \cos\phi = \frac{a}{b}.$$

Again, since the value of the *tangent* may range from *zero* to *infinity*, it is always legitimate to assume, for any real values of a and b, that

$$\tan \phi = \frac{a}{b}.$$

Still, as will be seen farther on, certain assumptions, although legitimate, are objectionable in particular cases. We shall not, however, enter upon this subject in the present chapter.

33. *To adapt to logarithmic computation the formula*

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

(i.) We have

$$c^2 = a^2 + b^2 - 2ab\left(2\cos^2\frac{C}{2} - 1\right)$$

$$= (a + b)^2 - 4ab\cos^2\frac{A}{2}$$

$$= (a + b)^2\left\{1 - \frac{4ab}{(a + b)^2}\cos^2\frac{A}{2}\right\} \ldots\ldots (1).$$

Now *assume*

$$\sin^2\theta = \frac{4ab}{(a + b)^2}\cos^2\frac{A}{2} \ldots\ldots\ldots\ldots\ldots (2).$$

(This assumption is allowable, for $(a + b)^2$ is never less than $4ab$.)

We then have

$$c^2 = (a + b)^2(1 - \sin^2\theta) = (a + b)^2\cos^2\theta.$$

$$\therefore\ c = (a + b)\cos\theta, \text{ the form required} \ldots\ldots\ldots\ldots (3).$$

It will be seen that θ may be obtained from (2) by means of logarithms, and then c from the equation just found.

Thus, from (2), $2\,\mathrm{L}\sin\theta$

$$= 2\log 2 + \log a + \log b - 2\log(a + b) + 2\,\mathrm{L}\cos\frac{A}{2},$$

And from (3),

$$\log c = \log (a + b) + \mathrm{L}\cos \theta - 10.$$

(ii.) *Or thus:*

$$c^2 = a^2 + b^2 - 2ab\left(1 - 2\sin^2 \frac{C}{2}\right)$$

$$= (a - b)^2 + 4ab \sin^2 \frac{C}{2}$$

$$= (a - b)^2 \left\{ 1 + \frac{4ab}{(a - b)^2} \sin^2 \frac{C}{2} \right\} \ldots\ldots\ldots (4).$$

Now $\dfrac{4ab}{(a - b)^2} \sin^2 \dfrac{C}{2}$ may have *any positive* value, and hence there is some angle θ, such that

$$\tan^2 \theta = \frac{4ab}{(a - b)^2} \sin^2 \frac{C}{2} \ldots\ldots\ldots \ldots (5).$$

Then, from (4),

$$c^2 = (a - b)^2 (1 + \tan^2 \theta) = (a - b)^2 \sec^2 \theta;$$

$$\therefore c = (a - b) \sec \theta \ldots\ldots\ldots\ldots\ldots (6).$$

We then have, from (5),

$$2\,\mathrm{L}\tan \theta$$

$$= 2\log 2 + \log a + \log b - 2\log (a - b) + 2\,\mathrm{L}\sin \frac{C}{2};$$

And from (6),

$$\log c = \log (a - b) + \mathrm{L}\sec \theta - 10.$$

(iii.)

$$c^2 = (a^2 + b^2)\left(\cos \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= (a + b)^2 \sin^2 \frac{C}{2} + (a - b)^2 \cos^2 \frac{C}{2}$$

$$= (a + b)^2 \sin^2 \frac{C}{2} \left\{ 1 + \left(\frac{a - b}{a + b}\right)^2 \cot^2 \frac{C}{2} \right\} \ldots\ldots\ldots\ldots (7),$$

$$\text{or} = (a - b)^2 \cos^2 \frac{C}{2} \left\{ 1 + \left(\frac{a + b}{a - b} \right)^2 \tan^2 \frac{C}{2} \right\} \ldots\ldots\ldots\ldots (8).$$

Either of these expressions may now be easily adapted to logarithmic computation.

34. *Adapt* $\sqrt{a + b} + \sqrt{a - b}$ *to logarithmic computation.*

Since b is not greater than a, we may assume $\cos \phi = \frac{b}{a}$.

Then

$$\sqrt{a + b} + \sqrt{a - b} = \sqrt{a} \left\{ \sqrt{1 + \cos \phi} + \sqrt{1 - \cos \phi} \right\}$$

$$= \sqrt{a} \left\{ \sqrt{2} \cos \frac{\phi}{2} + \sqrt{2} \sin \frac{\phi}{2} \right\} = 2 \sqrt{a} \sin \left(45^\circ + \frac{\phi}{2} \right).$$

35. *Given the formula*

$$(\cos \delta' \cos \alpha - \cos \delta) \cos h - \cos \delta' . \sin \alpha . \sin h$$
$$= \tan l \, (\sin \delta - \sin \delta'),$$

to find h *in a form adapted to logarithmic computation.*

We have

$$\cos \delta' \sin \alpha \left\{ \frac{\cos \delta' \cos \alpha - \cos \delta}{\cos \delta' \sin \alpha} \cos h - \sin h \right\}$$
$$= \tan l \, (\sin \delta - \sin \delta').$$

$$\text{Put } \cot \phi = \frac{\cos \delta' \cos \alpha - \cos \delta}{\cos \delta' \sin \alpha} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

then

$\cos \delta' \sin \alpha \, (\cot \phi \cos h - \sin h) = \tan l \, (\sin \delta - \sin \delta')$; or

$$\cos \delta' \sin \alpha . \frac{\cos (\phi + h)}{\sin \phi}$$
$$= \tan l \, . \, 2 \cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} (\delta - \delta').$$

$$\therefore \cos (\phi + h) = \frac{2 \sin \phi . \tan l}{\cos \delta' \sin \alpha} \cos \tfrac{1}{2} (\delta + \delta') \sin \tfrac{1}{2} (\delta - \delta'),$$

the form required.

Ex. III.

Miscellaneous Exercises.

1. Adapt $a + b + c + d +$ &c., to logarithmic computation.

2. Solve the equation $\sin 5\theta = \cos 2\theta$.

3. If t_r represent the sum of the products of $\tan \alpha$, $\tan \beta$, $\tan \gamma$, &c., taken r together, show that, when there are n angles,

$$\tan(\alpha + \beta + \gamma + \ldots \kappa)$$

$$= \frac{t_1 - t_3 + t_5 - \ldots . (-1)^{\frac{n}{2}-1} t_{n-1}}{1 - t_2 + t_4 - \ldots . (-1)^{\frac{n}{2}} t_n}, \; n \text{ even};$$

$$= \frac{t_1 - t_3 + t_5 - \ldots . (-1)^{\frac{n-1}{2}} t_n}{1 - t_2 + t_4 - \ldots . (-1)^{\frac{n-1}{2}} t_{n-1}}, \; n \text{ odd}.$$

4. In any triangle

$$(b+c)\sin\tfrac{1}{2}A \sin\tfrac{1}{2}(B-C) = (b-c)\cos\tfrac{1}{2}A \cos\tfrac{1}{2}(B-C).$$

5. Show that

$$\frac{\sin 3A - \cos 3A}{\sin 3A + \cos 3A} = \frac{2\sin 2A - 1}{2\sin 2A + 1}\tan(45° - A).$$

6. In any triangle the perpendiculars from the angles upon the opposite side are inversely proportional to the sides.

7. If $\tan 2\alpha = \frac{pr + qs}{pq + rs}$, and $\tan 2\beta = \frac{pr - qs}{pq + rs}$;

then $\tan(\alpha - \beta) = \frac{s}{p}$ or $-\frac{p}{s}$,

and $\tan(\alpha + \beta) = \frac{r}{q}$ or $-\frac{q}{r}$.

8. If $\tan u = \tan a \cos b$, and $\tan v = \tan b \cos a$,

then $\sec(u \pm v) = \sec a \sec b \pm \tan a \tan b$,

and $\tan(u \pm v) = \tan a \sec b \pm \sec a \tan b$.

9. If D be the middle point of BC in the triangle ABC, DE be drawn perpendicular to AC, and EF perpendicular to BC, then $C = \frac{1}{2} \sin^{-1} \frac{4\,EF}{a}$.

10. The lines joining the angles of a triangle to the middle points of the opposite sides, meet in a point.

11. Show that

(i.) $\log \sec 2\theta = 2\left\{\tan^2\theta + \frac{1}{3}\tan^6\theta + \frac{1}{5}\tan^{10}\theta + \&c\right\}$.

(ii.) $2 \log \sec \theta = \tan^2\theta - \frac{1}{2}\tan^4\theta + \frac{1}{3}\tan^6\theta - \&c.$

12. Show that

(i.) $\tan\theta + \cfrac{1}{2\tan\theta + \cfrac{1}{2\tan\theta + \ddots}} = \sec\theta.$

(ii.) $2\cot\theta + \cfrac{1}{2\cot\theta + \cfrac{1}{2\cot\theta + \ddots}} = \cot\frac{\theta}{2}.$

13. If the sides a, b, c of a triangle be in A. P., show that the area is to that of an equilateral triangle having the same perimeter as $\sqrt{1 - \frac{4(a-b)^2}{b^2}}$ to unity.

14. The area of a sector of a circle is half the product of the radius into the length of the arc.

15. Given that

$$a^2 \cos\alpha \cos\beta + a(\sin\alpha + \sin\beta) + 1 = 0,$$
$$a^2 \cos\alpha \cos\gamma + a(\sin\alpha + \sin\gamma) + 1 = 0,$$

show that

$$a^2 \cos\beta \cos\gamma + a(\sin\beta + \sin\gamma) + 1 = 0,$$

and $\cos\alpha + \cos\beta + \cos\gamma = \cos(\alpha + \beta + \gamma)$,

where β and γ are unequal and less than π.

16. Show that

$$\frac{1}{a + b\cos x} = \frac{\sec^2 \frac{1}{2}x}{(a+b) + (a-b)\tan^2 \frac{1}{2}x}.$$

17. If a point O be taken either within or without a triangle ABC then $\frac{\sin ABO \cdot \sin BCO \cdot \sin CAO}{\sin CBO \cdot \sin ACO \cdot \sin BAO} = 1.$

13. In any triangle ABC, show that

$$\tan \tfrac{1}{2} A \,.\, \tan \tfrac{1}{2} B \,.\, \tan \tfrac{1}{2} C = \frac{S}{s^2}.$$

19. If l be the length of the line drawn from the angle A to the opposite side so as to bisect the angle, then

$$\cos \tfrac{1}{2} A = \frac{l}{2}\left(\frac{1}{b} + \frac{1}{c}\right).$$

20. If p, q, r be the perpendiculars upon the sides of a triangle from the opposite angles, show that

$$\frac{a^2b^2c^2}{b^3c^3 + a^3c^3 + a^3b^3} = \frac{pqr}{p^3 + q^3 + r^3}.$$

21. If $\cos(a - b)\cos(b - c)\cos(c - a) = \frac{1}{4}$, then

$$\cos 3(a - b) \,.\, \cos 3(b - c) \,.\, \cos 3(c - a)$$
$$= 3\cos 2(a - b) \,.\, \cos 2(b - c) \,.\, \cos 2(c - a) + \tfrac{1}{4}.$$

22. If a point O be taken within or without an equilateral triangle ABC, and upon BO an equilateral triangle OBD be described so that the angles ABO and CBD are equal, then will

$$\cos \mathrm{DOC} = \frac{\mathrm{BO}^2 + \mathrm{CO}^2 - \mathrm{AO}^2}{2\,\mathrm{BO} \,.\, \mathrm{CO}}.$$

23. In the ambiguous case where a, b, A are given, and $b > a$, show that if C_1, C_2 are the values of the third angle, and c_1, c_2 the values of the third side, then

(i.) $C_1 + C_2 = 2\tan^{-1}(\cot A)$.

(ii.) The distance between the centres of the circumscribing circles of the triangles $= \dfrac{c_1 + c_2}{2\sin A}$.

24. If the centre of gravity of a triangle be joined to the angular points, and R_a, R_b, R_c be respectively the radii of the circles circumscribing the triangles thus formed, then

$$R_a^2 R_b^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + R_b^2 R_c^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right) + R_c^2 R_a^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right) = 0.$$

25. The area of the triangle formed by joining the centres of three circles of radii r_1, r_2, r_3 which touch each other *externally is* $\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}$.

26. If the middle points of the sides of a triangle be joined to the opposite angles, and r_1, r_2, r_3, &c., be the radii of the circles inscribed in the triangles thus formed, and R_1, R_2, R_3, &c., the radii of the circumscribed circles,

$$\text{then} \qquad R_1 R_2 R_5 = R_2 R_4 R_6,$$

$$\text{and} \qquad \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6}.$$

27. If r be the radius of the inscribed, and r_a, r_b, r_c those of the escribed circles, then $\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$.

28. If R be the radius of the circumscribed circle, show that

$$r_a + r_b + r_c - r = 4R = \frac{(r_a + r_b)(r_b + r_c)(r_c + r_a)}{r_a r_b + r_b r_c + r_c r_a}.$$

29. If p_a, p_b, p_c be the perpendiculars upon the sides a, b, c, then $\frac{1}{r_a} = \frac{1}{p_b} + \frac{1}{p_c} - \frac{1}{p_a}$.

30. If a point P be taken either within or without a triangle, and perpendiculars Pa, Pb, Pc be drawn upon the sides, show that the area of the triangle formed by joining the centres of the circles circumscribing the triangle Pab, Pac, Pbc $= \frac{1}{4}$ S.

31. Two circles have a common radius r, and a circle is described touching this radius, and the two circles; show that the radius of the circle touching the three circles is $\frac{3}{56} r$.

32. In any triangle, if $\frac{\tan C}{\tan B} = \frac{\tan B}{\tan A}$, show that each of these quantities $= \frac{\sin 2A}{\sin 2C}$.

33. If $\tan \frac{1}{2}\alpha = \tan^3 \frac{1}{2}\gamma$, and $\tan \gamma = 2 \tan \beta$, show that α, β, γ are in A. P.

34. Show, by the binomial theorem, that

$$(1 + \tfrac{1}{2}\sin\theta + \tfrac{3}{8}\sin^2\theta + \&c.)(1 - \tfrac{1}{2}\sin\theta + \tfrac{3}{8}\sin^2\theta - \&c.) = \sec\theta.$$

35. If a, β, γ be the distances between the centres of the escribed circles respectively, and a_1, β_1, γ_1, be the distances between the centre of the inscribed circle and the centres of the escribed, then $\frac{a}{a_1} = \cot\frac{A}{2}$, $\frac{\beta}{\beta_1} = \cot\frac{B}{2}$, $\frac{\gamma}{\gamma_1} = \cot\frac{C}{2}$.

36. Show that the area of the inscribed circle is to the area of the triangle as $\pi : \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

37. The area of a triangle $= \sqrt{rr_a r_b r_c}$.

38. If $\text{vers}^{-1}\frac{x}{a} - \text{vers}^{-1}\frac{bx}{a} = \text{vers}^{-1}(1-b)$;

$$\text{then } \frac{x}{a} = \pm\sqrt{\frac{2b}{1+b}}.$$

39. Show that

$$4R = \frac{(r_a+r_b)(r_a+r_c)(r_b+r_c)}{r_ar_b + r_ar_c + r_br_c} = \frac{a\beta\gamma}{\sqrt{\sigma(\sigma-a)(\sigma-\beta)(\sigma-\gamma)}},$$

when $2\sigma = a + \beta + \gamma$.

40. Three circles touch each other externally whose radii are a, b, c. Show that the tangents at the points of contact meet in a point, and that its distance from each of them is

$$\left(\frac{abc}{a+b+c}\right)^{\frac{1}{2}}.$$

CHAPTER V.

HEIGHTS AND DISTANCES.

36. In the first volume of this work we have explained at some length the methods to be followed in the simpler cases of heights and distances. We shall work out an example or two involving more complicated considerations as specimens of the general course to be pursued. First, however, we shall explain the Vernier Scale, the Sextant, and the Theodolite.

The Vernier Scale.

37. The vernier scale consists of a small scale CD which slides along the edge of any ordinary graduated scale AB.

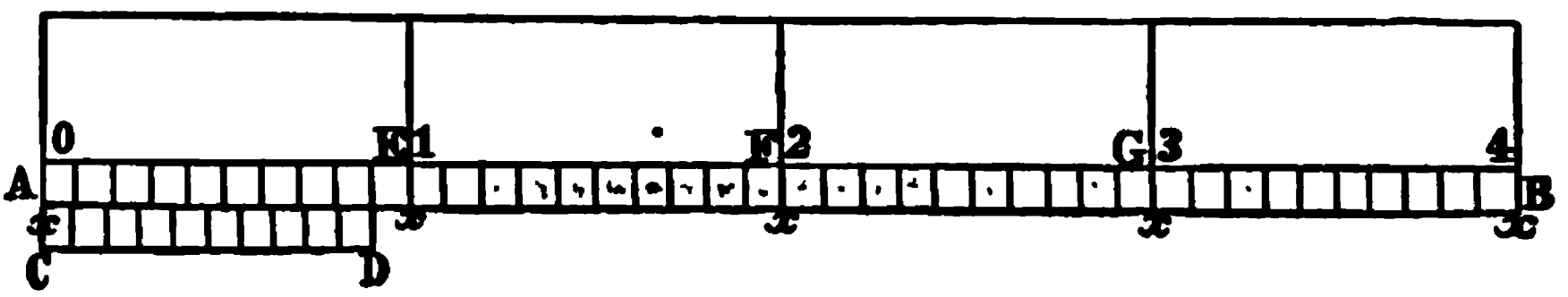


Fig 1.

With a straight scale, the vernier is straight; and with a graduated portion of the arc of a circle, the vernier is a small circular arc.

Suppose the scale AB to be divided into equal divisions (say inches), AE, EF, &c., and each of these divisions again into tenths of an inch. Then it would be possible to express any length in inches and tenths of an inch by the scale AB alone.

Let the length of CD be *nine* of these small divisions, and suppose we wish to measure the exact length of a line *ac* in the diagram below.

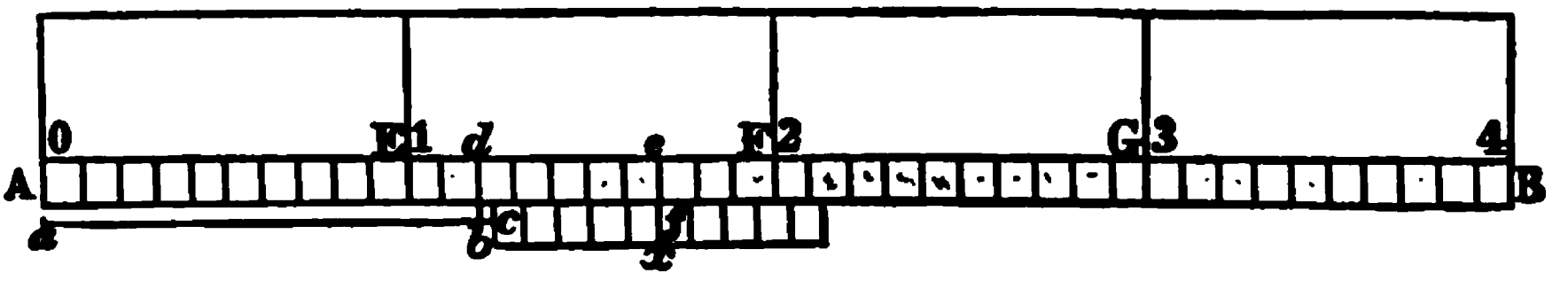


Fig 2.

We see, by the scale AB, that its length is something over *one* inch and *two tenths*. By means of the vernier CD we can express this excess in hundredths.

Let the xth *division* of the vernier correspond with a division of the scale AB, as at f.

$$\text{Now each division of the vernier} = \frac{1}{10} \text{ of } \frac{9}{10} \text{ in.} = \frac{9}{100} \text{ in.}$$

$$\text{Also } bc + cf = de.$$

$$\text{Hence } bc + \frac{9}{100}x = \frac{1}{10}x.$$

$$\therefore bc = \left(\frac{1}{10} - \frac{9}{100}\right)x = \frac{x}{100}.$$

Hence we learn that the excess over the length given by the scale AB is as many hundredths of an inch as corresponds to *the division* of the vernier which concides with a division of the scale.

We have supposed here the divisions of the vernier to be reckoned *from the left.* If, as in some verniers, they are reckoned from the right, we must take 11 divisions of the scale instead of 9, and divide it into 10 equal parts.

In that case, if the xth division correspond with the scale, we have, remembering that the divisions of the vernier are reckoned from the right, and that each division of the vernier

$$= \frac{1}{10} \text{ of } \frac{11}{10} \text{ in.} = \frac{11}{100} \text{ in.},$$

$$bc + (10 - x)\frac{11}{100} = (11 - x)\frac{1}{10}.$$

$$\therefore bc = \frac{x}{100}.$$

In the case of a graduated arc of a circle, where the arc is marked for degrees, then, if 59 divisions of the scale correspond to 60 divisions of the vernier, and the vernier is read from left to right, and if the xth division of the vernier corresponds with a division of the scale, we should learn that the arc ought to be increased to the amount of x'' above the scale reading.

The Sextant.

38. The sextant consists of a graduated arc AB, whose centre is C, formed upon a plate ABC. An arm CA carries a mirror fixed at its centre C in a plane at right angles to the plate. The arm is moveable about an axis through C, carrying the mirror with it, and terminates in an index which moves along the arc AB. When the index points to *zero, the mirror* C and the zero point are in the same *vertical plane.*

D is also a mirror, the lower half only of which is silvered; it is immoveable, and fixed parallel to the position of C, when the index points to zero.

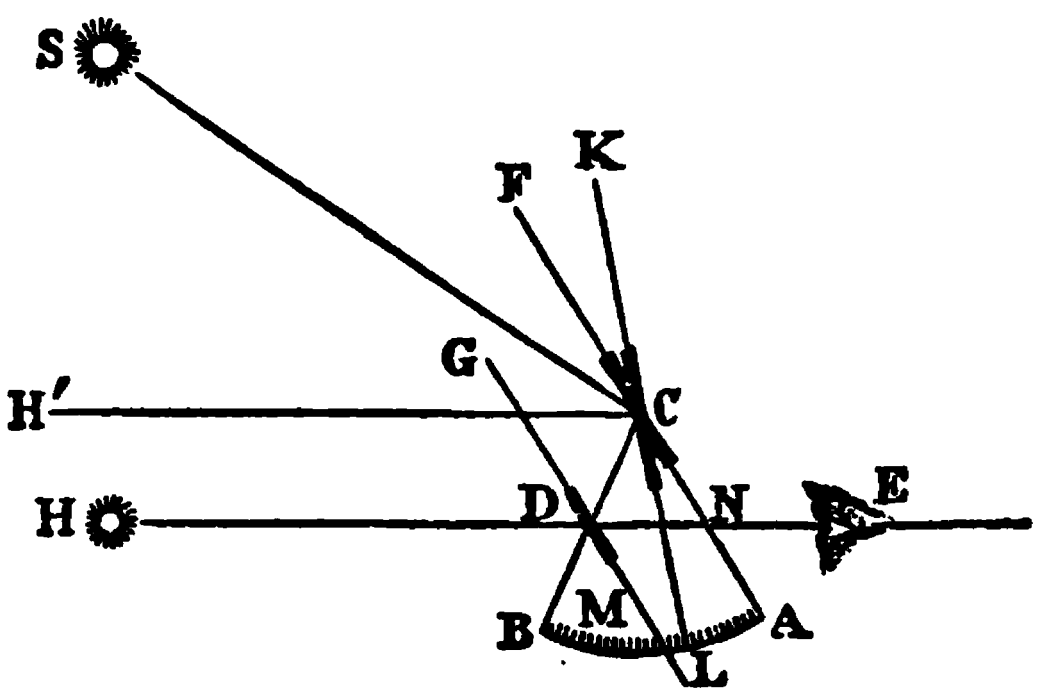


Fig. 3.

Let A be the zero point, and suppose the index to be at A. The instrument may be used either to observe the altitude of a star, or to obtain the angle which two stars subtend at the observer's eye.

Let H and S be two stars. If we require simply the altitude of S, we may suppose H to be a point on the horizon in the same meridian as S.

We shall now explain however the general principles involved in taking the angle subtended by any two stars.

H and S are both so distant that they may be supposed to emit parallel rays.

Let H be observed, by an eye placed at E, by direct vision through the centre D of the lower mirror.

Now hold the instrument so that a pencil of rays H′C parallel to HD may, after impinging upon the centre C of the upper mirror, be reflected sq as to fall upon the lower mirror at D.

Then FCH′ and ACD are respectively *the complements of the angles of incidence and reflection* with respect to the mirror C.

$$\therefore \angle FCH' = \angle ACD,$$

according to the law of reflection.

Again, CDG is the complement of the angle of incidence upon D.

Now, as the mirrors are parallel, $\angle ACD = \angle CDG$.

Hence $\angle FCH' = \angle CDG$.

But since CH′ is parallel to NH, we have also

$$\angle FCH' = \angle CNH = \angle NDM.$$

Hence, $\angle CDG = \angle NDM$.

But CDG is the complement of the angle of incidence of the pencil CD upon D. Hence, by the law of reflection, NDM is the complement of the angle of reflection.

Therefore, the reflected ray will proceed along the line DNE.

Hence the eye will see the object H both by direct vision, and by reflection, when the mirrors are parallel, or when the index points to zero.

Let the arm be now moved until the star S appears by reflection to coincide with H.

Suppose the index to point to L.

Then SCH′ is the angle subtended by the two stars at the eye.

Hence a pencil of rays SC will, after reflection, proceed along CD, and thence along DE.

$$\therefore \angle SCK = \angle DCL,$$

since they are respectively the complements of the angles of incidence and reflection.

$$\text{Now, } \angle SCK = \angle FCH' + \angle FCK - \angle SCH'$$
$$= \angle FCH' + \angle ACL - \angle SCH'.$$
$$\text{And } \angle DCL = \angle DCA - \angle ACL$$
$$= \angle FCH' - \angle ACL.$$

Then, equating these, we have

$$\angle FCH' - \angle ACL = \angle FCH' + \angle ACL - \angle SCH'.$$
$$\therefore \angle SCH' = 2\angle ACL.$$

Hence the angle subtended by the two stars is *twice* the angle through which the index has been turned.

It is usual to graduate the arc AB by dividing it into *twice the number of degrees* it actually contains, so that the reading from the instrument gives the *true value* of the angle required.

A vernier is attached to the graduated arc to enable the *observer to* take the reading with greater exactness.

The Theodolite.

39. This instrument is used for taking the angles of elevation of objects, and the horizontal angle between the objects.

It consists of a vertical circle carrying a telescope, whose axis is in the plane of the vertical circle, and moving about a horizontal axis, so that the line of vision can be directed to any point in the plane of the vertical circle; and also, of a horizontal circle carrying the vertical circle and telescope, and moveable in a horizontal plane, so that the vertical circle and telescope can be brought into any azimuth.

If the line of vision be successively directed to any two objects A and B, then an index upon the vertical circle gives the angles of elevation of the objects; and the difference of the indications of the index on the horizontal circle gives the horizontal angle between the objects.

The horizontal circle carries another, called a vernier circle, which has a horizontal motion independent of the horizontal circle. There are generally two verniers upon the horizontal circle, to enable the observer to correct for centreing and graduation. Two readings are made, and the mean of the two will give the corrected angle. The vertical circle also carries a vernier for a similar purpose.

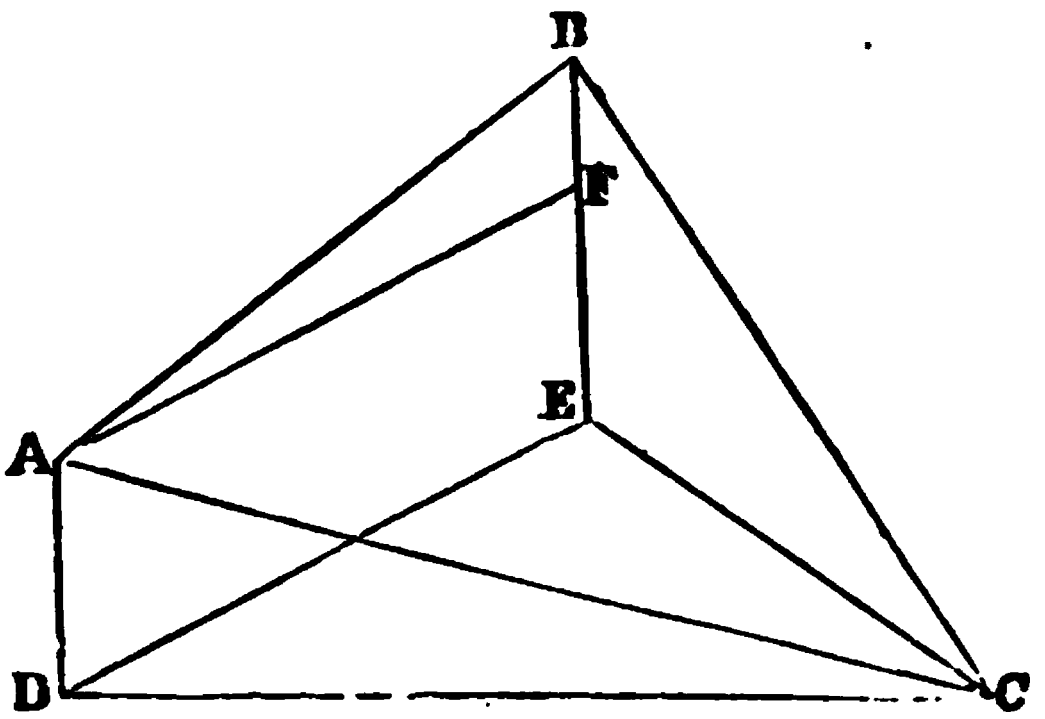
Before proceeding to use the instrument care must be taken that it stands firm, that the horizontal circle is truly horizontal, that the line of collimation is properly adjusted, and that the vertical circle moves in a truly vertical plane.

The theodolite often carries a compass, so that if the zero of the horizontal circle corresponds to the magnetic meridian, the horizontal angle between the object and the zero point gives the *bearing* of the object.

Ex. 1. Two places B, C, inaccessible from each other, are distant a from another place A on the side of a hill. From the lower place C there are observed the horizontal angle (α) between A and B, and also the elevations (λ, μ) of A and B. Find the distance BC.

Draw CD, CE horizontal, meeting verticals through A, B respectively in D, E.

Join DE, which is also horizontal; and draw AF parallel to DE.



Then we have:

$$AB = AC = a,$$
$$\angle DCE = \alpha,$$
$$\angle ACD = \lambda,$$
$$\angle BCE = \mu.$$

Now AC^2

$= AB^2 = AF^2 + BF^2$

$= DE^2 + (BE - AD)^2$

$= (EC^2 + DC^2 - 2\,EC \,.\, DC \cos ECD) + (BE - AD)^2$; or,

$a^2 = BC^2 \cos^2 \mu + AC^2 \cos^2 \lambda - 2\,BC \,.\, AC \cos \lambda \cos \mu \cos \alpha$
$+ (BC \sin \mu - AC \sin \lambda)^2$; or, simplifying,

$= BC^2 + AC^2 - 2\,AC.BC \cos \lambda \cos \mu \cos \alpha - 2\,AC.BC \sin \lambda \sin \mu$

$= BC^2 + a^2 - 2\,a\,BC\ (\cos \lambda \cos \mu \cos \alpha + \sin \lambda \sin \mu).$

Hence

$BC = 2\,a\,(\cos \lambda \cos \mu \cos \alpha + \sin \lambda \sin \mu)$; or, arranging,

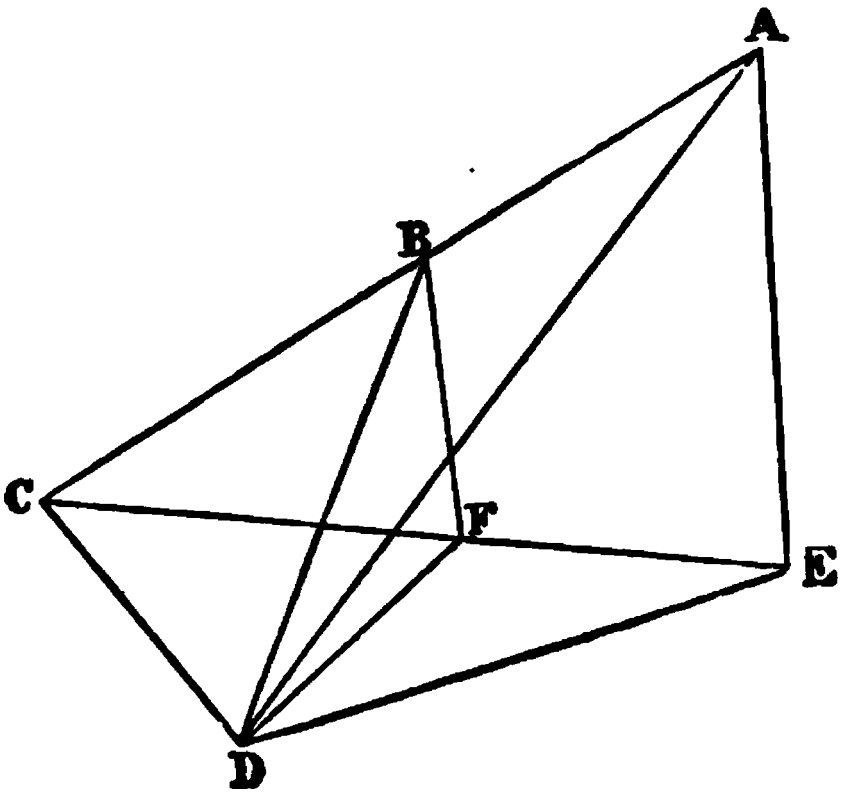
$$= 2\,a \left\{ \cos (\lambda - \mu) \cos^2 \frac{\alpha}{2} - \cos (\lambda + \mu) \sin^2 \frac{\alpha}{2} \right\}.$$

Ex. 2. An object A on the top of a hill is just seen from a point C behind the summit of hill B. A distance CD is measured along the road, and the following observations are made. At D the elevations of A, B are observed respectively to be α, β; and the angles ADC, ADB, ACD are found to be γ, δ, ϵ. Find the heights of A and B.

Draw CE, DE horizontal, and AE, BF vertical. Join DF.

We have :

$$\text{CD} = a,$$
$$\angle \text{ADE} = \alpha,$$
$$\angle \text{BDF} = \beta,$$
$$\angle \text{ADC} = \gamma,$$
$$\angle \text{ADB} = \delta,$$
$$\angle \text{ACD} = \epsilon.$$



Now

$$\text{AE} = \text{AD} \sin \text{ADE}$$

$$= \text{CD} \cdot \frac{\sin \text{ACD}}{\sin \text{CAD}} \sin \text{ADE}$$

$$= a \cdot \frac{\sin \epsilon \cdot \sin \alpha}{\sin \{180^\circ - (\gamma + \epsilon)\}} = a \cdot \frac{\sin \alpha \cdot \sin \epsilon}{\sin (\gamma + \epsilon)}.$$

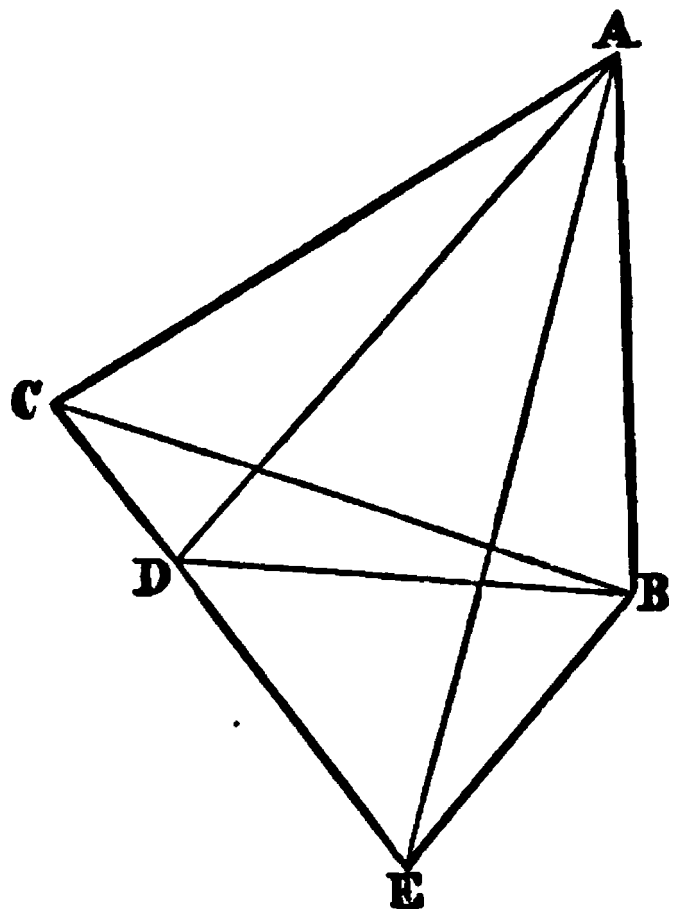
Again

$$\text{BF} = \text{BD} \sin \text{BDF} = \text{CD} \cdot \frac{\sin \text{ACD}}{\sin \text{CBD}} \sin \text{BDF}.$$

$$= a \cdot \frac{\sin \epsilon}{\sin \{180^\circ - (\gamma - \delta + \epsilon)\}} \sin \beta = a \cdot \frac{\sin \beta \sin \epsilon}{\sin (\gamma + \epsilon - \delta)}.$$

Ex. 3. Given the elevations of an object from three

points in a straight line in the horizontal plane, to find the height of the object.



Let A be the object; C, D, E the three points in the straight line CE.

Draw AB vertical, meeting the horizon in B.

Let CD = a, DE = b.....(1).

$\angle$ ACB = α, $\angle$ ADB = β, $\angle$ AEB = γ; and let h be the height required.

We have

$$\left.\begin{aligned} BC &= h \cot \alpha \\ BD &= h \cot \beta \\ BE &= h \cot \gamma \end{aligned}\right\} \text{.....(2).}$$

$$\text{Now cos BDC} = \frac{CD^2 + BD^2 - CB^2}{2\,CD \,.\, BD},$$

$$\text{And cos BDE} = \frac{BD^2 + DE^2 - BE^2}{2\,BD \,.\, DE}.$$

But cos BDC = $-$ cos (180° $-$ BDC) = $-$ cos BDE.

$$\text{Hence, } \frac{CD^2 + BD^2 - CB^2}{2\,CD \,.\, BD} = -\frac{BD^2 + DE^2 - BE^2}{2\,BD \,.\, DE}\text{; or}$$

$$DE\,(CD^2 + BD^2 - CB^2) = -\,CD\,(BD^2 + DE^2 - BE^2).$$

Or, substituting from (1) and (2), we have

$$b\,(a^2 + h^2 \cot^2 \beta - h^2 \cot^2 \alpha) = -a\,(h^2 \cot^2 \beta + b^2 - h^2 \cot^2 \gamma);$$

$$\text{or, } h^2\{a \cot^2 \gamma - (a + b) \cot^2 \beta + b \cot^2 \alpha\} = ab\,(a + b),$$

$$\therefore h = \sqrt{\frac{ab\,(a + b)}{a \cot^2 \gamma - (a + b) \cot^2 \beta + b \cot^2 \alpha}}.$$

Ex. IV.

1. An object was observed from the deck of a ship to have a bearing of $\alpha°$ from the N.; and after a and b hours' sailing in the same direction at a uniform rate, its bearing was found to be $\beta°$ and $\gamma°$ respectively. Find the direction of sailing.

2. A staff at the top of a tower subtends to the eye of a spectator on the ground an angle θ, and on going a feet nearer to the foot of the tower, it was found to subtend the same angle θ. What is the distance of the tower?

3. Three observers in a straight line, the middle one of which is at a distance a, b respectively from the other two, observe the elevation of a bird to be α, β, γ respectively at the same instant. Find the distance of each observer from the vertical through the bird.

4. There are two hills, the height of the second being h. From the top of the former the angles of depression of two objects in a straight line with both the hills and in the horizontal plane are α_1 and β_1. From the top of the second hill they are α_2 and β_2. Find the height of the former hill.

5. A gun is fired from a fort A, and the intervals between seeing the flash and hearing the report at two stations B, C are t_1, t_2 respectively. D is a point in the straight line BC whose distance from A is required, having given v the velocity of sound, BD $= a$, DC $= b$.

6. From a certain point on the slope of a hill the elevations α, β of two objects on the hill are observed, and angle γ they subtend at the eye. If l be the length of the hill in a direct road from the foot, find its height.

7. Two ships are sailing uniformly with velocities u, v. At a certain instant their distances from the point of intersection of their courses was a, b, respectively. Show that if d be their minimum distance from each other, and θ the inclination of their courses, then

$$(av - bu)^2 \sin^2\theta = d^2 (u^2 + v^2 - 2uv \cos\theta).$$

8. The shadow of a column whose height is h is found to be a in length. When the sun has attained double his altitude, it is found to be b. Find the relation between a, b, h.

9. From a ship a point of land was observed to bear $\alpha°$ from the N., and after sailing a miles in a N.W. direction the bearing was $\beta°$ from the N. Find the distance of the land from each point of observation.

10. From the top of a mountain whose height is h miles the depression of the horizon was $\alpha°$. Show that the radius of the earth $= h \cot \alpha \cot \frac{\alpha}{2}$.

11. Wishing to know the breadth of a river at an inaccessible point A, I observed from a point D at a distance a from the bank the angle θ subtended at my eye by the point A and the opposite point; and also, on walking direct to the bank, I observed the subtended angle to be ϕ. Find the breadth.

12. From a balloon the depressions of two towns whose distance apart is a, are found to be α and β respectively, and the angle they subtend at the balloon is γ. Find the height of the balloon.

CHAPTER VI.

DEMOIVRE'S THEOREM. CALCULATION OF THE VALUE OF π.

40. *To show that*

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots.$$
$$= \cos(\alpha + \beta + \gamma + \ldots.) + \sqrt{-1}\sin(\alpha + \beta + \gamma + \ldots.).$$

We have, by multiplication,

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)$$
$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + \sqrt{-1}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha + \beta) + \sqrt{-1}\sin(\alpha + \beta).$$

And hence also by multiplication,

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma) + \sqrt{-1}\sin(\alpha+\beta+\gamma).$$

Assume the law to hold for r factors corresponding to the angles $\alpha, \beta, \gamma \ldots . \kappa$.

Then we have

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma) \ldots . (\cos\kappa + \sqrt{-1}\sin\kappa)$$
$$= \cos(\alpha+\beta+\gamma+\ldots.\kappa) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots.\kappa).$$

Multiply each side of this equality by another factor $\cos\lambda + \sqrt{-1}\sin\lambda$.

Then

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma) \ldots . (\cos\lambda + \sqrt{-1}\sin\lambda)$$
$$= \{\cos(\alpha+\beta+\gamma+\ldots.\kappa) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots.\kappa)\}\{(\cos\lambda + \sqrt{-1}\sin\lambda)\};$$

or, by multiplication,

$$= \cos(\alpha+\beta+\gamma+\ldots.\lambda) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots.\lambda).$$

Hence, by assuming the law to hold for r factors, we have been able to prove it to hold for $(r+1)$ factors.

Now we know by actual multiplication that it is true for 2 and 3 factors. It is therefore true for 4, and hence for 5, 6, &c.

We may thus conclude it to be true for any number of factors.

Hence generally,

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots.$$
$$= \cos(\alpha + \beta + \gamma + \ldots.) + \sqrt{-1}\sin(\alpha + \beta + \gamma + \ldots.).$$

COR. Since $(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\alpha - \sqrt{-1}\sin\alpha) = \cos^2\alpha + \sin^2\alpha = 1$, it at once follows that we may write the above thus:

$$(\cos\alpha \pm \sqrt{-1}\sin\alpha)(\cos\beta \pm \sqrt{-1}\sin\beta)(\cos\gamma \pm \sqrt{-1}\sin\gamma)\ldots.$$
$$= \cos(\alpha + \beta + \gamma + \ldots.) \pm \sqrt{-1}\sin(\alpha + \beta + \gamma + \ldots.),$$

the upper or lower sign being taken throughout.

Demoivre's Theorem.

41. *To show that for all values of* n, *positive or negative, integral or fractional,*

$$(\cos\theta \pm \sqrt{-1}\sin\theta)^n = \cos n\theta \pm \sqrt{-1}\sin\theta.$$

As in the last Art., we need only trouble ourselves with the *upper sign.*

We have (Art. 40),

$$(\cos\alpha + \sqrt{-1}\sin\alpha)(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)\ldots.$$
$$\therefore = \cos(\alpha + \beta + \gamma + \ldots.) + \sqrt{-1}\sin(\alpha + \beta + \gamma \ldots.).$$

Let $\alpha = \beta = \gamma = \ldots\ldots = \theta$, and let there be n factors, then, since

$$\alpha + \beta + \gamma + \ldots\ldots = n\theta,$$

we have,

(i.) *When* n *is a positive integer,*

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta \quad \ldots\ldots(1).$$

Now let $n = -m$, where m is a positive integer.

We have

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \frac{1}{(\cos\theta + \sqrt{-1}\sin\theta)^m}; \text{ or by (1),}$$

$$= \frac{1}{\cos m\theta + \sqrt{-1}\sin m\theta}$$

$$= \frac{\cos m\theta - \sqrt{-1}\sin m\theta}{(\cos m\theta + \sqrt{-1}\sin m\theta)(\cos m\theta - \sqrt{-1}\sin m\theta)}$$

$$= \frac{\cos m\theta - \sqrt{-1}\sin m\theta}{1}$$

$$= \cos(-m)\theta + \sqrt{-1}\sin(-m)\theta$$

$$= \cos n\theta + \sqrt{-1}\sin n\theta.$$

Hence

(ii.) *The formula is true when* n *is a negative integer.*

Again, we have, when m and n are any integers,

$$(\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta)^n = \cos n\left(\frac{m}{n}\theta\right) + \sqrt{-1}\sin n\left(\frac{m}{n}\theta\right)$$

$$= \cos m\theta + \sqrt{-1}\sin m\theta$$

$$= (\cos\theta + \sqrt{-1}\sin\theta)^m.$$

Or, taking the nth root of each side,

$$\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta = (\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}.$$

Hence putting n for $\frac{m}{n}$, we have, where n is a fraction,

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta.$$

Therefore, *for all values of* n,

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta.$$

In the next Art. we shall show that $\cos\frac{m}{n}\theta + \sqrt{-1}\sin\frac{m}{n}\theta$ is only *one* of n values of the expression $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$, where $\frac{m}{n}$ is a fraction in its lowest terms.

42. *To find the* n *values of the expression* $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$, *where* $\frac{m}{n}$ *is a fraction in its lowest terms.*

Since $\cos\theta = \cos(2r\pi + \theta)$, and $\sin\theta = \sin(2r\pi + \theta)$, where r is an integer, we have

$$(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}} = \{\cos(2r\pi + \theta) + \sqrt{-1}\sin(2r\pi + \theta)\}^{\frac{m}{n}}$$

$$= \cos\frac{m}{n}(2r\pi + \theta) + \sqrt{-1}\sin\frac{m}{n}(2r\pi + \theta) \ldots\ldots\ldots(1).$$

By giving r the values 0, 1, 2, 3 .... $(n-1)$ successively, we obtain n *different* values of the expression $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$.

For, suppose $r = p$, and $r = q$, where p and q are each less than n, gave the *same values*,

Then evidently $\frac{m}{n}(2p\pi + \theta)$, and $\frac{m}{n}(2q\pi + \theta)$, must differ by a multiple of 2π.

Hence $\frac{m}{n}\{(2q\pi + \theta) - (2p\pi + \theta)\}$, or $\frac{m}{n}(p-q)\,2\pi$ is a multiple of 2π.

Therefore $\frac{m}{n}(p-q)$ is an integer, which is absurd, since $\frac{m}{n}$ is in its lowest terms, and p and q are each less than n.

Moreover, no value of r beyond the limits of the series 0, 1, 2, 3, ... $(n-1)$ will give any value of $(\cos\theta + \sqrt{-1}\sin\theta)^{\frac{m}{n}}$ which is not given by the values in this series.

For suppose $r = an + r'$, when a is an integer either positive or negative, and r' one of the values 0, 1, 2, 3 . . . . $(n - 1)$.

Then

$$\cos \frac{m}{n}(2r\pi + \theta) = \cos \frac{m}{n}\{2(an + r')\pi + \theta\} = \cos \{2am\pi + \frac{m}{n}(2r'\pi + \theta)\}$$

$$= \cos \frac{m}{n}(2r'\pi + \theta).$$

And so, $\sin \frac{m}{n}(2r\pi + \theta) = \sin \frac{m}{n}(2r'\pi + \theta)$.

Hence, when $r = an + r'$,

$(\cos \theta + \sqrt{-1} \cdot \sin \theta)^{\frac{m}{n}} = \cos \frac{m}{n}(2r'\pi + \theta) + \sqrt{-1} \sin \frac{m}{n}(2r'\pi + \theta)$, which is one of the values obtained from giving to r the series of values 0, 1, 2, 3 . . . . $(n - 1)$.

Hence,

$(\cos \theta + \sqrt{-1} \sin \theta)^{\frac{m}{n}} = \cos \frac{m}{n}(2r\pi + \theta) + \sqrt{-1} \sin \frac{m}{n}(2r\pi + \theta)$, where r may have each of the values 0, 1, 2, 3, ... $(n - 1)$.

43. *To expand* $\sin n\theta$, $\cos n\theta$, $\tan n\theta$ *in terms of the ratios of* θ.

We have $\cos n\theta + \sqrt{-1} \sin n\theta = (\cos \theta + \sqrt{-1} \sin \theta)^n$;

or, expanding by the Binomial Theorem and arranging,

$$\cos n\theta + \sqrt{-1}\sin n\theta$$
$$= \left\{\cos^n\theta - \frac{n(n-1)}{1.2}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\cos^{n-4}\theta\sin^4\theta - \&c.\right\}$$
$$+ \sqrt{-1}\left\{n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{1.2.3}\cos^{n-3}\theta\sin^3\theta + \&c.\right\}.$$

Equating the possible and impossible parts of this equality, we have

$$\cos n\theta = \cos^n\theta\left\{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - \&c.\right\} \ldots\ldots (1),$$

$$\text{and } \sin n\theta = \cos^n\theta\left\{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \&c.\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence (2) ÷ (1), we have

$$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\theta + \&c.}{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - \&c.} \ldots\ldots\ldots\ldots\ldots (3).$$

44. *To express* sin α, cos α, tan α *as series involving powers of* α.

We have (Art. 43),

$$\cos n\theta = \cos^n\theta\left\{1 - \frac{n(n-1)}{1.2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\tan^4\theta - \&c.\right\}.$$

Put $n\theta = a$, and therefore $n = \frac{a}{\theta}$; then, arranging,

$$\cos a = \cos^n \theta \left\{1 - \frac{a(a-\theta)}{1.2}\left(\frac{\tan\theta}{\theta}\right)^2 + \frac{a(a-\theta)(a-2\theta)(a-3\theta)}{1.2.3.4}\left(\frac{\tan\theta}{\theta}\right)^4 - \&c.\right\}.$$

Let n be infinite, and therefore the limit of θ be zero.

Then, limit of $\cos\theta = 1$, and of $\frac{\tan\theta}{\theta} = 1$.

Hence, we have

$$\cos a = \left\{1 - \frac{a \cdot a}{1.2}(1)^2 + \frac{a.a.a.a}{1.2.3.4}(1)^4 - \&c.\right\};$$

$$\text{or, } \cos a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

and so

$$\sin a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} - \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Dividing (2) by (1), then

$$\tan a = a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Exponential Expressions.

45. *To find exponential expressions for* $\sin\theta$, $\cos\theta$, $\tan\theta$.

We have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c.$$

Put successively $\theta\sqrt{-1}$ and $-\theta\sqrt{-1}$ for x, then

$$e^{\theta\sqrt{-1}} = 1 + \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} - \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} + \&c.$$

and $$e^{-\theta\sqrt{-1}} = 1 - \frac{\theta\sqrt{-1}}{1} - \frac{\theta^2}{1.2} + \frac{\theta^3\sqrt{-1}}{1.2.3} + \frac{\theta^4}{1.2.3.4} - \&c.$$

Hence, adding,

$$e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}} = 2\left\{1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \&c.\right\} = 2\cos\theta.$$

$$\therefore \cos\theta = \tfrac{1}{2}\left(e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}\right) \quad \ldots\ldots (1).$$

And so, subtracting,

$$\sin\theta = \frac{1}{2\sqrt{-1}}\left(e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}\right) \quad \ldots\ldots (2).$$

(2) ÷ (1), then

$$\tan\theta = \frac{1}{\sqrt{-1}} \cdot \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}} = \frac{1}{\sqrt{-1}} \cdot \frac{e^{2\theta\sqrt{-1}} - 1}{e^{2\theta\sqrt{-1}} + 1} \cdots\cdots\cdots (3).$$

Cor. From (1) and (2) we get

$$e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta,$$

$$e^{-\theta\sqrt{-1}} = \cos\theta - \sqrt{-1}\sin\theta.$$

Determination of the Value of π.

46. *Gregory's Series.* To show that

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \&c.$$

We have (Art. 45, Cor.),

$$\frac{e^{\theta\sqrt{-1}}}{e^{-\theta\sqrt{-1}}} = \frac{\cos\theta + \sqrt{-1}\sin\theta}{\cos\theta - \sqrt{-1}\sin\theta}, \text{ or}$$

$$e^{2\theta\sqrt{-1}} = \frac{1 + \sqrt{-1}\tan\theta}{1 - \sqrt{-1}\tan\theta}.$$

Taking log$_e$ of each side of this equality, then

$$2\theta\sqrt{-1} = \log_e \frac{1 + \sqrt{-1}\tan\theta}{1 - \sqrt{-1}\tan\theta}$$

$$= 2\left\{\sqrt{-1}\tan\theta - \frac{\sqrt{-1}\tan^3\theta}{3} + \frac{\sqrt{-1}\tan^5\theta}{5} - \&c.\right\}.$$

$$\therefore \theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \&c.$$

COR. Put $\theta = \tan^{-1}\alpha$, or $\tan\theta = \alpha$, then

$$\tan^{-1}\alpha = \alpha - \tfrac{1}{3}\alpha^3 + \tfrac{1}{5}\alpha^5 - \&c.$$

47. *To prove Euler's series for determining the value of* π, viz. :

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \&c.\right) + \left(\frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots.\right).$$

We have $$\tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} = \tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\,.\,\frac{1}{3}} = \tan^{-1}1.$$

But $$\tan^{-1}1 = \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$

Hence $\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$; or, by the last Art.,

$$= \left(\frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \ldots.\right) + \left(\frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots.\right).$$

48. *To prove Machin's series,* viz. :

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \ldots\right) - \left\{\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \ldots\right\}.$$

We have (Art. 15),

$$\frac{\pi}{4} = 4 \tan^{-1} \tfrac{1}{5} - \tan^{-1} \tfrac{1}{239}, \text{ which gives the series required.}$$

Miscellaneous Propositions.

49. *To prove Euler's series for* sin θ *and* cos θ, *without the aid of Demoivre's Theorem.*

Since sin $(-\theta) = -$ sin θ, it is evident that the value of sin θ changes sign with θ. Hence the series for sin θ in powers of θ can contain only odd powers of θ.

And since cos $(-\theta)$ = cos θ, it may be similarly reasoned that the series for cos θ in powers of θ contains only even powers.

Assume then $$\sin\theta = a_1\theta + a_3\theta^3 + a_5\theta^5 + \ldots \ldots \ldots \ldots \ldots \ldots (1),$$

and $$\cos\theta = a_0 + a_2\theta^2 + a_4\theta^4 + \ldots \ldots \ldots \ldots \ldots \ldots (2).$$

From (1), $$\frac{\sin\theta}{\theta} = a_1 + a_3\theta^2 + a_5\theta^4 + \ldots ;$$

Now, as this is true for all values of θ, it is true for the value $\theta = 0$; we then have

$$\frac{\sin\theta}{\theta} = 1 ; \text{ and hence } 1 = a_1.$$

And so from (2), when $\theta = 0$, we have $1 = a_0$.

We may therefore write

$$\sin\theta = \theta + a_3\theta^3 + a_5\theta^5 + \ldots \quad \ldots\ldots\ldots\ldots(3),$$

and $$\cos\theta = 1 + a_2\theta^2 + a_4\theta^4 + \ldots \quad \ldots\ldots\ldots\ldots(4).$$

(3) + (4), then

$$\cos\theta + \sin\theta = 1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \ldots. (5).$$

Put $\theta + \phi$ for θ; then, arranging,

$$(\cos\theta + \sin\theta)\cos\phi + (\cos\theta - \sin\theta)\sin\phi$$
$$= 1 + (\theta + \phi) + a_2(\theta + \phi)^2 + a_3(\theta + \phi)^3 + a_4(\theta + \phi)^4$$
$$+ a_5(\theta + \phi)^5 + \ldots. \quad \ldots\ldots\ldots\ldots(6).$$

But from (3), (4), and (5), we get

$$(\cos\theta + \sin\theta)\cos\phi + (\cos\theta - \sin\theta)\sin\phi$$
$$= (1 + \theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5 + \ldots.)$$
$$(1 + a_2\phi^2 + a_4\phi^4 + \ldots.)$$
$$+ (1 - \theta + a_2\theta^2 - a_3\theta^3 + a_4\theta^4 - a_5\theta^5 + \ldots.)$$
$$(\theta + a_3\theta^3 + a_5\theta^5 + \ldots.) \ldots\ldots\ldots\ldots(7).$$

Equating co-efficients of like powers of $\theta\phi$, $\theta^2\phi$, &c., in (6) and (7), then

$$2a_2 = -1, \therefore a_2 = -\frac{1}{1.2}.$$

$$3a_3 = a_2, \quad \therefore a_3 = -\frac{1}{1.2.3}.$$

$$4a_4 = -a_3, \therefore a_4 = \frac{1}{1.2.3.4}.$$

$$5a_5 = a_4, \quad \therefore a_5 = \frac{1}{1.2.3.4.5}.$$

$$\&c. = \&c.$$

Hence from (3) and (4) we get by substitution,

$$\sin\theta = \theta - \frac{\theta^3}{1\ 2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c.,$$

and $$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2\ 3.4} - \&c.$$

50. *To show that if* $2\cos\theta = x + \frac{1}{x}$, *then*

$$2\sqrt{-1}\sin\theta = x - \frac{1}{x};$$

and also that

$$2\cos n\theta = x^n + \frac{1}{x^n}, \text{ and } 2\sqrt{-1}\sin n\theta = x^n - \frac{1}{x^n}.$$

We have $2\cos\theta = x + \frac{1}{x}$; or, squaring,

$$4\cos^2\theta = \left(x + \frac{1}{x}\right)^2; \text{ or}$$

$$-4(1-\cos^2\theta) = \left(x + \frac{1}{x}\right)^2 - 4; \text{ or}$$

$$-4\sin^2\theta = \left(x - \frac{1}{x}\right)^2.$$

$$\therefore 2\sqrt{-1}\sin\theta = x - \frac{1}{x}.$$

Hence also,

$$2(\cos\theta + \sqrt{-1}\sin\theta) = \left(x + \frac{1}{x}\right) + \left(x - \frac{1}{x}\right) = 2x.$$

$$\therefore \cos\theta + \sqrt{-1}\sin\theta = x \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

And so, $$\cos\theta - \sqrt{-1}\sin\theta = \frac{1}{x} \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1), by Demoivre's Theorem,

$$\cos n\theta + \sqrt{-1}\sin n\theta = x^n;$$

and from (2), $\cos n\theta - \sqrt{-1}\sin n\theta = \frac{1}{x^n}$.

From these two equations, the required results at once follow.

51. *To expand* $\sin^n\theta$ *in terms of sines or cosines of multiples of* θ, *when* n *is a positive integer.*

Put $2\cos\theta = x + \frac{1}{x}$, and therefore $2\sqrt{-1}\sin\theta = x - \frac{1}{x}$.

Then $(2\sqrt{-1}\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{1\,.\,2}x^{n-4} - \ldots + \frac{n(n-1)}{1\,.\,2}x^2\left(-\frac{1}{x}\right)^{n-2} + nx\left(-\frac{1}{x}\right)^{n-1} + \left(-\frac{1}{x}\right)^n \ldots(1).$$

(i.) *Let* n *be even.*

Then, as there are $n+1$ terms in (1),

the middle term is the $\left(\frac{n}{2}+1\right)$th term

$$= \frac{n(n-1)\ldots\left(n-\frac{n}{2}+1\right)}{1\,.\,2\ldots\ldots\frac{n}{2}}x^{\frac{n}{2}}\left(-\frac{1}{x}\right)^{\frac{n}{2}}$$

$$= (-1)^{\frac{n}{2}}\,.\,\frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\ldots\ldots\frac{n}{2}}.$$

We have therefore from (1), collecting terms equidistant from the beginning and end of the series,

$$2^n(-1)^{\frac{n}{2}}\sin^n\theta = \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \frac{n(n-1)}{1\cdot 2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right)$$

$$+ \cdots\cdots + (-1)^{\frac{n}{2}} \cdot \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1\cdot 2\ldots\frac{n}{2}}$$

$$= 2\cos n\theta - n\cdot 2\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\,2\cos(n-4)\theta$$

$$+ \cdots\cdots + (-1)^{\frac{n}{2}} \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1\cdot 2\ldots\frac{n}{2}}.$$

$$\therefore 2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot 2}\cos(n-4)\theta$$

$$- \cdots\cdots + \tfrac{1}{2}(-1)^{\frac{n}{2}} \cdot \frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1\cdot 2\ldots\frac{n}{2}}.$$

(ii.) *Let* n *be odd.*

We may similarly show that

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\sin(n-4)\theta$$
$$- \cdots\cdots + (-1)^{\frac{n-1}{2}}\frac{n(n-1)\ldots\ldots\frac{1}{2}(n+3)}{1\,.\,2\ldots\ldots\frac{1}{2}(n-1)}\,.\sin\theta.$$

52. *To expand* $\cos^n\theta$ *in terms of the cosines of multiples of* θ.

We may show exactly as in the last Art. that:

(i.) *When* n *is even,*

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta$$
$$+ \ldots\ldots + \tfrac{1}{2}\cdot\frac{n(n-1)\ldots\ldots\left(\frac{n}{2}+1\right)}{1\,.\,2\ldots\ldots\frac{n}{2}}.$$

(ii.) *When* n *is odd,*

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta$$
$$+ \ldots\ldots + \frac{n(n-1)\ldots\ldots\frac{n+3}{2}}{1\,.\,2\ldots\ldots\frac{n-1}{2}}\cos 0.$$

53. *If* $\tan \alpha = \cos \omega \,.\, \tan l$, *to show that*

$$l - \alpha = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2l - \tfrac{1}{2} \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4l + \&c.\,;$$ *and that*

$$l - \alpha = \tan^2 \tfrac{1}{2}\omega \,.\, \sin 2\alpha + \tfrac{1}{2} \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4\alpha + \&c.$$

We have (Art 33),

$$\frac{1}{\sqrt{-1}} \cdot \frac{e^{2\alpha\sqrt{-1}} - 1}{e^{2\alpha\sqrt{-1}} + 1} = \cos\omega \,.\, \frac{1}{\sqrt{-1}} \frac{e^{2l\sqrt{-1}} - 1}{e^{2l\sqrt{-1}} + 1};\ \text{or,}$$

$$\frac{e^{2\alpha\sqrt{-1}} - 1}{e^{2\alpha\sqrt{-1}} + 1} = \frac{(e^{2l\sqrt{-1}} - 1)\cos\omega}{e^{2l\sqrt{-1}} + 1}.$$

Taking the sum and difference of numerator and denominator, then

$$e^{2\alpha\sqrt{-1}} = \frac{e^{2l\sqrt{-1}}(1 + \cos\omega) + (1 - \cos\omega)}{e^{2l\sqrt{-1}}(1 - \cos\omega) + (1 + \cos\omega)};\ \text{or, arranging, \&c.,}$$

$$= \frac{e^{2l\sqrt{-1}} + \tan^2 \tfrac{1}{2}\omega}{e^{2l\sqrt{-1}} \tan^2 \tfrac{1}{2}\omega + 1}$$

$$= e^{2l\sqrt{-1}} \frac{1 + \tan^2 \tfrac{1}{2}\omega \,.\, e^{-2l\sqrt{-1}}}{1 + \tan^2 \tfrac{1}{2}\omega \,.\, e^{2l\sqrt{-1}}}.$$

Taking logarithms of each side, and expanding, then

$$2a\sqrt{-1} = 2l\sqrt{-1} + \left\{\tan^2\tfrac{1}{2}\omega \quad e^{-2l\sqrt{-1}} - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \,.\, e^{-4l\sqrt{-1}} + \&c.\right\}$$
$$- \left\{\tan^2\tfrac{1}{2}\omega \,.\, e^{2l\sqrt{-1}} - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \,.\, e^{4l\sqrt{-1}} + \&c.\right\}$$

$$= 2l\sqrt{-1} - \tan^2\tfrac{1}{2}\omega\,(e^{2l\sqrt{-1}} - e^{-2l\sqrt{-1}}) + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega\,(e^{4l\sqrt{-1}} - e^{-4l\sqrt{-1}}) - \&c.$$

$$= 2l\sqrt{-1} - \tan^2\tfrac{1}{2}\omega \,.\, 2\sqrt{-1}\sin 2l + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \,.\, 2\sqrt{-1}\sin 4l - \&c.$$

Or, dividing each side by $2\sqrt{-1}$, then

$$a = l - \tan^2\tfrac{1}{2}\omega \,.\, \sin 2l + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \,.\, \sin 4l - \&c.$$

Hence, transposing,

$$l - a = \tan^2\tfrac{1}{2}\omega \,.\, \sin 2l - \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \,.\, \sin 4l + \&c.$$

And similarly, by taking the given equation in the form $\tan l = \dfrac{1}{\cos\omega} \,.\, \tan a$, we get the second required result.

54. *To resolve* $\sin\theta$ *and* $\cos\theta$ *into the product of quadratic factors in terms of* θ.

The equation $\sin\theta = 0$ is satisfied by $\theta = n\pi$, when n may be zero, or any integer, positive or negative.

Hence the values $\theta = 0, \pm \pi, \pm 2\pi, \pm 3\pi$, &c., represent the only possible solutions of the equation.

Therefore, by the Theory of Equations,

$$\sin\theta = a\theta\,(\theta+\pi)\,(\theta-\pi)\,(\theta+2\pi)\,(\theta-2\pi)\,(\theta+3\pi)\,(\theta-3\pi)\ldots.$$

$$= a\theta\,(\theta^2-\pi^2)\,(\theta^2-2^2\pi^2)\,(\theta^2-3^2\pi^2)\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where a is some constant quantity to be determined.

We may determine a thus:

Since the equation (1) may be written

$$\frac{\sin\theta}{\theta} = a\,(\theta^2-\pi^2)\,(\theta^2-2^2\pi^2)\,(\theta^2-3^2\pi^2)\ldots.,$$

and the equation is true when $\theta = 0$, we have, since $\frac{\sin\theta}{\theta} = 1$, when $\theta = 0$,

$$1 = a\,(-\pi^2)\,(-2^2\pi^2)\,(-3^2\pi^2)\ldots.$$

$$= \pm a\pi^3 \cdot 2^2\pi^2 \cdot 3^2\pi^2\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

But from (1) we get

$$\sin\theta = \pm a\pi^3 \cdot 2^2\pi^2 \cdot 3^2\pi^2\ldots.\,\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Hence, substituting from (2),

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots(A).$$

To find the expression for $\cos\theta$.

The equation $\cos\theta = 0$ is satisfied by $\theta = (2n + 1)\frac{\pi}{2}$, when n may be zero, or any integer, positive or negative.

Hence the values $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$, &c. represent the only possible solutions of the equation.

Therefore, by the Theory of Equations,

$$\cos\theta = a\left(\theta + \frac{\pi}{2}\right)\left(\theta - \frac{\pi}{2}\right)\left(\theta + \frac{3\pi}{2}\right)\left(\theta - \frac{3\pi}{2}\right)\left(\theta + \frac{5\pi}{2}\right)\left(\theta - \frac{5\pi}{2}\right)\ldots.$$

$$= a\left(\theta^2 - \frac{\pi^2}{2^2}\right)\left(\theta^2 - \frac{3^2\pi^2}{2^2}\right)\left(\theta^2 - \frac{5^2\pi^2}{2^2}\right)\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where a is some constant to be determined.

Put $\theta = 0$, then, since $\cos\theta = 1$, when $\theta = 0$, we have

$$1 = a\left(-\frac{\pi^2}{2^2}\right)\left(-\frac{3^2\pi^2}{2^2}\right)\left(-\frac{5^2\pi^2}{2^2}\right)\ldots.$$

$$= \pm a\,\frac{\pi^2}{2^2}\cdot\frac{3^2\pi^2}{2^2}\cdot\frac{5^2\pi^2}{2^2}\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

But from (3)

$$\cos\theta = \pm a \cdot \frac{\pi^2}{2^2} \cdot \frac{3\pi^2}{2^2} \cdots \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\cdots$$

Hence, substituting from (4),

$$\cos\theta = \left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\left(1 - \frac{4\theta^2}{5^2\pi^2}\right)\cdots \quad \ldots(B).$$

Cor. From the result for sin θ, we may obtain *Wallis's expression for* π.

In (A), put $\theta = \frac{\pi}{2}$, then since $\sin\frac{\pi}{2} = 1$, and $\frac{\pi}{\theta} = 2$, we have

$$1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{2^2 \cdot 2^2}\right)\left(1 - \frac{1}{3^2 \cdot 2^2}\right)\cdots$$

$$= \frac{\pi}{2} \cdot \frac{1.3}{2^2} \cdot \frac{3.5}{4^2} \cdot \frac{5.7}{6^2} \cdots$$

$$\therefore \frac{\pi}{2} = \frac{2^2}{1.3} \cdot \frac{4^2}{3.5} \cdot \frac{6^2}{5.7} \cdots$$

55. *To resolve* $x^n - 1$ *into factors.*

We have, from the equation $x^n - 1 = 0$,

$$x^n = 1 = \cos 2m\pi \pm \sqrt{-1}\sin 2m\pi,$$

where m is any integer, including zero.

$$\therefore x = (\cos 2m\pi \pm \sqrt{-1}\sin 2m\pi)^{\frac{1}{n}}$$

$$= \cos\frac{2m\pi}{n} \pm \sqrt{-1}\sin\frac{2m\pi}{n}.$$

(i.) *Let* n *be even.*

Then by giving m al. integral positive values from 0 to $\frac{n}{2}$,

we get + 1 for each of the first pair of values, and − 1 for each of the last pair. For all other values of m, the values of x are different.

For suppose p and q to be two values of m between the limits 0 and $\frac{n}{2}$, such that

$$\cos\frac{2p\pi}{n} + \sqrt{-1}\sin\frac{2p\pi}{n} = \cos\frac{2q\pi}{n} \pm \sqrt{-1}\sin\frac{2q\pi}{n}.$$

Then we have $$\cos\frac{2p\pi}{n} = \cos\frac{2q\pi}{n},$$

and $$\sin\frac{2p\pi}{n} = \pm\sin\frac{2q\pi}{n}.$$

Hence (Art. 21), we must therefore have

$$\frac{2p\pi}{n} = 2r\pi \pm \frac{2q\pi}{n},$$

where r is any integer.

$$\text{or,}\quad \frac{p}{n} = r \pm \frac{q}{n};\quad \text{or,}\quad \frac{p \mp q}{n} = r, \text{ an integer.}$$

But p and q are by supposition each $< \frac{n}{2}$, and are different.

Hence $\frac{p \mp q}{n}$ cannot be integral.

Hence the n values of x obtained from the equation

$$x = \frac{\cos 2m\pi}{n} \pm \sqrt{-1}\sin\frac{2m\pi}{n},$$

by giving m all integral values from 0 to $\frac{n}{2}$, are all different.

It must be remembered that we get really $(n + 2)$ values, the first pair of which are equal, viz. + 1, and the last pair equal, viz. − 1.

Further, no value of m beyond the limits 0 to $\frac{n}{2}$ will give any other value for x.

For any integer greater than $\frac{n}{2}$ may be put in the form $sn + t$, where t is not greater *in magnitude* than $\frac{n}{2}$.

Put then $m = sn + t$, where s and t are integral, and t not $> \frac{n}{2}$, but may be negative.

Then $\cos \frac{2m\pi}{n} \pm \sqrt{-1} \sin \frac{2m\pi}{n} = \cos\left(2s + \frac{2t}{n}\right)\pi \pm \sqrt{-1} \sin\left(2s + \frac{2t}{n}\right)\pi$

$$= \cos \frac{2t\pi}{n} \pm \sqrt{-1} \sin \frac{2t\pi}{n},$$

which is *equal to* one of the values before obtained.

The only possible values of x which satisfy the equation $x^n - 1 = 0$, when n is even, are hence the following series:

$$\pm 1, \cos \frac{2\pi}{n} \pm \sqrt{-1} \sin \frac{2\pi}{n}, \cos \frac{4\pi}{n} \pm \sqrt{-1} \sin \frac{4\pi}{n}, \text{ \&c., } \cos \frac{(n-2)\pi}{n} \pm \sqrt{-1} \sin \frac{(n-2)\pi}{n}.$$

Hence, by the Theory of Equations, since the coefficient of x^n in the equation $x^n - 1 = 0$, is unity, we must have

$$x^n - 1 = (x+1)(x-1) \cdot \left\{x - \left(\cos \frac{2\pi}{n} + \sqrt{-1} \sin \frac{2\pi}{n}\right)\right\} \left\{x - \left(\cos \frac{2\pi}{n} - \sqrt{-1} \sin \frac{2\pi}{n}\right)\right\}$$

$$\ldots \left\{x - \left(\cos \frac{n-2 \cdot \pi}{n} + \sqrt{-1} \sin \frac{n-2 \cdot \pi}{n}\right)\right\} \left\{x - \left(\cos \frac{n-2 \cdot \pi}{n} - \sqrt{-1} \sin \frac{n-2 \cdot \pi}{n}\right)\right\}.$$

Or, simplifying, we have, when n is *even*, $x^n - 1$

$$= (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right) \ldots (1).$$

(ii.) And so, *when* n *is odd*, we similarly get : $x^n - 1$

$$= (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right) \ldots (2).$$

Cor. 1. By an exactly similar process, we get,

When n *is even,*

$$x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-1}{n}\pi + 1\right) \ldots\ldots (3).$$

And, *when* n *is odd,*

$$x^n + 1 = (x + 1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right) \ldots \left(x^2 - 2x\cos\frac{n-2}{n}\pi + 1\right) \ldots\ldots (4).$$

Cor. 2. Many factorials may be obtained from the above expressions.

(i.) In (2), dividing each side by $x - 1$, and puttting $x = 1$, then

since $\dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \text{\&c.} + x + 1 = n$, when $x = 1$, we have

$$n = \left(2 - 2\cos\frac{2\pi}{n}\right)\left(2 - 2\cos\frac{4\pi}{n}\right) \ldots\ldots \left(2 - 2\cos\frac{n-1}{n}\pi\right), \textit{ when n is odd;}$$

$$= 2^2 \sin^2\frac{\pi}{n} \cdot 2^2 \sin^2\frac{2\pi}{n} \ldots\ldots 2^2 \sin^2\frac{n-1}{2n}\pi;$$

or, $$n = 2^{n-1} \cdot \sin^2\frac{\pi}{n} \sin^2\frac{2\pi}{n} \ldots\ldots \sin^2\frac{n-1}{2n}\pi.$$

$$\therefore \sqrt{n} = 2^{\frac{n-1}{2}} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \ldots\ldots \sin\frac{n-1}{2n}\pi, \textit{ when n is odd.}$$

We take the positive sign of the square root, because every sine is positive, every angle being between 0 and $\frac{\pi}{2}$.

(ii.) So from (1), by putting $2n$ for n, and similarly reasoning, we get for all integral values of n:

$$\sqrt{n} = 2^{n-1} \cdot \sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \ldots\ldots \sin\frac{n-1}{2n}\pi.$$

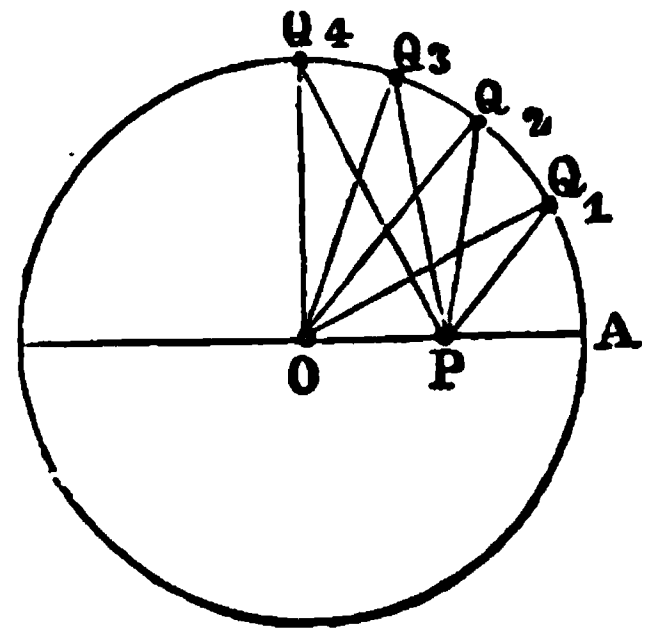
with many similar results.

56. *Demoivre's Property of the Circle.*

Let P be any point within or without a circle whose centre is O.

And let the whole circumference be divided into n equal parts Q_1Q_2, Q_2Q_3, &c., so that

$$Q_1Q_2 = Q_2Q_3 = \&c. = \frac{2\pi}{n}.$$



Join O and P with the points Q_1, Q_2, &c.;

And let $\angle POQ_1 = \theta$.

Then will

$$OP^{2n} - 2\,OP^n \cdot OQ_1^n \cos n\theta + OQ_1^n = PQ_1^2 \cdot PQ_2^2 \ldots\ldots PQ_n^2.$$

For we have

$$\angle POQ_1 = \theta,\ \angle POQ_2 = \theta + \frac{2\pi}{n},\ \angle POQ_3 = \theta + \frac{4\pi}{n}, \&c.$$

Hence

$$PQ_1^2 = OP^2 - 2\,OP \cdot r\cos\theta + r^2,$$

$$PQ_2^2 = OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{2\pi}{n}\right) + r^2,$$

$$PQ_3^2 = OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{4\pi}{n}\right) + r^2,$$

. . . . . .

$$PQ_n^2 = OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{n-1 \cdot 2\pi}{n}\right) + r^2.$$

Therefore, multiplying these equations together,

$$PQ_1^2 \cdot PQ_2^2 \cdot PQ_3^2 \ldots PQ_n^2$$

$$= (OP^2 - 2\,OP \cdot r\cos\theta + r^2)\left\{OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{2\pi}{n}\right) + r^2\right\}$$

$$\times \left\{OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{4\pi}{n}\right) + r^2\right\} \ldots \left\{OP^2 - 2\,OP \cdot r\cos\left(\theta + \frac{n-1 \cdot 2\pi}{n}\right) + r^2\right\}.$$

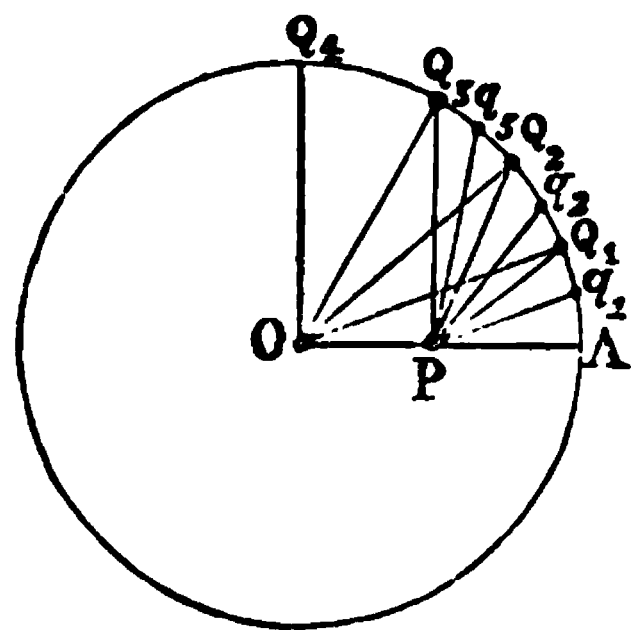
Now it may be easily shown, as in Art. 54, that the resolution of $OP^{2n} - 2\,OP^n \cdot r^n \cos n\theta + r^{2n}$ gives the expression on the right side of this equation.

Hence $$OP^{2n} - 2\,OP^n \cdot r^n \cos n\theta + r^{2n} = PQ_1^2 \cdot PQ_2^2 \cdot PQ_3^2 \ldots PQ_n^2,$$

which proves the proposition.

57. *Cotes's Property of the Circle.*

Let OP meet the circle in A so that A corresponds with Q_n.
Also let the arcs AQ_1, Q_1Q_2, &c., be bisected in q_1, q_2, q_3, &c.



Then shall (i.) $OP^n \sim OQ_n^n = PQ_1 \cdot PQ_2 \ldots PQ_n$; and

(ii.) $OP^n + OQ_n^n = Pq_1 \cdot Pq_2 \ldots Pq_n$.

For we have $$PQ_1^2 = OP^2 - 2\,OP \cdot r\cos\frac{2\pi}{n} + r^2,$$

$$PQ_2^2 = OP^2 - 2\,OP \cdot r\cos\frac{4\pi}{n} + r^2, \text{ &c.}$$

Hence $$PQ_1^2 . PQ_2^2 \ldots . PQ_n^2$$

$$= (OP^2 - 2\,OP . r\cos\frac{2\pi}{n} + r^2)\,(OP^2 - 2\,OP . r\cos\frac{4\pi}{n} + r^2)$$

$$\ldots . (OP^2 - 2\,OP . r\cos\frac{n . 2\pi}{n} + r^2).$$

Or, as in last Art.,

$$PQ_1^2 . PQ_2^2 \ldots . PQ_n^2 = OP^{2n} - 2\,OP^n . r\cos n . \frac{2\pi}{n} + r^{2n} = (OP^n \sim r^n)^2.$$

Taking the square roots, and putting OQ_n for r, we have

$$\therefore OP^n \sim OQ_n^n = PQ_1 . PQ_2 \ldots . PQ_n, \text{ which proves (i).}$$

Put 2 n for n, then, since q_1, q_2, &c., are the intermediate divisions of the arcs, and the last two points of division are q_n, Q_n, we have

$$OP^{2n} \sim OQ_n^{2n} = PQ_1 . Pq_1 . PQ_2 . Pq_2 \ldots . Pq_n . PQ_n.$$

Dividing this result by the former, then

$$OP^n + OQ_n^n = Pq_1 . Pq_2 \ldots . Pq_n, \text{ which proves (ii.)}$$

Ex. VI.

1. Find a formula which gives all the values of $\sqrt[n]{1}$, where n is a positive integer.

2. Find the nth roots of -1.

3. Prove that

$$\cos n\theta = (-1)^{\frac{n}{2}} \left\{ 1 - \frac{n^2}{1 \cdot 2} \cos^2\theta + \frac{n^2(n^2-2^2)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^4\theta - \text{\&c.} \right\}, \qquad (n \text{ even});$$

$$= (-1)^{\frac{n-1}{2}} \left\{ n\cos\theta - \frac{n(n^2-1^2)}{1 \cdot 2 \cdot 3} \cos^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cos^5\theta - \text{\&c.} \right\}, (n \text{ odd}).$$

4. Show that

$$\sin n\theta = (-1)^{\frac{n}{2}-1} \sin\theta \left\{ n\cos\theta - \frac{n(n^2-2^2)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^3\theta + \text{\&c.} \right\}, \qquad (n \text{ even});$$

$$= (-1)^{\frac{n-1}{2}} \sin\theta \left\{ 1 - \frac{n^2-1^2}{1 \cdot 2} \cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{1 \cdot 2 \cdot 3 \cdot 4} \cos^4\theta - \text{\&c.} \right\}, (n \text{ odd}).$$

5. Having given the series for $\sin\theta$ and $\cos\theta$ in powers of θ, and the factorial expressions of Art. 53, show that

$$(1), \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{\&c.} = \frac{\pi^2}{6},$$

$$(2), \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \text{\&c.} = \frac{\pi^2}{8}.$$

6. Show that, when $\alpha = \frac{\pi}{2n}$,

$$2^{n-1} \sin\alpha \sin 3\alpha \sin 5\alpha \ldots\ldots \sin(2n-1)\alpha = 1,$$

$$\text{and } 2^{n-1} \sin 2\alpha \sin 4\alpha \sin 6\alpha \ldots\ldots \sin(2n-2)\alpha = n.$$

Prove that

7. $\text{Sin}\,\theta = 2^{n-1} \sin\frac{\theta}{n} \sin\frac{\pi+\theta}{n} \cdot \sin\frac{2\pi+\theta}{n} \ldots \sin\frac{(n-1)\pi+\theta}{n}$.

8. $2^{n-1} \cos\frac{\theta}{n} \cos\frac{\pi+\theta}{n} \cdot \cos\frac{2\pi+\theta}{n} \ldots \cos\frac{(n-1)\pi+\theta}{n} = \pm \sin\theta, n \text{ even},$
$= \pm \cos\theta, n \text{ odd}.$

9. $\text{Tan}\,\frac{\pi}{4n} \tan\frac{5\pi}{4n} \tan\frac{9\pi}{4n} \ldots \tan\frac{4n-3}{4n}\pi = \pm 1.$

10. Show, by substituting the values of $\cos 2\theta$, and $\cos\theta$, when expressed as series in powers of θ, that $\cos 2\theta = 2\cos^2\theta - 1$.

11. Show that

$$e^x + 2\cos\theta + e^{-x} = 4\cos\frac{\theta + x\sqrt{-1}}{2} \cos\frac{\theta - x\sqrt{-1}}{2},$$

And hence show how to resolve $e^x + 2\cos\theta + e^{-x}$ into real quadratic factors.

12. Prove that

$$\log\sec\theta = \frac{\tan^2\theta}{2} - \frac{\tan^4\theta}{4} + \frac{\tan^6\theta}{6} - \ldots.$$

13. Show that, when x is very large,

$$\sqrt{\pi x} = \frac{2 \cdot 4 \cdot 6 \ldots 2x}{1 \cdot 3 \cdot 5 \ldots (2x-1)}.$$

14. Prove that

$$e^x + e^{-x} = 2\left(1 + \frac{2^2 x^2}{\pi^2}\right)\left(1 + \frac{2^2 x^2}{3^2 \pi^2}\right)\ldots\ldots,$$

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2 \pi^2}\right)\ldots\ldots$$

15. If $\sin\theta = \sin m \,.\, \sin(n + \theta)$, then

$$\theta = \sin m \sin n + \tfrac{1}{2}\sin^2 m \,.\, \sin 2n + \tfrac{1}{3}\sin^3 m \,.\, \sin 3n + \ldots\ldots$$

16. If $\tan m = a \,.\, \tan n$, show that

$$m = n + \frac{a-1}{a+1}\sin 2n + \tfrac{1}{2}\left(\frac{a-1}{a+1}\right)^2 \sin 4n + \text{\&c.},$$

$$n = m - \frac{a-1}{a+1}\sin 2m + \tfrac{1}{2}\left(\frac{a-1}{a+1}\right)^2 \sin 4m - \text{\&c.}$$

17. If $\sin(\alpha + \beta\sqrt{-1}) = a + b\sqrt{-1}$, then

$$a = (e^{\beta} + e^{-\beta})\sin\alpha,$$

$$\text{and} \quad b = (e^{\beta} - e^{-\beta})\cos\alpha.$$

18. Find the roots of the equation $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

CHAPTER VII.

SUMMATION OF TRIGONOMETRICAL SERIES.

58. *To find the sums of a series of sines and cosines of angles in A. P.*

Let $\alpha,\ \alpha + \beta,\ \alpha + 2\beta,\ \ldots\ldots,\ \alpha + \overline{n-1}\,.\,\beta$ be the angles.

We have

$$2\sin\frac{\beta}{2}\sin\alpha = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\sin(\alpha+\beta) = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right),$$

$$2\sin\frac{\beta}{2}\sin(\alpha+2\beta) = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right),$$

. . . . . .

$$2\sin\frac{\beta}{2}\sin(\alpha + \overline{n-1}\,.\,\beta) = \cos(\alpha + \frac{2n-3}{2}\beta) - \cos(\alpha - \frac{2n-1}{2}\beta).$$

Hence, adding, we get

$$2\sin\frac{\beta}{2}\left\{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots. + \sin(\alpha + \overline{n-1}\,.\,\beta)\right\}$$

$$= \cos\left(\alpha - \frac{\beta}{2}\right) - \cos(\alpha + \frac{2n-1}{2}\beta) = 2\sin(\alpha + \frac{n-1}{2}\beta)\sin\frac{n\beta}{2}.$$

$$\therefore \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots + \sin(\alpha + \overline{n-1}\,.\,\beta)$$

$$= \frac{\sin(\alpha + \frac{n-1}{2}\beta)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \quad \ldots\ldots\ldots\

$$\therefore\ 16\{\sin^5\alpha + \sin^5(\alpha+\beta) + \sin^5(\alpha+2\beta) + \ldots. + \sin^5(\alpha + \overline{n-1}\,.\,\beta)\}$$

$$= \{\sin 5\alpha + \sin(5\alpha + 5\beta) + \ldots. + \sin(5\alpha + \overline{n-1}\,.\,5\beta)\}$$

$$- 5\{\sin 3\alpha + \sin(3\alpha + 3\beta) + \ldots. + \sin(\alpha + \overline{n-1}\,.\,\beta)\}$$

$$+ 10\{\sin\alpha + \sin(\alpha+\beta) + \ldots. + \sin(\alpha + \overline{n-1}\,.\,\beta)\}.$$

From which the required sum is easily found by (A) above.

59. *To sum* $\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) -$ &c. *to* n *terms,*

and $\cos\alpha - \cos(\alpha+\beta) + \cos(\alpha+2\beta) -$ &c. *to* n *terms.*

$$2\cos\frac{\beta}{2}\sin\alpha = \sin\left(\alpha + \frac{\beta}{2}\right) + \sin\left(\alpha - \frac{\beta}{2}\right),$$

$$-2\cos\frac{\beta}{2}\sin(\alpha+\beta) = -\sin\left(\alpha + \frac{3\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right),$$

$$2\cos\frac{\beta}{2}\sin(\alpha+2\beta = \sin\left(\alpha + \frac{5\beta}{2}\right) + \sin\left(\alpha + \frac{3\beta}{2}\right),$$

. . . . . .

$$(-1)^{n-1}\,2\cos\frac{\beta}{2}\sin(\alpha + \overline{n-1}\,.\,\beta) = (-1)^{n-1}\left\{\sin\left(\alpha + \frac{2n-1}{2}\beta\right) + \left(\alpha + \frac{2n-3}{2}\beta\right)\right\}.$$

Hence, adding, we get

$$2\cos\frac{\beta}{2}\{\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots + (-1)^{n-1}\sin(\alpha+\overline{n-1}\,.\,\beta)\}$$

$$= \sin\left(\alpha - \frac{\beta}{2}\right) + (-1)^{n-1}\sin\left(\alpha + \frac{2n-1}{2}\beta\right).$$

$$\therefore \sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \ldots + (-1)^{n-1}\sin(\alpha+\overline{n-1}\,.\,\beta)$$

$$= \frac{\sin\left(\alpha - \frac{\beta}{2}\right) + (-1)^{n-1}\sin\left(\alpha + \frac{2n-1}{2}\beta\right)}{2\cos\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\cos\frac{n\beta}{2}}{\cos\frac{\beta}{2}}, \; n \text{ odd; and}$$

$$= -\frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\cos\frac{\beta}{2}}, \; n \text{ even.}$$

60. *To sum the series*

$$\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots + x^{n-1}\sin(\alpha+\overline{n-1}\,\beta),$$

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots + x^{n-1}\cos(\alpha+\overline{n-1}\,\beta).$$

We have

$$2\sqrt{-1}\left\{\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots\ldots \text{ to } n \text{ terms}\right\}$$

$$= (e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}) + x\left\{e^{(\alpha+\beta)\sqrt{-1}} - e^{(\alpha+\beta)\sqrt{-1}}\right\} + x^2\left\{e^{(\alpha+2\beta)\sqrt{-1}} - e^{-(\alpha+2\beta)\sqrt{-1}}\right\}$$

$$+ \ldots\ldots \text{ to } n \text{ terms.}$$

$$= e^{\alpha\sqrt{-1}}\left\{1 + xe^{\beta\sqrt{-1}} + x^2e^{2\beta\sqrt{-1}} + \ldots\ldots x^{n-1}.e^{(n-1)\beta\sqrt{-1}}\right\}$$

$$- e^{-\alpha\sqrt{-1}}\left\{1 + xe^{-\beta\sqrt{-1}} + x^2e^{-2\beta\sqrt{-1}} + \ldots + x^{n-1}e^{-(n-1)\beta\sqrt{-1}}\right\},$$

$$= e^{\alpha\sqrt{-1}}.\frac{x^ne^{n\beta\sqrt{-1}} - 1}{xe^{\beta\sqrt{-1}} - 1} - e^{-\alpha\sqrt{-1}}.\frac{x^ne^{-n\beta\sqrt{-1}} - 1}{xe^{-\beta\sqrt{-1}} - 1}.$$

Or, bringing to a common denominator, and arranging, &c., we easily get

$$\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots\ldots + x^{n-1}\sin(\alpha + \overline{n-1}.\beta)$$

$$= \frac{\sin\alpha - x\sin(\alpha-\beta) - x^n\sin(\alpha+n\beta) + x^{n+1}\sin(\alpha+\overline{n-1}.\beta)}{1 - 2x\cos\beta + 2x^2} \ldots\ldots\ldots\ldots(1).$$

And similarly we have

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots\ldots x^{n-1}\cos(\alpha+\overline{n-1}\,\beta)$$

$$= \frac{\cos\alpha - x\cos(\alpha-\beta) - x^n\cos(\alpha+n\beta) + x^{n+1}\cos(\alpha+\overline{n-1}\,.\,\beta)}{1-2x\cos\beta+x^2} \quad \ldots\ldots\ldots(2).$$

Cor. By making n infinite, we get, when $x<1$, the limits of the corresponding infinite series.

And similarly a variety of series of this kind may be summed.

The student who is acquainted with the Differential Calculus will at once see that, by differentiating the result in (1) with respect to α, we get the result in (2).

And similarly, and with the aid of the Integral Calculus, the sums of very complicated series may be easily obtained.

61. *To sum the series*

$$\operatorname{cosec}\alpha + \operatorname{cosec}2\alpha + \operatorname{cosec}2^2\alpha + \ldots\ldots + \operatorname{cosec}2^{n-1}\alpha.$$

We have

$$\begin{aligned}
\operatorname{cosec}\alpha &= \cot\frac{\alpha}{2} - \cot\alpha,\\
\operatorname{cosec}2\alpha &= \cot\alpha - \cot 2\alpha,\\
\operatorname{cosec}2^2\alpha &= \cot 2\alpha - \cot 2^2\alpha,\\
&\ldots\ldots\ldots\\
\operatorname{cosec}2^{n-1} &= \cot 2^{n-2}\alpha - \cot 2^{n-1}\alpha.
\end{aligned}$$

Therefore adding,

$$\operatorname{cosec} a + \operatorname{cosec} 2a + \operatorname{cosec} 2^2 a + \ldots + \operatorname{cosec} 2^{n-1} a$$
$$= \cot \frac{a}{2} - \cot 2^{n-1} a.$$

62. *To sum the series*

$$\tan a + \frac{1}{2} \tan \frac{a}{2} + \frac{1}{2^2} \tan \frac{a}{2^2} + \ldots + \frac{1}{2^{n-1}} \tan \frac{a}{2^{n-1}}.$$

We have

$$\tan a = \cot a - 2 \cot 2a$$
$$\frac{1}{2} \tan \frac{a}{2} = \frac{1}{2} \cot \frac{a}{2} - \cot a$$
$$\frac{1}{2^2} \tan \frac{a}{2^2} = \frac{1}{2^2} \cot \frac{a}{2^2} - \frac{1}{2} \cot \frac{a}{2}$$
$$\ldots\ldots\ldots\ldots$$
$$\frac{1}{2^{n-1}} \tan \frac{a}{2^{n-1}} = \frac{1}{2^{n-1}} \cot \frac{a}{2^{n-1}} - \frac{1}{2^{n-2}} \cot \frac{a}{2^{n-2}}.$$

Hence, adding,

$$\text{Sum to } n \text{ terms} = \frac{1}{2^{n-1}} \cot \frac{a}{2^{n-1}} - 2 \cot 2a.$$

Cor. Let n be infinite,

Then, $$\text{limit of } \frac{1}{2^{n-1}} \cot \frac{\alpha}{2^{n-1}} = \text{limit of } \frac{\frac{\alpha}{2^{n-1}}}{\tan \frac{\alpha}{2^{n-1}}} \cdot \frac{1}{\alpha} = \frac{1}{\alpha}.$$

$$\therefore \Sigma = \frac{1}{\alpha} - 2 \cot 2\alpha.$$

Ex. VII.

Show that

1. $x \sin \theta + x^2 \sin 2\theta + x^3 \sin 3\theta + \ldots$ to n terms
$$= \frac{x \sin \theta + x^{n+2} \sin n\theta - x^{n+1} \sin (n+1)\theta}{x^2 - 2x \cos \theta + 1}.$$

2. $(\text{Tan } \alpha + \cot \alpha) + (\tan 2\alpha + \cot 2\alpha) + (\tan 2^2 \alpha + \cot 2^2 \alpha) + \ldots$ to n terms
$$= 2 \cot \alpha - 2 \cot 2^n \alpha.$$

3. $\text{Tan } \alpha + 2 \tan 2\alpha + 2^2 \tan 2^2 \alpha + \ldots$ to n terms $= \cot \alpha - 2^n \cot 2^n \alpha$.

4. $\text{Sin } \alpha + \sin 3\alpha + \sin 5\alpha + \ldots + \sin (2n-1)\alpha = \frac{\sin^2 n\alpha}{\sin \alpha}$.

5. $\text{Cos } \alpha + \cos 3\alpha + \cos 5\alpha + \ldots + \cos (2n-1)\alpha = \frac{\sin 2n\alpha}{2 \sin \alpha}$.

6. $\text{Cos}^3\theta + \cos^3(\theta+\alpha) + \cos^3(\theta+2\alpha) + \ldots$ to n terms

$$= \frac{3}{4}\cdot\frac{\cos\left\{\theta+\frac{n-1}{2}\alpha\right\}\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}} + \frac{1}{4}\cdot\frac{\cos\left\{3\theta+\frac{n-1}{2}3\alpha\right\}\sin\frac{3n\alpha}{2}}{\sin\frac{3\alpha}{2}}.$$

7. $\text{Cos}\,\alpha\cos(\alpha+\beta) + \cos(\alpha+\beta)\cos(\alpha+2\beta) + \cos(\alpha+3\beta)\cos(\alpha+4\beta) + \ldots$ to n terms

$$= \frac{n}{2}\cos\beta + \frac{\cos(2\alpha+n\beta)\sin n\beta}{2\sin\beta}.$$

8. $\text{Cot}^{-1}(1+1+1^2) + \cot^{-1}(1+2+2^2) + \ldots$ to n terms $= \cot^{-1}\frac{n+2}{n}$.

9. $\text{Sec}\,\alpha\sec 2\alpha + \sec 2\alpha\sec 3\alpha + \sec 3\alpha\sec 4\alpha + \ldots$ to n terms

$$= \text{cosec}\,\alpha\left\{\tan(n+1)\alpha - \tan\alpha\right\}.$$

10. $\left(\frac{2}{\cos 2\theta}\right)^2 + \left(\frac{2^2}{\cos 2^2\theta}\right)^2 + \left(\frac{2^3}{\cos 2^3\theta}\right)^3 + \ldots$ to n terms

$$= \left(\frac{2^{n+1}}{\cos 2^{n+1}\theta}\right)^2 - \left(\frac{2}{\sin 2\theta}\right)^2.$$

11. $\left(\frac{1}{2}\text{cosec}\frac{\theta}{2}\right)^2 + \left(\frac{1}{2^2}\text{cosec}\frac{\theta}{2^2}\right)^3 + \left(\frac{1}{2^3}\text{cosec}\frac{\theta}{2^3}\right)^3 + \ldots$ to n terms

$$= (\text{cosec}\,\theta)^2 - \left(\frac{1}{2^n}\text{cosec}\frac{\theta}{2^n}\right)^2.$$

12. $\text{Sin}\,\frac{\theta}{2}\left(\frac{1}{2}\sec\frac{\theta}{2}\right)^3 + \sin\frac{\theta}{2^2}\left(\frac{1}{2}\sec\frac{\theta}{2^2}\right)^3 + \text{to } n \text{ terms}$

$$= \cos\frac{\theta}{2^n}\left(\frac{1}{2^n}\operatorname{cosec}\frac{\theta}{2^n}\right)^3 - \cos\alpha\,.\,\operatorname{cosec}^3\theta.$$

The sum to infinity of

13. $\text{Sin}\,\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + \ldots. = \frac{\sin\alpha - x\sin(\alpha-\beta)}{1 - 2x\cos\beta + x^2}.$

14. $x\cos(\alpha+\beta) + \frac{x^2}{1\,.\,2}\cos(\alpha+2\beta) + \frac{x^3}{1\,.\,2\,.\,3}\cos(\alpha+3\beta) + \ldots.$

$$= e^{x\cos\beta}\cos(\alpha + x\sin\beta) - \cos\alpha.$$

15. $x\sin(\alpha+\beta) + \frac{x^2}{1\,.\,2}\sin(\alpha+2\beta) + \frac{x^3}{1\,.\,2\,.\,3}\cos(\alpha+3\beta) + \ldots..$

$$= e^{x\cos\beta}\sin(\alpha + x\sin\beta) - \sin\alpha.$$

16. $x\cos(\theta+\alpha) + \frac{x(x-1)}{1\,.\,2}\cos(\theta+2\alpha) + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\cos(\theta+3\alpha) + \ldots.$

$$= 2^x\cos(\theta + \tfrac{1}{2}x\alpha)\cos^x\tfrac{1}{2}\alpha - \cos\theta.$$

17. $x\sin(\theta+\alpha) + \frac{x(x-1)}{1\,.\,2}\sin(\theta+2\alpha) + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}\sin(\theta+3\alpha) + \ldots.$

$$= 2^x\sin(\theta + \tfrac{1}{2}x\alpha)\cos^x\tfrac{1}{2}\alpha - \sin\theta.$$

18. Show that $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8} = \frac{3}{2}$.

19. If $\alpha = \frac{\pi}{17}$, show that

$$\cos \alpha + \cos 9\alpha + \cos 13\alpha + \cos 15\alpha = \tfrac{1}{4}(1 + \sqrt{17}),$$
$$\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \cos 11\alpha = \tfrac{1}{4}(1 - \sqrt{17}).$$

20. If $\alpha = \frac{\pi}{17}$, show that

$$(\cos \alpha + \cos 13\alpha)(\cos 9\alpha + \cos 15\alpha)$$
$$= (\cos 3\alpha + \cos 5\alpha)(\cos 7\alpha + \cos 11\alpha) = -\tfrac{1}{4}.$$

21. If $\phi = \frac{\pi}{13}$, show that

$$\cos 5\phi + \cos 7\phi + \cos 11\phi = \frac{1 - \sqrt{13}}{4},$$

and

$$\cos \phi + \cos 3\phi + \cos 9\phi = \frac{1 + \sqrt{13}}{4}.$$

22. If a circle be inscribed in an equilateral triangle, and in the angular points other circles be described, and so on, *ad infinitum;* then the sum of the areas of this series of circles $= \frac{1}{9}\frac{1}{6}\alpha^2\pi$, where α is the side of the triangle.

23. If a circle is inscribed in a square, and between this circle and the angular points circles be described, and so on *ad infinitum*: show that the sum of the areas of this series of circles $= \frac{1}{4}(44\sqrt{2} - 61)\alpha^2\pi$, where α is the side of the square.

24. Circles are described in triangles whose bases are the sides of a regular polygon of n sides, and whose vertices are the angular points of the polygon; show that:

(i.) The sum of the radii of the circles $= 2r\left(1 - n\sin^2\frac{\pi}{2n}\right)$,

(ii.) The sum of the areas of the circles $= 4\sin^2\frac{\pi}{2n}\left\{n\sin^2\frac{\pi}{2n} + \frac{n-4}{2}\right\} r^2\pi$,

where r is the radius of the circle circumscribing the polygon.

CHAPTER VIII.

CONSTRUCTION OF TRIGONOMETRICAL TABLES.

63. By means of the formula $\sin\theta < \theta$, and $> \theta - \frac{\theta^3}{4}$, it may be easily shown that

$$\sin 10'' = \cdot 000048481368\ldots;$$ and from this, that

$$\cos 10'' = \cdot 9999999988248\ldots.$$

Knowing these, we may find the sines and cosines of all angles which are multiples of $10''$.

Thus we have :

$$\sin (n + 1) A + \sin (n - 1) A = 2 \sin nA \cos A.$$

$$\therefore \sin (n + 1) A = 2 \cos A \,.\, \sin nA - \sin (n - 1) A.$$

Put $A = 10''$, and $n = 1, 2, 3$, &c., successively, then

$$\sin 20'' = 2 \cos 10'' \,.\, \sin 10'',$$

$$\sin 30'' = 2 \cos 10'' \,.\, \sin 20'' - \sin 10''$$

$$\sin 40'' = 2 \cos 10'' \,.\, \sin 30'' - \sin 20''$$

$$\text{&c.} \quad = \quad \text{&c.}$$

In this manner we may find the sines of the series of angles whose common difference is $10''$.

64. When the sines of all such angles up to $45°$ have been thus found, we may calculate the sines of the angles from $45°$ to $90°$, by means of the following formula, which is easily proved :

$$\sin (45° + A) = \sin (45° - A) + \sqrt{2} \,.\, \sin A.$$

Or, having found the sines of all such angles up to $60°$, we may find the remainder up to $90°$ more easily by the formula—

$$\sin (60° + A) = \sin (60° - A) + \sin A.$$

Having found the *sines* it is unnecessary to find the *cosines* independently, since we have

$$\text{ccs } A = \sin (90° - A).$$

The *tangents*, *secants*, *cotangents*, *cosecants* are, of course, easily obtained from the sines and cosines.

65. *Formulæ of Verification.*

As the trigonometrical ratios are found from each other, any error made is continued. It is therefore necessary to test the accuracy of the results. This may be done by what are called *formulæ of verification.* Amongst others we may mention the following:

(i.) $\text{Sin A} = \frac{1}{2}\left\{\sqrt{1 + \sin 2\text{A}} - \sqrt{1 - \sin 2\text{A}}\right\}$,

(ii.) $\text{Cos A} = \frac{1}{2}\left\{\sqrt{1 + \sin 2\text{A}} + \sqrt{1 - \sin 2\text{A}}\right\}$, where A in each case is less than 45°.

(iii.) *Euler's formula.*

$$\text{Sin A} + \sin(36° - \text{A}) + \sin(72° + \text{A}) = \sin(36° + \text{A}) + \sin(72° - \text{A}).$$

We may prove this formula thus:

$$\sin(72° + \text{A}) - \sin(72° - \text{A}) = 2\cos 72° . \sin \text{A} = 2\sin 18° . \sin \text{A}$$

$$= \frac{\sqrt{5} - 1}{2}\sin \text{A} \quad \ldots\ldots(1).$$

And so

$$\sin(36° + \text{A}) - \sin(36° - \text{A}) = 2\cos 36° . \sin \text{A} = \frac{\sqrt{5} + 1}{2}\sin \text{A} \quad \ldots\ldots(2).$$

(1) − (2), then,

$$\sin(72° + \text{A}) - \sin(72° - \text{A}) - \sin(36° + \text{A}) + \sin(36° - \text{A}) = \sin \text{A}; \text{ or}$$

$$\sin \text{A} + \sin(36° - \text{A}) + \sin(72° + \text{A}) = \sin(36° + \text{A}) + \sin(72° - \text{A}).$$

And similarly we may prove the following: viz.,

(iv.) *Legendre's formula.*

$$\text{Cos A or } \sin(90° - \text{A}) = \sin(54° + \text{A}) + \sin(54° - \text{A}) - \sin(18° + \text{A}) - \sin(18° - \text{A}).$$

As an example, suppose we make in (iii) and (iv), A = 11°, we have

$$\sin 11° + \sin 25° + \sin 83° = \sin 47° + \sin 61°; \text{ and}$$
$$\sin 79° = \sin 65° + \sin 43° - \sin 29° - \sin 7°.$$

By taking from the tables each of these *sines*, it is an easy matter to test the accuracy of the results there tabulated.

Logarithmic Tables of Trigonometrical Ratios.

66. Having determined the natural sines, cosines, &c., of angles, we may obtain from them the corresponding logarithmic trigonometrical ratios.

It is usual, however, to determine these independently.

67. *To find the logarithmic sine and cosine of an angle.*

We have (Art. 42),

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\ldots.$$

Put $\theta = \frac{m}{n} \cdot \frac{\pi}{2}$, or $\frac{\theta}{\pi} = \frac{m}{2n}$; then, taking the logarithms of each side, we have

$$\log_{10}\sin\left(\frac{m}{n}\ \frac{\pi}{2}\right) = \log_{10}\left(\frac{m}{n}\cdot\frac{\pi}{2}\right) + \log_{10}\left(1 - \frac{m^2}{2^2n^2}\right) + \log_{10}\left(1 - \frac{m^2}{4^2n^2}\right) + \log_{10}\left(1 - \frac{m^2}{6^2n^2}\right) + \ldots.$$

Or expanding, arranging, &c., we have,

$$\log_{10}\sin\left(\frac{m}{2}\cdot\frac{\pi}{2}\right) = \log_{10} m + \log_{10}(2n+m) + \log_{10}(2n-m) + \log_{10}\pi - 3(\log_{10} n + \log_{10} 2)$$

$$-\frac{1}{\log_e 10}\left\{\begin{array}{l} \left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots.\right)\frac{m^2}{n^2} \\ +\frac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ldots.\right)\frac{m^4}{n^4} \\ +\frac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ldots.\right)\frac{m^6}{n^6} \\ +\ldots. \end{array}\right\}.$$

From which, by giving m and n different values, so that $m < n$, we obtain logarithmic sines of all angles $< 90°$.

So from the formula

$$\cos\theta = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta}{5^2\pi^2}\right)\ldots.$$

we may directly obtain the logarithmic cosines.

The values of the other logarithmic trigonometrical ratios are then easily found. As explained in Vol. I., the tabular values are found by adding 10 to the values thus obtained.

Theory of Proportional Parts.

68. *The increment of the sine of an angle is in general approximately proportional to the small increment of the angle.*

Let the angle A receive an increment h, then the increment of sin A will be

$$\sin (A + h) - \sin A.$$

Denoting this by $\Delta \sin A$, we have

$$\begin{aligned}\Delta \sin A &= \sin (A + h) - \sin A \\ &= \sin A \,.\, \cos h + \cos A \,.\, \sin h - \sin A ; \\ &= \sin h \,.\, \cos A - (1 - \cos h) \sin A \\ &= \sin h \cos A \left\{ 1 - \frac{1 - \cos h}{\sin h} \,.\, \tan A \right\} \\ &= \sin h \cos A \left\{ 1 - \tan \frac{h}{2} \,.\, \tan A \right\}.\end{aligned}$$

If h be a small increment, we may put $\sin h = h$.

And unless tan A be very large, that is, unless A be very nearly 90°, the value of $\tan \frac{h}{2} \,.\, \tan A$ will be also very small, and may be neglected.

Hence, *unless* A *be very nearly a right angle*, we have

$$\Delta \sin A = h \cos A, \text{ very nearly.}$$

Hence, in determining any angle from its *sine*, we may adopt the principle of proportional parts, unless the angle be very nearly a right angle.

69. *The increment of the cosine of an angle is in general approximately proportional to the small increment of the angle.*

We have

$$\begin{aligned}\Delta \cos A &= \cos (A + h) - \cos A \\ &= \cos A \cos h - \sin A \sin h - \cos A \\ &= - \sin h \sin A - (1 - \cos h) \cos A\end{aligned}$$

$$= -\sin h \sin A \left\{ 1 + \frac{1 - \cos h}{\sin h} \cot A \right\}$$

$$= -\sin h \sin A \left(1 + \tan \frac{h}{2} \cot A \right).$$

Now, when A is very small, cot A is very large.

Hence, when the increment h is small, we find that,
Unless A be a very small angle,

$$\Delta \cos h = -h \sin A, \text{ very nearly.}$$

It will be noticed that the cosine diminishes as the angle increases.

70. *The increment of the tangent of an angle is in general proportional to the small increment of the angle.*

We have

$$\Delta \tan A = \tan (A + h) - \tan A$$

$$= \frac{\sin (A + h)}{\cos (A + h)} - \frac{\sin A}{\cos A} = \frac{\sin (A + h - A)}{\cos (A + h) \cos A}$$

$$= \frac{\sin h}{\cos (A + h) \cos A}$$

$$= \frac{\tan h}{\cos^2 A (1 - \tan h . \tan A)}.$$

Hence, as before, if h be small, and A be not nearly 90°, we have approximately

$$\Delta \tan A = \frac{h}{\cos^2 A} = h \sec^2 A \ldots\ldots\ldots\ldots (1).$$

And so we may show that,
Unless A be a very small angle,

$$\Delta \cot A = -h \operatorname{cosec}^2 A \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Also, that,
When A is *neither very small, nor nearly equal to* 90°,

$$\Delta \sec A = h \sin A \sec^2 A \ldots\ldots\ldots\ldots\ldots (3),$$

and $$\Delta \operatorname{cosec} A = -h \cos A \operatorname{cosec}^2 A \ldots\ldots\ldots (4).$$

71. *The increment of the values of the logarithmic trigonometrical ratios is in general proportional to the small increment of the angle.*

(i.) Δ L sin A.

We have $\sin (A + h) = \sin A \cos h + \cos A \sin h$

$= \sin A + h \cos A$, approximately.

$$\therefore \frac{\sin (A + h)}{\sin A} = 1 + h \cot A.$$

Taking the logarithms, we have

$$\log \sin (A + h) - \log \sin A = \log (1 + h \cot A).$$

But $\log \sin (A + h) - \log \sin A = L \sin (A + h) - L \sin A$

$= \Delta L \sin A.$

And

$$\log (1 + h \cot A) = \mu \left\{ h \cot A - \tfrac{1}{2} h^2 \cot^2 A + \&c. \right\}$$

$$= \mu h \cot A, \text{ approximately.}$$

Hence $\Delta L \sin A = \mu h \cot A$,

except in cases to be discussed in the next article.

So we may easily show that

(ii.) $\Delta L \cos A = - \mu h \tan A.$

And also that

(iii.) $\Delta L \tan A = 2 \mu h \operatorname{cosec} 2A.$

(iv.) $\Delta L \cot A = - 2 \mu h \operatorname{cosec} 2A.$

(v.) $\Delta L \operatorname{cosec} A = - \mu h \cot A.$

(vi.) $\Delta L \sec A = \mu h \tan A.$

72. The limits of our space compel us to omit the purely trigonometrical investigations as to the amount of error, and the cases of failure in the principle of proportional parts.

To the student who is acquainted with the Differential Calculus we will indicate how this may be done by Taylor's Theorem.

We have, by Taylor's Theorem.

$$\sin (A + h) = \sin A + h \cos A - \frac{h^2}{1 \cdot 2} \sin (A + \theta h),$$

where θ *is some* proper fraction.

Hence $\Delta \sin A = h \cos A - \frac{h^2}{1 \cdot 2} \sin (A + \theta h)$ .....(1).

Hence, by taking $\Delta \sin A = h \cos A$, *the error is less than* $\frac{h^2}{1 \cdot 2}$.

Again, when A is a small angle, $\frac{h^2}{1 \cdot 2} \sin (A + \theta h)$ is extremely small.

But when A is nearly equal to 90°,

$\sin (A + \theta h)$ *is very nearly unity.*

Hence the value of $\frac{h^2}{1 \cdot 2} \sin (A + \theta h)$ is sensibly comparable with $h \cos A$, as cos A is then very small.

Hence the differences between the sines of angles which are nearly 90° are very small, or, as it is called, *insensible;* and the second term of the series in (1) cannot be neglected, so that the increments of the sines are not proportional to the increments of the angle. They are in this case said to be *irregular.*

We hence conclude that:

The differences of consecutive sines, when the angle is nearly 90°, are *insensible* and *irregular.*

And so we may show that for cosines they are *insensible* when the angles are very small; for tangents they are *irregular* when the angles are nearly 90°; for cotangents *irregular* when the angles are very small; for secants *insensible* when the angles are small, and *irregular* when nearly 90°; and for cosecants *irregular* when the angles are small, and *insensible* when nearly 90°. *Irregularity* accompanies *insensibility.*

Again, by Taylor's Theorem,

$$\log \sin (A + h) = \log \sin A + \mu h \cot A - \frac{\mu h^3}{2} \operatorname{cosec}^2 (A + \theta h),$$

where θ *is* some proper fraction.

$$\therefore \Delta \text{ L sin A} = \mu h \cot \text{A} - \frac{\mu h^2}{2} \operatorname{cosec}^2 (\text{A} + \theta h).$$

Hence, if we may neglect the second term, we have approximately,

$$\Delta \text{ L sin A} = \mu h \cot \text{A}.$$

When A is very small, $\operatorname{cosec}^2 (\text{A} + \theta h)$ is very large, and the second term may not be neglected.

Hence the differences of consecutive logarithmic sines are irregular *when the angle is very small.*

Again, when A is very nearly equal to 90°,

$\operatorname{cosec}^2 (\text{A} + \theta h)$ is very nearly equal to unity.

Hence the value of $\frac{\mu h^2}{2} \operatorname{cosec}^2 (\text{A} + \theta h)$ is comparable to $\mu h \cot \text{A}$, for cot A is then very small.

Hence the differences are *insensible* when the angles are nearly 90°.

And in a similar way it may be shown that the principle of proportional parts fails for every logarithmic trigonometrical ratio, when the angle is either very small or very nearly equal to a right angle. We may express these results as follows :

	Angles Small.	Angles nearly 90°.
Logarithmic :		
(1.) Sines and cosecants, ...	irregular,...	insensible.
(2.) Tangents and cotangents,	irregular,...	irregular.
(3.) Cosines and secants, ...	insensible,	irregular.

73. Hence, when an angle is very small or nearly 90°, it is

impracticable to use the principle of proportional parts in the case of logarithmic trigonometrical functions taken from ordinary tables.

We sometimes however require to know the *number of seconds* in a very small angle. The methods which may be followed will be now explained.

First Method.

Tables are constructed for a few degrees to intervals of a second, instead of to intervals of 10″, or of a minute.

Here h must be less than one second, and the greatest value of $\frac{h^2}{2}\operatorname{cosec}^2(A + \theta h)$ is $\frac{h^2}{2}$, where h is the circular measure of *one second*. It is therefore small enough to be neglected.

Second Method. Maskelyne's Method.

We have $\sin\alpha = \theta - \frac{\theta^3}{1.2.3} + \ldots = \alpha - \frac{\alpha^3}{1.2.3}$ nearly.

And $\cos\alpha = 1 - \frac{\alpha^2}{1.2} + \ldots = 1 - \frac{\alpha^2}{2}$ nearly.

$$\therefore \frac{\sin\alpha}{\alpha} = 1 - \frac{\alpha^2}{6} = \left(1 - \frac{\alpha^2}{2}\right)^{\frac{1}{3}}, \text{ nearly.}$$

$$= (\cos\alpha)^{\frac{1}{3}}, \text{ nearly ; or } \sin\alpha = \alpha(\cos\alpha)^{\frac{1}{3}}.$$

Let α be an angle containing n'', then

Circular measure of the angle $= n \sin 1''$

or $\alpha = n \sin 1''$.

$\therefore \sin\alpha = n \sin 1'' (\cos\alpha)^{\frac{1}{3}}$, or

$\sin n'' = n \sin 1'' (\cos n'')^{\frac{1}{3}}$; or, taking logarithms.

$$\begin{aligned} \mathrm{L}\sin n'' &= \log n + \mathrm{L}\sin 1'' + \tfrac{1}{3}(\mathrm{L}\cos n'' - 10) \\ &= \log n + \mathrm{L}\sin 1'' - \tfrac{1}{3}(10 - \mathrm{L}\cos n'') \ldots\ldots (1). \end{aligned}$$

And therefore

$$\log n = \mathrm{L}\sin n'' + \tfrac{1}{3}(10 - \mathrm{L}\cos n'') - \mathrm{L}\sin 1'' \ldots (2).$$

Since the angle n'' is small, we may, in finding n from L sin n'' or in finding L sin n'' from n, make use of an

approximate value of L cos n'' from the tables; for its variation is very small when the variation of n is small.

We may find a similar value for the *tangent* of a small angle.

$$\tan \alpha^2 = \frac{\sin n}{\cos \alpha} = \left(\alpha - \frac{\alpha^3}{6}\right)\left(1 - \frac{\alpha^2}{2}\right)^{-1}$$

$$= \left(\alpha - \frac{\alpha^3}{6}\right)\left(1 + \frac{\alpha^2}{2}\right) = \alpha + \frac{\alpha^3}{3}.$$

$$\therefore \frac{\tan \alpha}{\alpha} = 1 + \frac{\alpha^3}{3} = \left(1 - \frac{\alpha^2}{2}\right)^{-\frac{2}{3}} = (\cos \alpha)^{-\frac{2}{3}}$$

Hence, as before

$$\text{L tan } n'' = \log n + \text{L tan } 1'' - \tfrac{2}{3}\,(\text{L cos } n'' - 10).$$

Third Method—Delambre's Method.

Delambre constructed a table for *every second* to a degree for $\log \frac{\sin \alpha}{\alpha} + \text{L sin } 1''$.

We have, if α be the circular measure of an angle of n''.

$$\log \frac{\sin \alpha}{\alpha} = \log \frac{\sin n''}{n \sin 1''}$$

$$= \text{L sin } n'' - \log n - \text{L sin } 1''.$$

$$\therefore \text{L sin } n'' = \log n + \left(\log \frac{\sin \alpha}{\alpha} + \text{L sin } 1''\right) \ldots\ldots\ldots\ldots (3),$$

$$\text{and } \log n = \text{L sin } n'' - \left(\log \frac{\sin \alpha}{\alpha} + \text{L sin } 1''\right) \ldots\ldots\ldots\ldots (4).$$

Suppose the L sin of the angle is known, and we require to find n.

We can find *approximately*, to the nearest integer, the value of n, and then making use of this to obtain from the tables the value of $\left(\log \frac{\sin \alpha}{\alpha} + \text{L sin } 1''\right)$, we can at once find $\log n$, and then n from a table of logarithms of numbers.

By making use of the approximate value of n we are of course liable to error. That the error is not sensible will be seen as follows:

We have

$$\frac{\sin \alpha}{\alpha} = 1 - \frac{\alpha^2}{1.2.3} + \frac{\alpha^4}{1.2.3.4.5} - \ldots = 1 - \frac{\alpha^2}{6}, \text{ nearly.}$$

$$\therefore \log_{10} \frac{\sin \alpha}{\alpha} = \mu \log \left(1 - \frac{\alpha^2}{6}\right)$$

$$= \mu \left\{ - \frac{\alpha^2}{6} - \tfrac{1}{2} \left(\frac{\alpha^2}{6}\right) - \&c. \right\}.$$

$$\therefore \Delta \log_{10} \frac{\sin \alpha}{\alpha}$$

$$= - \mu \left\{ \tfrac{1}{6} \left\{ (\alpha + h)^2 - \alpha^2 \right\} + \tfrac{1}{72} \left\{ (\alpha + h)^4 - \alpha^4 \right\} + \&c. \right\}$$

$$= - \mu \left(\frac{\alpha}{3} + \frac{\alpha^3}{18} + \ldots \right) h = - \frac{\mu h}{3} \alpha \text{ very nearly.}$$

$$\therefore \Delta \log_{10} \frac{\sin \alpha}{\alpha} \div h = - \frac{\mu}{3} \alpha.$$

Hence, the variation of $\log \frac{\sin \alpha}{\alpha}$ is much less rapid than that of α, and the error will therefore be insensible

74. Since, Art. 70, $\left.\begin{aligned} \Delta \sin A &= h \cos A, \\ \text{and} \quad \Delta \cos A &= - h \sin A, \end{aligned}\right\}$ approximately,

we see that the differences for consecutive sines vary approximately as the cosine, and the differences for consecutive cosines vary approximately as the sine.

Now, when the angle is $< 45°$, *sine* is $<$ *cosine*.
and „ „ $> 45°$, *sine* is $>$ *cosine*.

Hence, for angles less than 45°, the differences for consecutive sines will be greater than the differences for consecutive cosines, and *vice versâ*.

Hence, for angles less than 45°, it is better to determine them from their *sines* than from their cosines; and for angles greater than 45° it is better to determine them from their *cosines*. And the same rule should be observed in the use of *logarithmic* sines and cosines.

SECTION III.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

DEFINITIONS, AND INTRODUCTORY PROPOSITIONS.

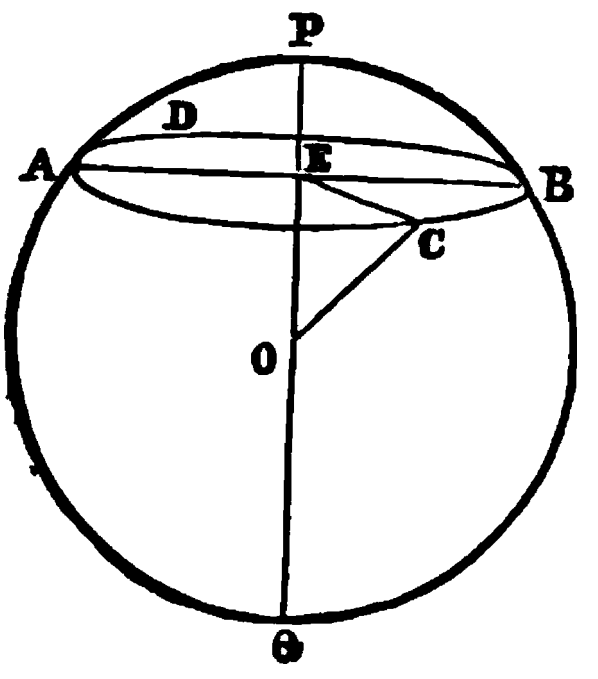
1. A sphere is a body, every point of the surface of which is equidistant from a certain point within it, called the centre. We may also define a sphere as the solid generated by the revolution of a semicircle about its diameter or axis.

2. *Every section of a sphere made by a plane is a circle.*

Let ABCD be the section made by a plane passing through the points A and B, of the sphere whose centre is O.

From O draw OE at right angles to the plane, meeting it in E.

Take any point C in the circumference of the section, and draw the line EC.



Then (Euc. XI., Def. 3), OE is perpendicular to EC.

Hence, $EC^2 = OC^2 - OE^2$.

$\therefore EC = \sqrt{OC^2 - OE^2}$, a constant quantity, since

OC is the radius of the sphere, and OE the distance of the plane from the centre of the sphere.

Hence, it follows that all straight lines drawn from the point E to the circumference of the section, are equal to one another.

The section is, therefore, a circle, whose centre is E.

Cor. 1. When the plane passes through the centre of the sphere, the section is a circle, whose radius is that of the sphere.

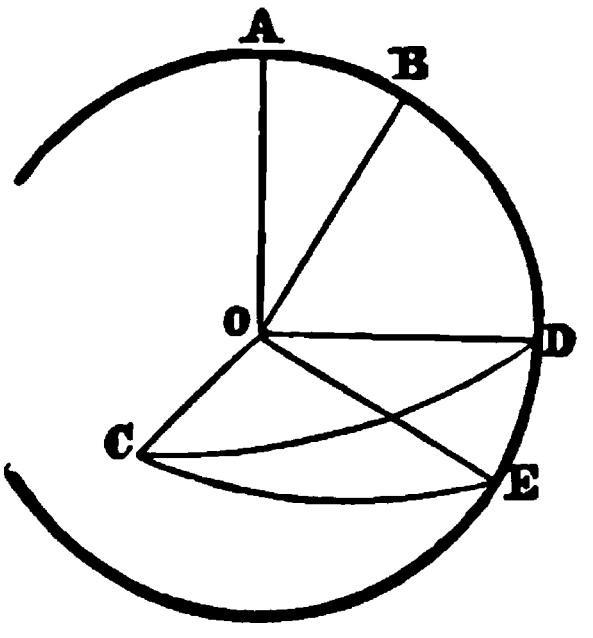
Def. A **great circle** is a section of a sphere made by a plane passing through the centre of the sphere, and has its radius equal therefore to the radius of the sphere.

All other sections are termed **small circles.**

If OE be produced both ways, to meet the sphere in P and Q, the points P and Q, where the perpendicular to the plane of the section meets the surface of the sphere, are called **the poles of the circle** ACDB.

Cor. 2. It follows from this definition that all arcs of great circles, and all chords, drawn from a pole to the circumference of the corresponding circle, are equal to each other.

3. *The angle subtended at the centre of the sphere by the arc of a great circle, which joins the poles of two great circles, is equal to the angle of inclination of the planes of the circles.*



Let CD, CE be the two given great circles;

A, B, their poles respectively, O the centre of the sphere, and ABDE the great circle passing through the poles A, B, and meeting CD, CE, respectively in DE.

Then AO is perpendicular to CO, and OB is also perpendicular to CO.

Hence CO is perpendicular to both AO and OB. It is therefore perpendicular to every line drawn from O in the plane AOB (Euc. XI. 4).

Hence CO is perpendicular to OD and OE.

Now OC is the intersection of the planes OCD and OCE, and OD and OE are lines drawn respectively in those planes.

Hence (Euc. XI., Def. 6) DOE is the angle of inclination of the planes OCD and OCE.

Now $\angle$ AOB = $\angle$ AOD − $\angle$ BOD

= $\angle$ BOE − $\angle$ BOD = $\angle$ DOE.

DEF. The angle between the arcs of two great circles is the **angle of inclination** of the planes of the circles.

Hence $\angle$ DCE = $\angle$ DOE = $\angle$ AOB.

COR. Hence by Art. 2, any circle of a sphere is at right angles to the great circles which pass through its poles.

4. *Two great circles bisect each other.*

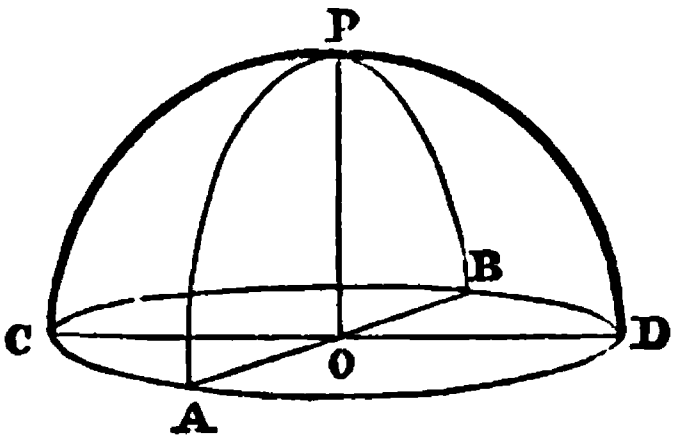
As the planes of all great circles pass through the centre of the sphere, it follows that a diameter of the sphere is the line of intersection of any two circles. Each segment of the great circles must therefore be a semicircle.

5. *Arcs of great circles drawn from a pole of a great circle to points in its circumference are quadrants.*

Let P be the pole of the great circle ACBD, and let O be the centre of the sphere, which therefore lies in the plane of ACBD.

Then PO is at right angles to the plane ACBD.

Draw PA, PB, arcs of great circles, from P to the points A, B.



Then, since PO is perpendicular to the plane ACBD,

$\angle$ AOP is a right angle.

Hence, the arc AP, which subtends the angle AOP, must be a quadrant. And similarly, PB is a quadrant.

DEF. Great circles drawn through the pole of a great circle are called **secondaries** to that great circle.

Thus, in the figure, Art. 3, where C is a pole of ABDE, the arcs CD, CE are arcs of secondaries to the great circle ABDE.

Cor. Since the arc DE measures the angle of inclination DOE, it follows that *the angle between any two great circles is measured by the arc intercepted on the great circle to which they are secondaries.*

6. *If from a point on the surface of a sphere there can be drawn two arcs of great circles, at right angles to a given circle, which two arcs are not parts of the same circle, that point is a pole of the given circle.*

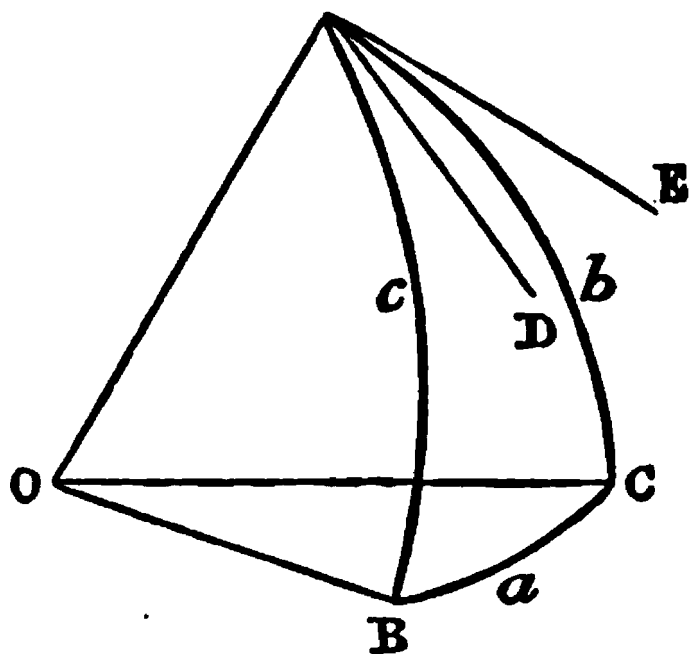
Since the arcs are at right angles to the given circle, their planes must intersect in a line which is perpendicular to the plane of the given circle, and this line must pass through the centre of the given circle and sphere.

Hence, the line is the axis of the given circle, and the point from which the arcs spring is a pole of the given circle.

CHAPTER II.

SPHERICAL TRIANGLES.

7. **Spherical Trigonometry** investigates the relations which subsist between the angles of the plane faces of a solid angle, and the inclinations of the plane faces to each other.



Let O be the centre of a sphere, and let a solid angle be formed at O, by the three planes, BOC, AOC, AOB.

These planes will cut the surface of the sphere in the arcs of great circles, BC, AC, AB.

Then ABC is called a spherical triangle.

In Spherical Trigonometry the arcs of great circles only are concerned, and it must be understood, that when arcs

are in future mentioned, they are arcs of great circles unless it be otherwise expressed.

BC, AC, AB are called the sides of the spherical triangle ABC, and are usually expressed respectively by the letters *a*, *b*, *c*.

If AD, AE be drawn respectively tangents to the arcs AB, AC, then OA is perpendicular to both AD, AE.

Hence (Euc. XI., Def. 6),

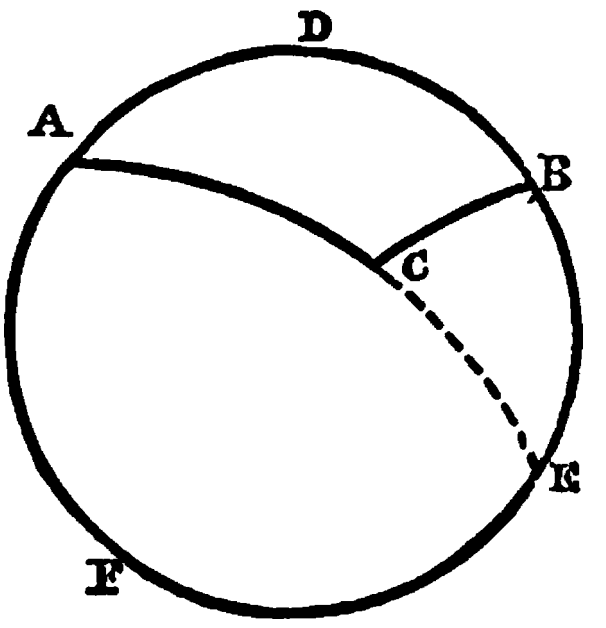
$\angle$ DAE is the angle of inclination of the planes AOB, AOC.

Hence, *the angle between two sides*, AB, AC, *of a spherical triangle, is the angle between the tangents drawn at the point of intersection of the arcs.*

It is convenient in Spherical Trigonometry to limit the sides of a triangle to less than a semicircle.

8. *Every angle of a spherical triangle is less than two right angles.*

For suppose, if possible, that a triangle is formed of the arcs AC, BC, AEB, where the angle ACB is greater than two right angles.



Since the angle ACB is greater than two right angles, if we produce AC it will meet the circle ACBD in some point between A and B on the side remote from D.

Let it be produced and meet ACB in E.

Then (Art. 4), arc AFE is a semicircle, and therefore AEB is greater than a semicircle, and consequently the triangle we have supposed is not one considered in Spherical Trigonometry.

9. *Any two sides of a spherical triangle are together greater than the third side, and the sum of the three sides is less than the circumference of a great circle.*

Using the figure of Art. 7, we have, by Euc. XI., 20,

Any two of the angles AOB, AOC, BOC, are together greater than the third.

Now these angles are respectively subtended by the arcs AB, AC, BC.

Hence any two of the sides AB, AC, BC of the triangle ABC, are together greater than the third.

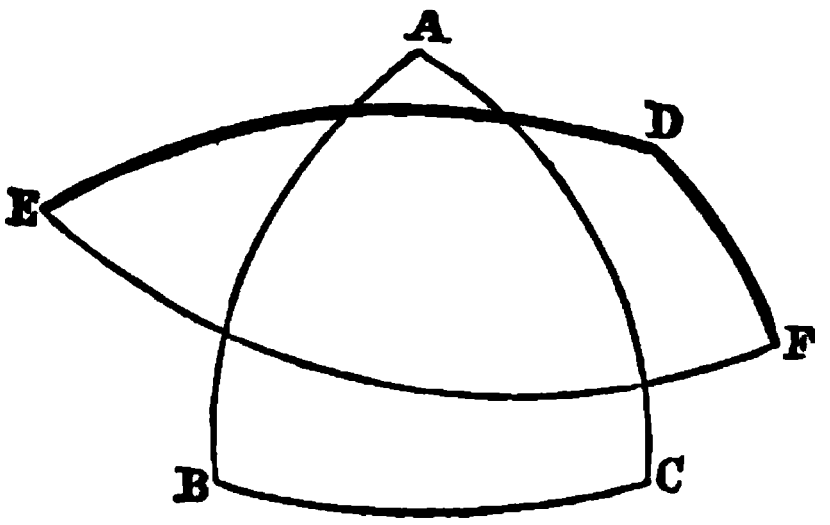
Again, by Euc. XI., 21,

The three angles AOB, AOC, BOC, forming the solid angle at O, are together less than four right angles.

And four right angles is the angle subtended at the centre by the circumference of a great circle.

Hence, AB, AC, BC, are together less than the circumference of a great circle.

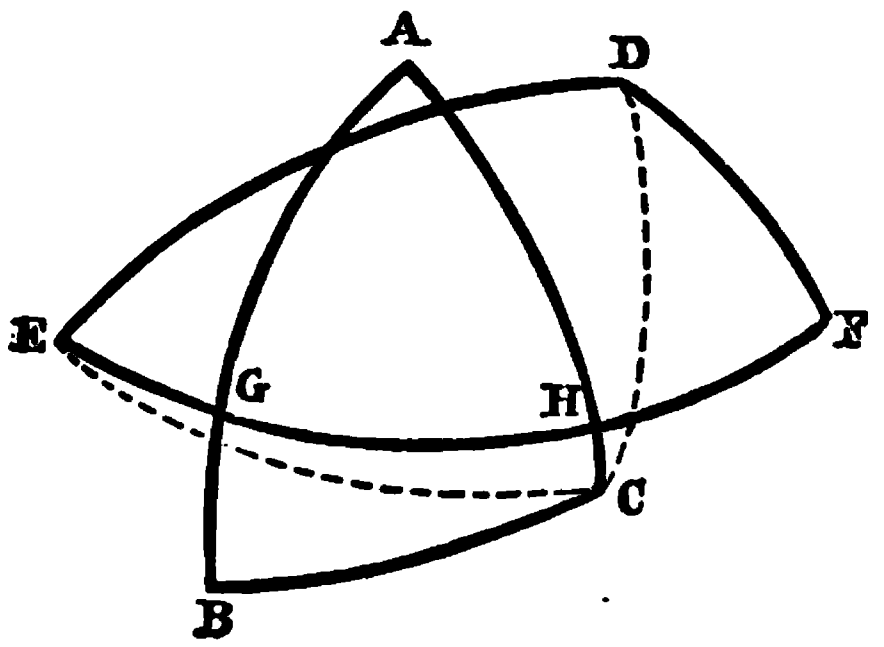
The Polar Triangle.



10. DEF. If the points D, E, F be respectively the poles of the sides BC, AC, AB of the triangle ABC, then the triangle DEF is the **polar triangle** of the *primitive triangle* ABC.

As the sides of the triangle ABC have each two poles, it is evident we may form *eight* triangles having for their angular points poles of the triangle ABC. Thus, if D′, E′, F′ be the other poles corresponding to D, E, F, we may form the triangle DEF, D′E′F′, DE′F′, D′EF′, ED′F′, E′DF, FD′E′, F′DE. The triangle DEF is, however, the only one when the poles D, E, F lie *towards the same parts* with the corresponding angles A, B, C. This is the triangle meant in the above definition.

11. *If one triangle is the polar triangle of another, the latter is the polar triangle of the former.*



Let DEF be the polar triangle of ABC, then shall ABC be the polar triangle of DEF.

Since D is the pole of BC, then DC is a quadrant, and since E is the pole AC, then EC is a quadrant.

Hence the arcs CE, CD, drawn from the point C to

the great circle ED, are quadrants; and hence C is the pole of ED.

And, similarly, we may show that A, B are respectively the poles of EF, DF.

Hence ABC is the polar triangle of DEF.

The polar triangle is often called the **supplemental triangle**, from the property proved in the next article.

12. *The sides and angles of the polar triangle are respectively the supplements of the angles and sides of the primitive triangle.*

For suppose the arc EF to meet the arcs AB, AC in the points G, H, producing the arc EF, if necessary.

Then (Art. 5),

$$\begin{aligned} \angle A &= GH = EH - EG \\ &= EH - (EF - GF) \\ &= EH + GF - EF. \end{aligned}$$

But EH and GF are quadrants, and subtend, therefore, angles each equal to $\frac{\pi}{2}$. And EF is the side of the polar triangle corresponding to the angle A.

Hence, if a', b', c', represent respectively the sides of the polar triangle corresponding to the sides a, b, c of the primitive triangle, we have

$$A = \pi - a',$$

And so $$B = \pi - b', \; C = \pi - c'.$$

And again, similarly, if A′, B′, C′, be the angles of the polar triangle,

$$a = \pi - A', \; b = \pi - B', \; c = \pi - C'.$$

Cor. Hence, *if a general relation be established between the sides and angles of a spherical triangle, we shall still have a true relation, if, for the angles, we substitute the supplements of the respective sides; and for the sides, the supplements of the respective angles.*

Thus, it will be shown farther on that

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad \ldots\ldots\ldots\ldots (1).$$

Hence, in the polar triangle,

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'} \quad \ldots\ldots\ldots\ldots (2).$$

But

$$A' = \pi - a, \; a' = \pi - A, \; b' = \pi - B, \; c' = \pi - C.$$

Hence, substituting in (2), we have

$$\cos(\pi - a) = \frac{\cos(\pi - A) - \cos(\pi - B)\cos(\pi - C)}{\sin(\pi - B)\sin(\pi - C)}.$$

This is just what we at once obtain by replacing in (1) the angle and sides by the supplements of the corresponding side and angles respectively.

The result, when simplified, gives

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

The proposition here established is one of fundamental importance.

13. *The sum of the angles of a spherical triangle is greater than two right angles, and less than six right angles.*

By Art. 9, $a + b + c < 2\pi$.

Put $\pi - A$, $\pi - B$, $\pi - C$ for a, b, c respectively, then

$$(\pi - A) + (\pi - B) + (\pi - C) < 2\pi.$$

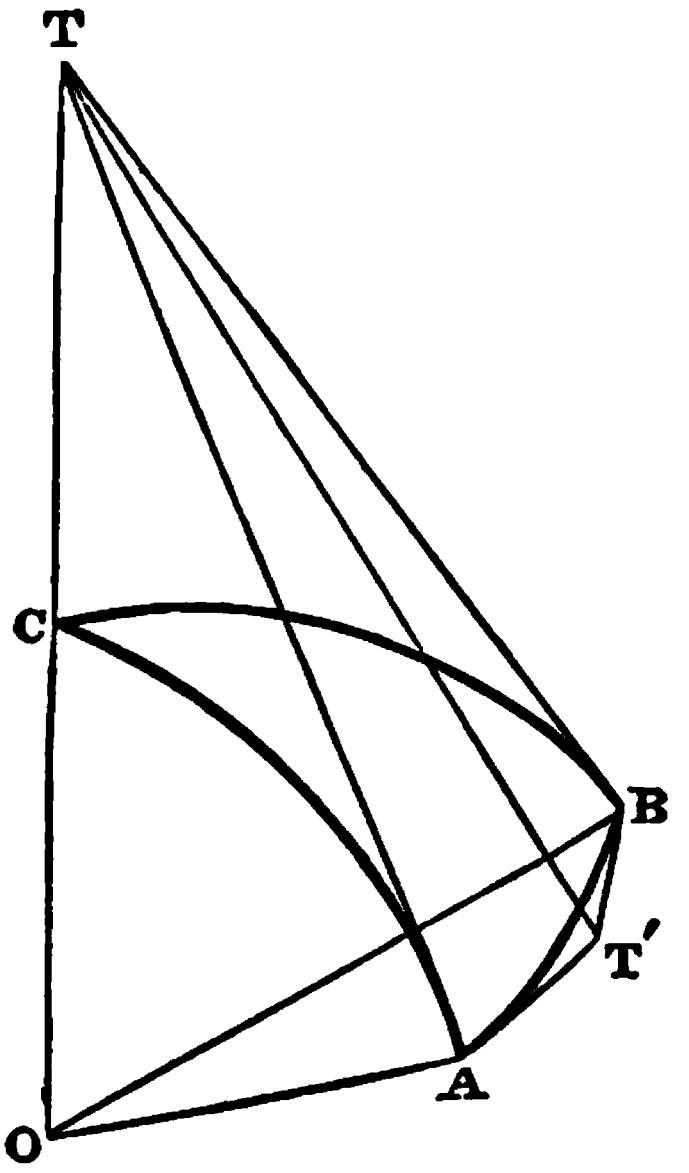
Hence $A + B + C > \pi$.

Again, Art. 8, each of the angles A, B, C is $< \pi$.

Hence $A + B + C < 3\pi$.

$\therefore A + B + C > \pi$, and $< 3\pi$. Q.E.D.

14. *The angles at the base of an isosceles spherical triangle are equal to one another.*



Let ABC be a spherical triangle, having the side BC equal to the side AC.

Then shall $\angle A = \angle B$.

First, let the two given sides be less than quadrants.

At A, B draw tangents AT, BT. These will meet OC produced in the same point T, since CA = CB.

Again, draw AT′, BT′, tangents to the arc AB, then AT′ = BT′.

Also, we have AT = BT.

Hence, in the two triangles ATT′, BTT′ the sides AT, AT′ are respectively equal to the sides BT, BT′, and TT′ is common.

$$\therefore \angle TAT' = \angle TBT'.$$

But (Art. 7) these are respectively the angles A and B.

$$\therefore \angle A = \angle B.$$

If the sides AC, BC are greater than quadrants, the tangents AT, BT will meet OC produced in the opposite direction, and the proof will be similar.

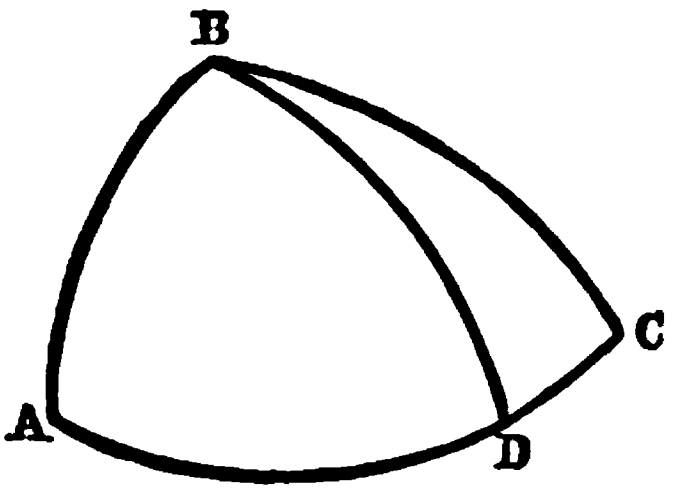
If the sides AC, BC are quadrants, then (Art. 3) the angles A, B are right angles.

Cor. Conversely, from the same figure it may be shown that if the angles A and B are equal, then the side BC is equal to the side AC.

Hence, also, *every equilateral triangle is equiangular, and every equiangular triangle is equilateral.*

15. *If two angles of a triangle be unequal, the side opposite to the greater angle is greater than the side opposite to the less.*

Let the angle B be greater than the angle A.



Then shall AC be greater than BC.

At B make the angle ABD equal to the angle at A.

Then (Art. 14,) $BD = AD$.

Now (Art. 9,) $BD + DC > BC$.

Hence, $AD + DC > BC$;

that is, $AC > BC$.

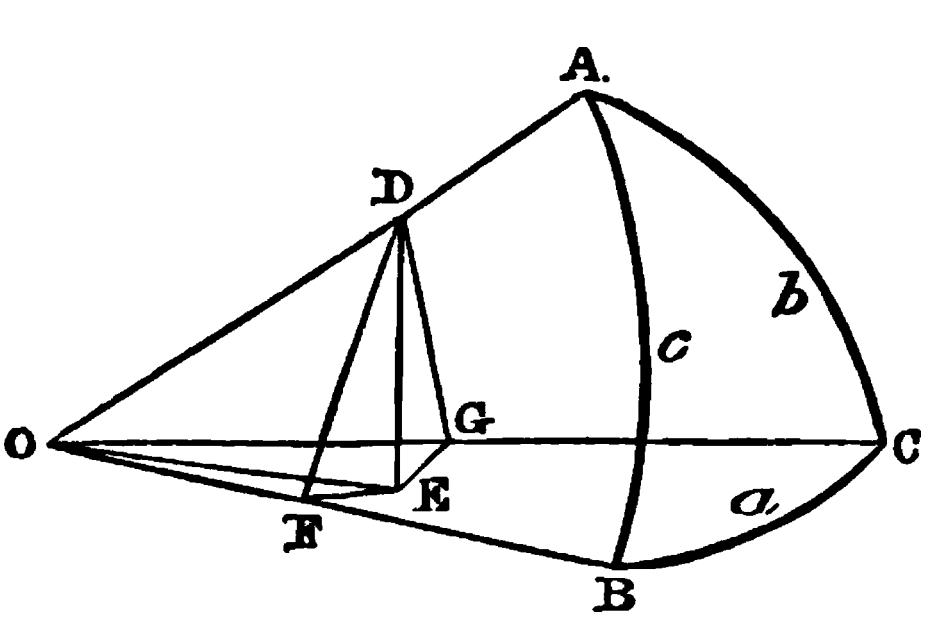
And similarly, the converse proposition may be proved, viz. :

If two sides of a triangle be unequal, the angle opposite to the greater side is greater than the angle opposite to the less.

CHAPTER III.

RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SIDES AND ANGLES OF SPHERICAL TRIANGLES.

16. *The sines of the angles are proportional to the sines of the opposite sides.*



Let ABC be the spherical triangle, O the centre of the sphere.

Join OA, OB, OC, and from any point D in OA, draw DE perpendicular to the plane BOC; also, from D draw DF, DG perpendicular respectively to OB, OC. Join OE, EF, EG.

Then, DEG, DEF are right angles since DE is perpendicular to the plane BOC.

$$\therefore EF^2 = DF^2 - DE^2$$
$$= (OD^2 - OF^2) - (OD^2 - OE^2) = OE^2 - OF^2.$$

Hence (Euc. I., 48), $\angle$ OFE is a right angle. And so $\angle$ OGE is a right angle.

Hence DF, EF are each at right angles to OF.

Therefore, $\angle$ DFE is the inclination of the planes AOB, BOC (Euc. XI., Def. 6).

$$\therefore \angle \text{DFE} = \text{B}.$$

And so, $$\angle \text{DGE} = \text{C}.$$

Now $$\text{DE} = \text{DF} \sin \text{DFE} = \text{OD} \cdot \sin \text{AOB} \cdot \sin \text{DFE}$$
$$= \text{OD} \sin c \sin \text{B}.$$

And so $$\text{DE} = \text{OD} \sin b \sin \text{C}.$$

Hence $$\sin c \sin \text{B} = \sin b \sin \text{C}.$$
$$\therefore \frac{\sin \text{B}}{\sin b} = \frac{\sin \text{C}}{\sin c}.$$

It at once follows from the form of this result, that

$$\frac{\sin \text{A}}{\sin a} = \frac{\sin \text{B}}{\sin b} = \frac{\sin \text{C}}{\sin c}.$$

In the figure the sides are all less than quadrants, and the perpendicular DE falls within the triangle BOC. Any other case will be found to admit of a similar proof.

17. *To express the cosine of an angle in terms of the sines and cosines of the sides.*

Let ABC be the triangle, O the centre of the sphere.

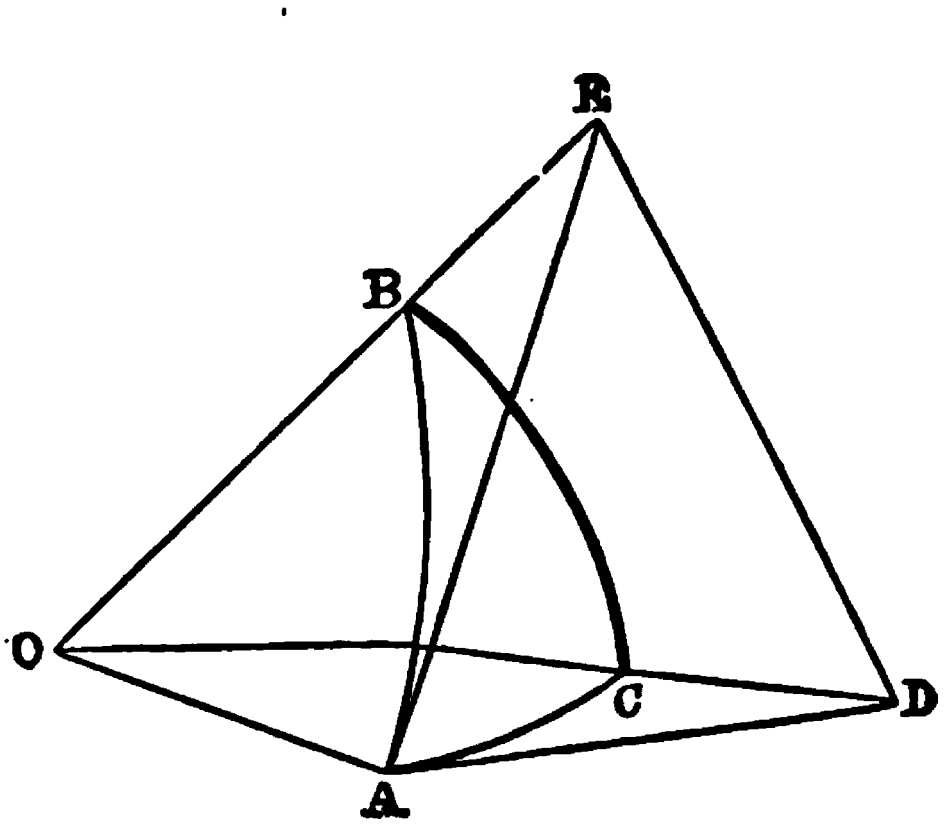
Join OA, OB, OC.

At A draw the lines AD, AE tangents to AC, AB respectively, and meeting OC, OB in D, E.

Join DE.

Then (Art. 7), $\angle$ DAE $= \angle$ BAC $=$ A.

From the triangle EOD, we have



$$\text{DE}^2 = \text{OE}^2 + \text{OD}^2 - 2\,\text{OE} \cdot \text{OD} \cos \text{DOE}$$
$$= \text{OE}^2 + \text{OD}^2 - 2\,\text{OE} \cdot \text{OD} \cos a.$$

And from the triangle EAD, we have

$$DE^2 = AE^2 + AD^2 - 2\,AE.\,AD\cos DAE$$
$$= AE^2 + AD^2 - 2\,AE.\,AD\cos A.$$

Hence, equating these results,

$$AE^2 + AD^2 - 2\,AE.\,AD\cos A$$
$$= OE^2 + OD^2 - 2\,OE.\,OD\cos a.$$

$$\therefore\ 2\,OE.\,OD\cos a$$
$$= (OE^2 - AE^2) + (OD^2 - AD^2) + 2\,AE.\,AD\cos A$$
$$= OA^2 + OA^2 + 2\,AE.\,AD\cos A.$$

$$\therefore\ \cos a = \frac{OA}{OD}\cdot\frac{OA}{OE} + \frac{AE}{OD}\cdot\frac{AD}{OE}\cos A,$$

or $$\cos a = \cos b\cos c + \sin b\sin c\cos A.$$

And so

$$\cos b = \cos a\cos c + \sin b\sin c\cos B,$$

and $$\cos c = \cos a\cos b + \sin a\sin c\cos C.$$

From these formulæ, the cosines of the angles may be expressed in terms of the sines and cosines of the sides (Art. 12).

18. *To express the cosine of a side in terms of the sines and cosines of the angles.*

We have, by the last Art.,

$$\cos a = \cos b\cos c + \sin b\sin c\cos A.$$

Hence (Art. 12), replacing A by $\pi - a$, and a, b, c respectively by $\pi - A$, $\pi - B$, $\pi - C$, we get

$$\cos(\pi - A)$$
$$= \cos(\pi - B)\cos(\pi - C) + \sin(\pi - B)\sin(\pi - C)\cos(\pi - a);$$

or, simplifying

$$\cos A = -\cos B\cos C + \sin B\sin C\cos a.$$

And so $$\cos B = -\cos A\cos C + \sin A\sin C\cos b,$$

$$\cos C = -\cos A\cos B + \sin A\sin B\cos c.$$

From these formulæ we have at once, cos a, cos b, cos c expressed in terms of the angles A, B, C; thus

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

19. *To express the sine of an angle in terms of the functions of the sides.*

We have $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$.

$$\therefore \sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c)}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Hence, taking the square root of each side, and taking the positive sign of the radical, as sin A is always positive, we have

$$\sin A = \frac{1}{\sin b \sin c} \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}.$$

Cor. This result, by taking the expression in (1) and breaking into factors, may be easily expressed in the following form:

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)},$$

where $2s = a + b + c$.

And so

$$\sin B = \frac{2}{\sin a \sin c}\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s -}$$

$$\sin C = \frac{2}{\sin a \sin b}\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s -}$$

20. *To express the sines, cosines, and tangents of* $\frac{A}{2}, \frac{B}{2},$ *in terms of the functions of the sides.*

We have, Art. 17,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence

$$2 \sin^2 \frac{A}{2} = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

From which

$$\sin \frac{A}{2} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

And so

$$\cos \frac{A}{2} = \sqrt{\frac{\cos s \sin (s - a)}{\sin b \sin c}}.$$

Hence also, since

$$\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}, \text{ we have}$$

$$\tan \frac{A}{2} = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

We have taken the positive sign of the radicals becaus $\sin \frac{A}{2}, \cos \frac{A}{2}, \tan \frac{A}{2}$ are always positive.

From the form of these results we at once obtain similar formulæ for $\frac{B}{2}, \frac{C}{2}$.

The student cannot fail to see the similarity between the expressions obtained in the last two articles, and the expressions for sin A, sin $\frac{A}{2}$, &c., in Plane Trigonometry.

21. *To express the functions of the sides and half sides in terms of the functions of the angles.*

We have (Art. 18),

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

And by proceeding as in Arts. 19, 20, we can obtain sin a, sin $\frac{a}{2}$, cos $\frac{a}{2}$, &c., as wethe n obtained sin A, sin $\frac{A}{2}$, cos $\frac{A}{2}$, &c.

We may get these results however by means of the properties of the supplemental triangle (Art. 12).

The student may easily see that we then get

$$\sin a = \frac{2}{\sin B \sin C} \sqrt{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)} \quad (1),$$

$$\sin \frac{a}{2} = \sqrt{-\frac{\cos S \cos (S-A)}{\sin B \sin C}} \quad \ldots\ldots\ldots\ldots (2),$$

$$\cos \frac{a}{2} = \sqrt{\frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}} \quad \ldots\ldots\ldots (3),$$

$$\tan \frac{a}{2} = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}} \quad \ldots\ldots (4),$$

where $2S = A + B + C$.

We have taken here again the positive signs of the radicals, because a is less than two right angles, and therefore $\sin a$, $\sin \frac{a}{2}$, &c., are all positive.

We can show that the expression in (1), (2), (4) are all real, thus :

By Art. 13, $2\,S > \pi$, and $< 3\,\pi$.

$$\therefore S > \frac{\pi}{2}, \text{ and } < \frac{3\,\pi}{2}.$$

Hence cos S is negative.

Again (Art. 9), $a < b + c$.

$\therefore$ (Art. 12), $\pi - A < \pi - B + \pi - C$, or $B + C - A < \pi$.

$$\therefore S - A < \frac{\pi}{2}.$$

Again B + C − A is always greater than $-\pi$, for B and C must have some positive value, and A cannot be greater than π.

Therefore S − A is always algebraically $> -\frac{\pi}{2}$.

Hence, cos (S − A) is always positive, and similarly cos (S − B), cos (S − C) are positive.

Hence the expressions for $\sin a$, $\sin \frac{a}{2}$, &c., are real quantities.

22. *To show that*

$$\cot a \sin b = \cot A \sin C + \cos b \cos C.$$

We have $\cos a = \cos b \cos c + \sin b \sin c \cos A$;

also $\cos c = \cos a \cos b + \sin a \sin b \cos C$,

and $\sin c = \sin a \cdot \frac{\sin C}{\sin A}$.

Substituting for these values of sin c and cos c, then

$$\cos a = \cos b (\cos a \cos b + \sin a \sin b \cos C) + \sin b \,.\, \sin a \frac{\sin C}{\sin A} \,.\, \cos A ;$$

or, $$\cos a (1 - \cos^2 b) = \sin a \sin b \cot A \sin C + \sin a \sin b \cos b \cos C ;$$

or, putting $\sin^2 b$ for $1 - \cos^2 b$, and dividing each side by $\sin a \sin b$, then

$$\cot a \sin b = \cot A \sin C + \cos b \cos C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From the form of this result, we also have

$$\cot b \sin a = \cot B \sin C + \cos a \cos C \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\cot b \sin c = \cot B \sin A + \cos c \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$\cot c \sin b = \cot C \sin A + \cos b \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\cot c \sin a = \cot C \sin B + \cos a \cos B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

$$\cot a \sin c = \cot A \sin B + \cos c \cos B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

In these results we may of course replace the angles by the supplements of the sides, if we also replace sides by the supplements of the angles. But it will be found that by so doing we obtain no new formulæ.

We thus obtain (6) from (1), (3) from (2), and (5) from (4).

23. *Napier's Analogies—To show that*

$$\text{(i.)} \quad \tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2},$$

$$\text{(ii.)}\ \tan\tfrac{1}{2}(A - B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2},$$

$$\text{(iii.)}\ \tan\tfrac{1}{2}(a + b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2},$$

$$\text{(iv.)}\ \tan\tfrac{1}{2}(a - b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

We have $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b}.$$

Hence it is easily shown (see Vol. I., page 215), that each of the quantities

$$\frac{\sin A}{\sin a} \text{ or } \frac{\sin B}{\sin b} = \frac{\sin A \pm \sin B}{\sin a \pm \sin b} \quad \ldots\ldots\ldots\

Hence, simplifying, we have

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

And so, by taking the lower sign in (1), we have

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

Now writing $\pi - A$ for a, $\pi - B$ for b, &c., we have, on simplifying,

$$\tan \tfrac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

and

$$\tan \tfrac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{c}{2}.$$

The last two may be of course obtained directly by commencing with the identity

$$\cos a - \cos b \cos c = \sin b \sin c \cos A,$$

instead of as in (2).

Cor. Hence $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ *are of the same affection, that is, are either both less or both greater than a right angle.*

For we learn from (i.) that $\tan \frac{1}{2}(A + B)$ and $\cos \frac{1}{2}(a + b)$ *are of the same sign,* since $\cos \frac{1}{2}(a - b)$ and $\cot \frac{C}{2}$ are necessarily positive, $\frac{1}{2}(a - b)$ and $\frac{C}{2}$ being each less than $\frac{\pi}{2}$.

Hence $\frac{1}{2}(A + B)$ and $\frac{1}{2}(a + b)$ are both less or both greater than a right angle together.

24. *Formulæ of Gauss.* To show that:

$$\text{(i.)}\quad \cos\tfrac{1}{2}c\cos\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \sin\tfrac{1}{2}\mathrm{C}\cos\tfrac{1}{2}(a+b),$$

$$\text{(ii.)}\quad \sin\tfrac{1}{2}c\cos\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \sin\tfrac{1}{2}\mathrm{C}\sin\tfrac{1}{2}(a+b),$$

$$\text{(iii.)}\quad \cos\tfrac{1}{2}c\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = \cos\tfrac{1}{2}\mathrm{C}\cos\tfrac{1}{2}(a-b),$$

$$\text{(iv.)}\quad \sin\tfrac{1}{2}c\sin\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) = \cos\tfrac{1}{2}\mathrm{C}\sin\tfrac{1}{2}(a-b).$$

Now $\cos c = \cos a\cos b + \sin a\sin b\cos \mathrm{C}$.

Hence $1+\cos c = 1+\cos a\cos b+\sin a\sin b\cos \mathrm{C}$; or

$$2\cos^2\tfrac{1}{2}c = (1+\cos a\cos b)\left(\cos^2\frac{\mathrm{C}}{2}+\sin^2\frac{\mathrm{C}}{2}\right)+\sin a\sin b\left(\cos^2\frac{\mathrm{C}}{2}-\sin^2\frac{\mathrm{C}}{2}\right)$$

$$= \left\{1+\cos(a-b)\right\}\cos^2\frac{\mathrm{C}}{2}+\left\{1+\cos(a+b)\right\}\sin^2\frac{\mathrm{C}}{2}$$

$$= 2\cos^2\tfrac{1}{2}(a-b)\cos^2\frac{\mathrm{C}}{2}+2\cos^2\tfrac{1}{2}(a+b)\sin^2\frac{\mathrm{C}}{2}.$$

$$\therefore \cos^2\tfrac{1}{2}c = \cos^2\tfrac{1}{2}(a-b)\cos^2\frac{\mathrm{C}}{2}+\cos^2\tfrac{1}{2}(a+b)\sin^2\frac{\mathrm{C}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}.$$

And so

$$\sin^2\tfrac{1}{2}c = \sin^2\tfrac{1}{2}(a-b)\cos^2\tfrac{1}{2}\mathrm{C}+\sin^2\tfrac{1}{2}(a+b)\sin^2\frac{\mathrm{C}}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

But (Art. 23), by Napier's Analogies,

$$\tan^2 \tfrac{1}{2}(A + B) = \frac{\cos^2 \tfrac{1}{2}(a - b) \cos^2 \frac{C}{2}}{\cos^2 \tfrac{1}{2}(a + b) \sin^2 \frac{C}{2}}$$; adding unity to each side, then

$$\sec^2 \tfrac{1}{2}(A + B) = \frac{\cos^2 \tfrac{1}{2}(a - b) \cos^2 \frac{C}{2} + \cos^2 \tfrac{1}{2}(a + b) \sin^2 \frac{C}{2}}{\cos^2 \tfrac{1}{2}(a + b) \sin^2 \frac{C}{2}}$$; or by (1),

$$\frac{1}{\cos^2 \tfrac{1}{2}(A + B)} = \frac{\cos^2 \tfrac{1}{2} c}{\cos^2 \tfrac{1}{2}(a + b) \sin^2 \frac{C}{2}}$$; or

$$\cos^2 \tfrac{1}{2} c \cos^2 \tfrac{1}{2}(A + B) = \sin^2 \tfrac{1}{2} C \cos^2 \tfrac{1}{2}(a + b).$$

Taking the square root of each side, and remembering (Art. 23) that $\frac{1}{2}(A + B)$, $\frac{1}{2}(a + b)$ are of the same affection, and that $\frac{1}{2} c$, $\frac{1}{2} C$ are each less than $\frac{\pi}{2}$, we have

$$\cos \tfrac{1}{2} c \cos \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2} C \cos \tfrac{1}{2}(a + b) \quad \ldots\ldots(3).$$

And similarly

$$\sin \tfrac{1}{2} c \,.\, \cos \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2} C \,.\, \sin \tfrac{1}{2}(a + b) \quad \ldots\ldots(4).$$

But (Art. 23),

(i.) $$\tan \tfrac{1}{2}(A + B) = \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cot \frac{C}{2} \quad \ldots\ldots(5),$$

and

(ii.) $$\tan \tfrac{1}{2}(A - B) = \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)} \cot \frac{C}{2} \quad \ldots\ldots(6).$$

(3) × (5), then

$$\cos \tfrac{1}{2} c \,.\, \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = \cos \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a - b).$$

And (4) × (6), then

$$\sin \tfrac{1}{2} c \,.\, \sin \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = \cos \tfrac{1}{2} \mathrm{C} \,.\, \sin \tfrac{1}{2} (a - b).$$

Ex. I.

In any right-angled triangle ABC, right angled at C, show that

1. $2 \cos \dfrac{c}{2} = \sqrt{2 + 2 \cos a \cos b}.$

2. $\tan^2 \dfrac{a}{2} = \tan \dfrac{c + b}{2} \tan \dfrac{c - b}{2}.$

3. $\sin^2 \dfrac{a}{2} + \sin^2 \dfrac{b}{2} - \sin^2 \dfrac{c}{2} = 2 \sin^2 \dfrac{a}{2} \sin^2 \dfrac{b}{2}.$

4. $\sin (c - b) = \tan^2 \dfrac{\mathrm{A}}{2} \sin (c + b).$

5. $\sin (a - b) \cot \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = \sin (a + b) \tan \tfrac{1}{2} (\mathrm{A} + \mathrm{B}).$

6. $\tan^2 \left(\dfrac{\pi}{4} + \dfrac{c}{2}\right) = \tan \tfrac{1}{2} (\mathrm{A} + a) \cot \tfrac{1}{2} (\mathrm{A} - a).$

$$\tan^2 \left(\frac{\pi}{4} + \frac{\mathrm{A}}{2}\right) = \cot \tfrac{1}{2} (\mathrm{B} - b) \cot \tfrac{1}{2} (\mathrm{B} + b).$$

7. In any right-angled triangle, if α, β be the arcs respectively drawn perpendicular to the hypothenuse and bisecting it, then

$$\cot \alpha = \sqrt{\cot^2 a + \cot^2 b}, \cot \beta = \frac{\cos a + \cos b}{\sqrt{\sin^2 a + \sin^2 b}}.$$

8. The perpendicular drawn from the pole of any circle upon the great circle chord, bisects the chord.

9. If perpendiculars be drawn from the angles of a triangle ABC, meeting the opposite sides in D, E, F, show that

$$\frac{\tan \mathrm{BD}}{\tan \mathrm{CD}} \cdot \frac{\tan \mathrm{EC}}{\tan \mathrm{EA}} \cdot \frac{\tan \mathrm{FA}}{\tan \mathrm{FB}} = 1.$$

10. If the sides BC, CA, AB be cut by the arc of a great circle respectively in a, b, c, show that

$$\sin Ac \sin Ba \sin Cb = \sin cB \sin aC \sin bA.$$

11. Let P be any point on the sphere, and draw through P arcs of great circles to the angular points, meeting the sides of a triangle in D, E, F respectively, then

$$\sin AF \sin BD \sin CE = \sin CD \sin BF \sin AE.$$

12. If great circles be drawn from any point on a sphere to the angles of a polygon, the products of the sines of the alternate angles are equal.

13. In any equilateral triangle whose side is A,

$$\cot^2 \frac{A}{2} = 2 \cos a + 1,$$

$$\text{and } \tan^2 \frac{a}{2} = 1 - 2 \cos A.$$

14. In any equilateral triangle whose sides are quadrants, if α, β, γ be the lengths of arcs of great circles from a point in the triangle to the angular points, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

15. If δ, z be the declination and zenith distance of a heavenly body, and l the latitude, show that the cosine of the azimuth $= \dfrac{\sin \delta - \sin l \cos z}{\cos l \sin z}$.

16. If l be the latitude of a place on the earth's surface, δ the sun's declination; then if $2t$ be the length of a day in solar hours, show that

$$\cos 15^\circ . t = \tan l \tan \delta.$$

17. If l_1, l_2 be the latitudes of two stars, and λ their difference in longitude, show that their distance apart can be found from the formula

$$\cos \theta = \frac{\sin l_1 \cos (l_2 - \omega)}{\sin \omega},$$

where θ is the distance, and $\cot \omega = \cot l_1 \cos \lambda$.

18. If a, l, δ be respectfully the sun's right ascension,

longitude, and declination; and ω be the obliquity of the ecliptic, prove the following relations:

$$\tan a = \cos \omega \tan l,$$
$$\tan \delta = \sin a \tan \omega,$$
$$\sin \delta = \sin \omega \sin l,$$
$$\cos l = \cos a \cos \delta.$$

CHAPTER IV.

SOLUTION OF SPHERICAL TRIANGLES.

25. In general when any three elements of a spherical triangle are given, the other three can be determined. For the angles and sides are connected together by six equations of the form

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ or } \cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

When the trangle is *right-angled*, then the solution is generally possible if we know two other elements; and so also, when the triangle is *quadrantal*, or has one of its sides a quadrant, two other elements are generally sufficient for a complete solution.

I.—Right-Angled Triangles.

26. We have:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \ldots\ldots\ldots (1).$$
$$\cos c \sin A \sin B = \cos C + \cos A \cos B \ldots\ldots (2).$$
$$\cos A + \cos B \cos C = \cos a \sin B \,.\, \sin C \ldots\ldots (3).$$
$$\sin a = \frac{\sin c \sin A}{\sin C} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$
$$\cot a \sin b = \cot A \sin C + \cos b \cos C \ldots\ldots\ldots (5).$$
$$\cot C \sin B + \cos a \cos B = \cot c \sin a \ldots\ldots\ldots (6).$$

Let C be the right angle of the spherical triangle ABC.

Then $C = 90°$, $\sin C = 1$, $\cos C = 0$, $\cot C = 0$.

Hence from (1),

$$\cos c = \cos a \cos b;$$

from (2), $\cos c = \cot A \cot B;$

from (3), $\cos A = \cos a \sin B;$

and so, $\cos B = \cos b \sin A.$

Also from (4),

$$\sin a = \sin c \sin A;$$

and so, $\sin b = \sin c \sin B.$

And from (5),

$$\sin b = \tan a \cot A$$

and so, $\sin a = \tan b \cot B.$

And again, from (6),

$$\cos B = \cot c \tan a;$$

and so, $\cos A = \cot c \tan b.$

We may write these results as follows:

(i.) $\sin\left(\frac{\pi}{2} - c\right) = \cos a \cos b = \tan\left(\frac{\pi}{2} - A\right) \tan\left(\frac{\pi}{2} - B\right).$

(ii.) $\sin\left(\frac{\pi}{2} - A\right) = \cos a \cos\left(\frac{\pi}{2} - B\right) = \tan\left(\frac{\pi}{2} - c\right) \tan b.$

(iii.) $\sin\left(\frac{\pi}{2} - B\right) = \cos b \cos\left(\frac{\pi}{2} - A\right) = \tan\left(\frac{\pi}{2} - c\right) \tan a.$

(iv.) $\sin a = \cos\left(\frac{\pi}{2} - c\right) \cos\left(\frac{\pi}{2} - A\right)$

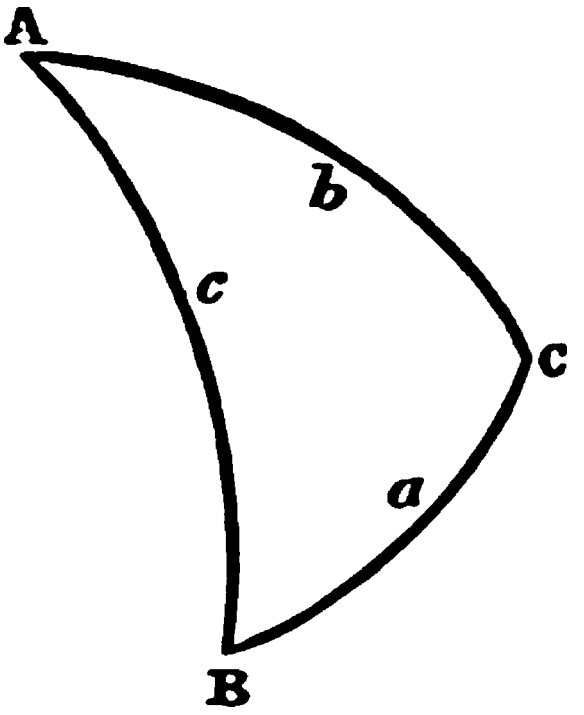
$= \tan b \tan\left(\frac{\pi}{2} - B\right).$

(v.) $\sin b = \cos\left(\frac{\pi}{2} - c\right) \cos\left(\frac{\pi}{2} - B\right)$

$= \tan a \tan\left(\frac{\pi}{2} - A\right).$

We can now explain *Napier's Rules.*

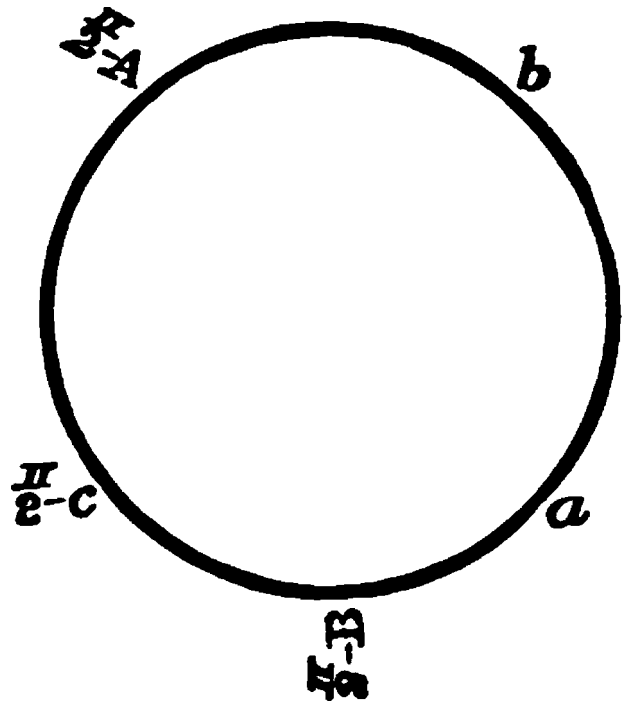
Leaving the right angle C out of consideration, we have *five* remaining elements.



Again, if we read the remaining elements in order *round the triangle* commencing with any one of them, there will always be two adjacent parts, and two opposite parts.

Instead however of reading A, c, B, we shall read the *complements of the angles* and *the complement of the hypothenuse.*

We shall then have what Napier called *the circular parts*, viz., a, b, $\frac{\pi}{2} - A$, $\frac{\pi}{2} - c$, $\frac{\pi}{2} - B$, any one of which may be the starting point, or what he called *the middle part.*



The formulæ proved above may then be expressed by the following rules, which the student can easily verify for himself.

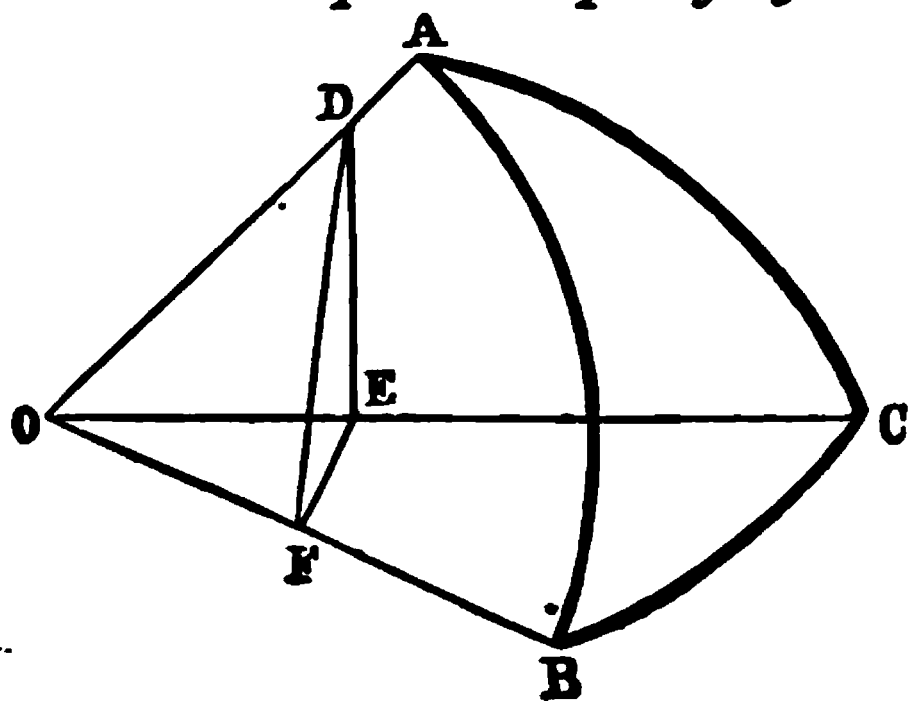
Napier's Rules:

(1.) *Sine of the middle part = product of cosines of opposite parts.*

(2.) *Sine of the middle part = product of tangent of adjacent parts.*

It will be seen that all the possible cases of these rules are exemplified in the formula just proved.

27. *Independent proof of the relations in the last Article.*



Let ABC be a spherical triangle having the right angle at C, and let O be the centre of the sphere.

In OA take any point D; draw DE perpendicular to OC, and from E draw EF perpendicular to OB.

Join DF.

Then, because C is a right angle, and DE is perpendicular to OC, DE is perpendicular to EF.

We shall now show that DFO is a right angle.

We have

$$DF^2 = DE^2 + EF^2 = (OD^2 - OE^2) + (OE^2 - OF^2)$$
$$= OD^2 - OF^2.$$

Hence (Euc. I., 48), DF is perpendicular to OF.

Hence also,

$\angle$ DFE is the inclination of the planes AOB and BOC.

$$\therefore \angle DFE = B.$$

Now $\frac{OF}{OD} = \frac{OF}{OE} \cdot \frac{OE}{OD}$; $\therefore \cos c = \cos a \cos b$..... (1).

$$\left.\begin{aligned} \frac{DE}{OD} = \frac{DF}{OD} \cdot \frac{DE}{DF}; \therefore \sin b = \sin c \sin B; \\ \text{and so,} \quad \sin a = \sin c \sin A \end{aligned}\right\} \text{..... (2).}$$

$$\left.\begin{aligned} \frac{EF}{OE} = \frac{DE}{OE} \cdot \frac{EF}{DE}; \therefore \sin a = \tan b . \cot B; \\ \text{and so,} \quad \sin b = \tan a \cot A \end{aligned}\right\} \text{..... (3).}$$

$$\left.\begin{aligned} \frac{EF}{DF} = \frac{EF}{OF} \cdot \frac{OF}{DF}; \therefore \cos B = \tan a . \cot c; \\ \text{and so,} \quad \cos A = \tan b \cot c \end{aligned}\right\} \text{..... (4).}$$

From (3), $\sin a \sin b = \tan a \tan b \,.\, \cot A \cot B$; or

$$1 = \frac{1}{\cos a \cos b} \cot A \cot B; \text{ or, by (1),}$$

$$= \frac{1}{\cos c} \cot A \cot B.$$

$$\therefore \cos c = \cot A \cot B \text{ ..... (5).}$$

Again (2) $\times$ (4), we have

$\sin a \cos B = \sin c \sin A \,.\, \tan a \cot c = \sin A \tan a \cos c$; or,

$$\cos B = \frac{\sin A \tan a \cos c}{\sin a} = \frac{\cos c}{\cos a} \sin A; \text{ or by (1),}$$

$$\left.\begin{aligned} \cos B = \cos b \sin A; \\ \text{and so,} \\ \cos A = \cos a \sin B \end{aligned}\right\} \text{..... (6).}$$

Cor. (i.) From (1) we learn that cos c has the same sign as the product of the cosines of a and b. Hence either all the cosines of the sides are positive, or only one is positive.

Hence, in any right-angled spherical triangle either *all the sides are quadrants*, or *all the sides are less than quadrants, or else one side is less than a quadrant, and the other two greater than quadrants.*

(ii.) *Either angle and the opposite side are of the same affection;* that is, A and a are both less, equal to, or greater than $\frac{\pi}{2}$, and so are B and b.

For we have from (3), $\sin b = \tan a \cot A = \frac{\tan a}{\tan A}$.

Now sin b must be positive, and hence tan a and tan A *must have the same sign.*

(iii.) *From a point of a sphere the shortest great circle that can be drawn to a given great circle is the perpendicular.*

For we have $\sin a = \sin c \sin A$.

Hence sin a is less than sin c, unless $\sin A = 1$, or $A = 90°$.

Hence sin c is greater than sin a, or c greater than a, unless $A = 90°$, when $c = a$. In this case both a and c are perpendiculars.

(iv.) *The hypothenuse is acute if the sides,* a, b *are of the same affection, or the angles* A, B *opposite to them are of the same affection, but otherwise obtuse.*

For we have $\cos c = \cos a \cos b$.

Hence if a is $< \frac{\pi}{2}$ and $b > \frac{\pi}{2}$, cos c is negative and $c > \frac{\pi}{2}$.

Also if a is $> \frac{\pi}{2}$ and $b > \frac{\pi}{2}$, or $a < \frac{\pi}{2}$, and $b < \frac{\pi}{2}$, cos c is *positive,* and $c < \frac{\pi}{2}$.

And hence also, by (ii.), c is $< \frac{\pi}{2}$ when A, B are of the same affection, but otherwise $< \frac{\pi}{2}$.

28. *When in a right-angled triangle we have given an angle and the side opposite to it, the solution is generally ambiguous.*

Let A, a be given.

We have

$$\sin a = \sin c \sin A, \text{ or } \sin c = \frac{\sin a}{\sin A} \ldots\ldots\ldots\ldots (1).$$

$$\sin b = \tan a \cot A \ldots\ldots\ldots\ldots (2).$$

$$\cot A = \cos a \sin B, \text{ or } \sin B = \frac{\cos A}{\cos a} \ldots\ldots\ldots\ldots (3).$$

Hence c, b, B are determined from their sines, and generally therefore we may expect two values of each.

(i.) If sin a *be greater than* sin A, then C is impossible; and hence, since (Art. 27) a and A are of the same affection, we must have $a <$ A *when both are acute*, and $a >$ A *when both are obtuse.*

Otherwise $\sin a <$ sin A, and the solution impossible.

(ii.) *If* sin a *be less than* sin A, then c has two values.

Each of these values of c however gives *only one* value for b, and *only one* for B, since

$$\cos c = \cos a \cos b, \text{ or } \cos b = \frac{\cos c}{\cos a};$$

and $$\cos c = \cot A \cot B, \text{ or } \cot B = \frac{\cos c}{\cos A}.$$

Hence there are in this case *two triangles* which answer the conditions, but only two.

(iii.) If $\sin a = \sin A$, and therefore $a = A$ (since a and A are of the same affection), we have

$$\sin c = 1 = \sin \frac{\pi}{2}, \therefore c = \frac{\pi}{2}.$$

Hence also, from (2) and (3),

$$b = \frac{\pi}{2}, \text{ and } B = \frac{\pi}{2}.$$

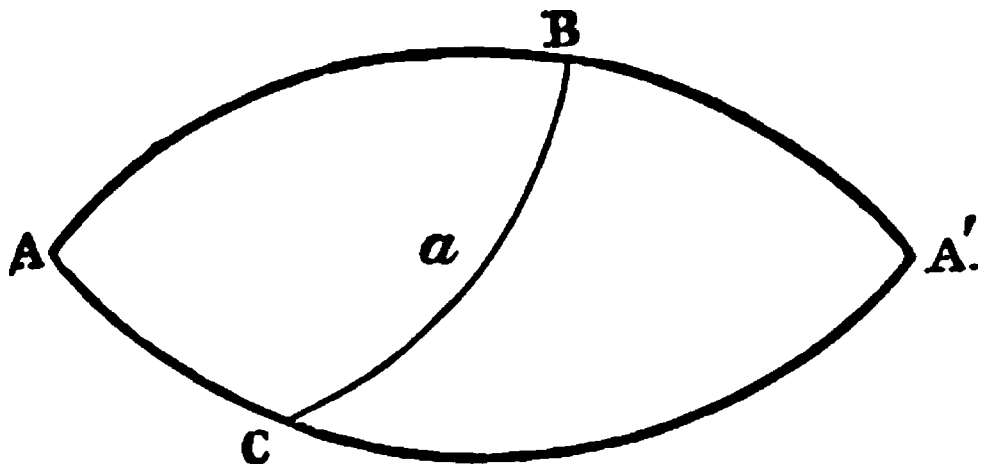
Therefore c, b, B are each right angles. And hence, since C is a right angle, A is the pole of a, and the two triangles ABC, A′BC (see below) will be *symmetrically* equal.

In the case where $a = A = \frac{\pi}{2}$, we have from (2) $\sin b = \frac{0}{0}$, and from (3) $\sin B = \frac{0}{0}$.

Hence b, B are indeterminate.

29. We may illustrate the above remarks geometrically thus :

We have given A, a; and the triangle ABC is right angled at C.



Produce AB, AC to meet at A′.

Then evidently A′ = A.

Hence we have two triangles ABC, A′BC answering the given conditions, for they each have an angle equal to A, and they have a common side a, and a right angle at C.

If A and a are both right angles, then evidently B is the pole of AC.

Hence (Art. 5), B = b, but the values are indeterminate.

There is no other ambiguous case in the solution of right-angled triangles, as we now proceed to show.

30. *Given a side* a *and an adjacent angle* B.

We have, by Napier's Rules,

$$\cos B = \tan a \cot c, \therefore \tan c = \frac{\tan a}{\cos B}.$$

Also, $$\sin a = \tan b \cot B.$$

$$\therefore \tan b = \sin a \tan B.$$

And $$\cos A = \cos a \sin B.$$

Hence b, c, A are determined without any ambiguity, and *there is no* impossible case.

31. *Given the two sides* a, b.

By Napier's Rules, we have

$$\cos c = \cos a \cos b,$$

$$\sin a = \tan b \cot \mathrm{B}, \therefore \cot \mathrm{B} = \frac{\sin a}{\tan b}.$$

and $$\sin b = \tan a \cot \mathrm{A}, \therefore \cot \mathrm{A} = \frac{\sin b}{\tan a}.$$

Hence there is no ambiguity in determining the values of *c*, A, B, and the triangle is always possible.

32. *Given the hypothenuse* c *and an angle* A.

By Napier's Rules,

$$\cos \mathrm{A} = \tan b \cot c, \therefore \tan b = \tan c \cos \mathrm{A} \ldots\ldots\ldots(1).$$

$$\text{Also } \cos c = \cot \mathrm{A} \cot \mathrm{B}, \therefore \cot \mathrm{B} = \cos c \tan \mathrm{A} \ldots\ldots(2).$$

$$\text{And } \sin a = \sin c \sin \mathrm{A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1) and (2) we obtain *b*, B without any ambiguity, and their values are always possible.

In (3) there is an *apparent ambiguity* in the determination of *a*. We must remember however that *a* and A *are of the same affection*. Hence, if A be given greater than a right angle, we must take the value of *a* which is such; and so when A is less than a right angle, the value of *a* must be taken less than a right angle.

Hence there is no ambiguity, and the triangle is always possible.

From (1) and (2), when A, *c* are both right angles, we find B, *b* are indeterminate, for we have tan *b* and cot B each taking the form $\infty \times 0$.

33. *Given the hypothenuse* c *and a side* a.

By Napier's Rules, we have

$$\sin a = \sin c \sin \mathrm{A}, \text{ or } \sin \mathrm{A} = \frac{\sin a}{\sin c} \ldots\ldots\ldots\ldots(1).$$

$$\text{Also, } \cos \mathrm{B} = \tan a \cot c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$\text{And } \cos c = \cos a \cos b, \text{ or } \cos b = \frac{\cos c}{\cos a} \ldots\ldots\ldots\ldots(3).$$

From (2) and (3) we obtain B, b without ambiguity ; and from (1) we obtain A without ambiguity, since A and a are of the same affection.

Again, we see that the solution is impossible if we obtain from (1), (2), (3) values of sin A, cos B, or cos b greater than unity.

Hence, for a possible solution, we cannot have

$$\sin c < \sin a, \quad \cos c > \cos a, \quad \tan c < \tan a.$$

And hence for a possible solution we must have c between the limits a and $\pi - a$.

34. *Given the two angles* A, B.

By Napier's Rules, we have

$$\cos \mathrm{A} = \cos a \sin \mathrm{B}, \quad \therefore \cos a = \frac{\cos \mathrm{A}}{\sin \mathrm{B}} \ldots\ldots\ldots(1),$$

$$\cos \mathrm{B} = \cos b \sin \mathrm{A}, \quad \therefore \cos b = \frac{\cos \mathrm{B}}{\sin \mathrm{A}} \ldots\ldots\ldots(2),$$

$$\cos c = \cot \mathrm{A} \cot \mathrm{B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence the values of a, b, c can be determined without ambiguity.

We learn however from (1), (2), and (3) that the solution is not always possible.

For we cannot have cos B > sin A where sin A may be positive or negative, and the inequality is with regard to magnitude only.

(i.) *Let* A *be less than* $\frac{\pi}{2}$.

Then since cos B *cannot be* > $\cos\left(\frac{\pi}{2} - \mathrm{A}\right)$,

and cannot be > $\cos\left(\frac{\pi}{2} + \mathrm{A}\right)$ in magnitude,

B must lie between $\frac{\pi}{2} - \mathrm{A}$ and $\frac{\pi}{2} + \mathrm{A}$.

(ii.) Let A *be greater than* $\frac{\pi}{2}$.

Then $\pi - A$ is less than $\frac{\pi}{2}$, and we have, as in the last case,

B must lie between $\frac{\pi}{2} - (\pi - A)$ and $\frac{\pi}{2} + (\pi - A)$.

$\therefore$ B must lie between $A - \frac{\pi}{2}$ and $\frac{3\pi}{2} - A$.

When the side or angle required is small, or nearly equal to one right angle, or to two right angles, we must be careful as to what formula we use, and proceed conformably to the principles laid down in Plane Trigonometry, Art. 70. If a convenient formula cannot be obtained by Napier's Rules, we may transform the formula.

Thus, knowing that $\cos c = \cot A \cot B$, it follows easily that

$$\sin \frac{c}{2} = \sqrt{\frac{-\cos(A + B)}{2 \sin A \sin B}},$$

and

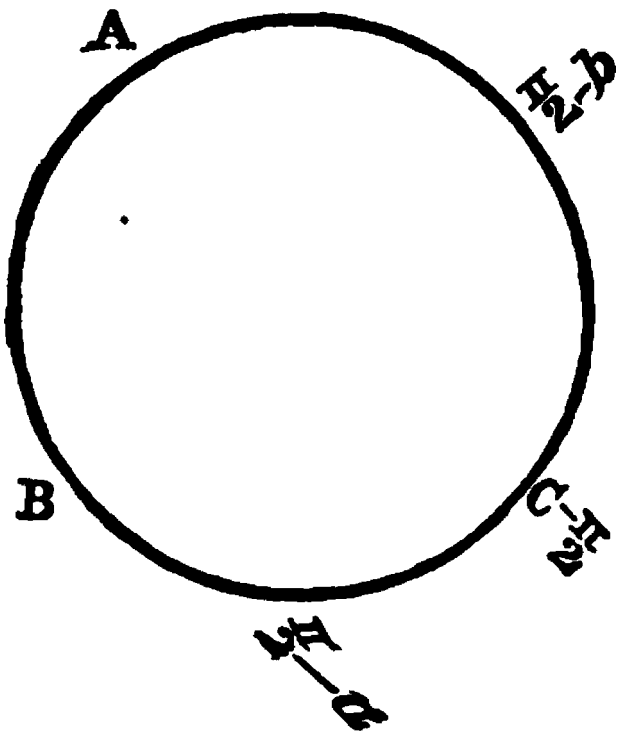
$$\cos \frac{c}{2} = \sqrt{\frac{\cos(A - B)}{2 \sin A \sin B}}.$$

And so on.

II.—Quadrantal Triangles.

35. *A quadrantal triangle* ABC, *of which* c *is the quadrantal side, may be solved by Napier's Rules, if we take as the circular parts (the quadrantal side being neglected),*

$$\frac{\pi}{2} - a, \frac{\pi}{2} - b, A, B, C - \frac{\pi}{2}.$$



Let A′B′C′ be the polar triangle; then

$$A' = \pi - a, B' = \pi - b, C' = \pi - c = \frac{\pi}{2},$$

$$a' = \pi - A, b' = \pi - B, c' = \pi - C.$$

Applying Napier's Rules to the polar right-angled triangle A′B′C′, we have

$$\cos c' = \cos a' \cos b' = \cot A' \cot B',$$

$$\sin a' = \sin c' \sin A' = \tan b' \cot B',$$

$$\sin b' = \sin c' \sin B' = \tan a' \cot A',$$

$$\cos A' = \cos a' \sin B' = \tan b' \cot c',$$

$$\cos B' = \cos b' \sin A' = \tan a' \cot c'.$$

Substituting the above values of A′, B′, &c., in these equations, and arranging, we have

$$\sin\left(C - \frac{\pi}{2}\right) = \cos A \cos B$$

$$= \tan\left(\frac{\pi}{2} - a\right) \tan\left(\frac{\pi}{2} - b\right),$$

$$\sin A = \cos\left(\frac{\pi}{2} - a\right) \cos\left(C - \frac{\pi}{2}\right)$$

$$= \tan\left(\frac{\pi}{2} - b\right) \tan B,$$

$$\sin B = \cos\left(\frac{\pi}{2} - b\right) \cos\left(C - \frac{\pi}{2}\right)$$

$$= \tan\left(\frac{\pi}{2} - a\right) \tan A,$$

$$\sin\left(\frac{\pi}{2} - a\right) = \cos\left(\frac{\pi}{2} - b\right) \cos A$$

$$= \tan B \,.\, \tan\left(C - \frac{\pi}{2}\right),$$

$$\sin\left(\frac{\pi}{2} - b\right) = \cos\left(\frac{\pi}{2} - a\right) \cos B$$

$$= \tan A \,.\, \tan\left(C - \frac{\pi}{2}\right).$$

These results prove the proposition.

III.—Oblique-Angled Triangles.

36. CASE I. *Given the three sides* (a, b, c).

We may determine the angles from any of the formulæ in Art. 17. In certain cases the particular one chosen must depend upon the principles laid down in Plane Trigonometry, Art. 70.

37. CASE II. *Given the three angles* (A, B, C).

We may determine the sides from the formulæ in Art. 18, with the precautions mentioned in last Art.

38. CASE III. *Given two sides and the included angle* (a, C, b).

By Napier's Analogies

$$\tan \tfrac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{C}{2},$$

$$\tan \tfrac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{C}{2}.$$

Hence A, B are determined.

Then, since $\sin c = \sin a \cdot \dfrac{\sin C}{\sin A}$, c is also determined.

As c is determined from its *sine*, there may be some uncertainty as to which of the two values obtainable we are to take. We may generally determine this point by remembering that the greater side of a spherical triangle is opposite to the greater angle.

We may, however, determine c from formulæ which give no ambiguous values. We do this in the next article.

39. *To find* c *without previously finding the values of* A *and* B.

We have

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$
$$= \cos b (\cos a + \sin a \tan b \cos C) \quad \ldots\ldots (1).$$

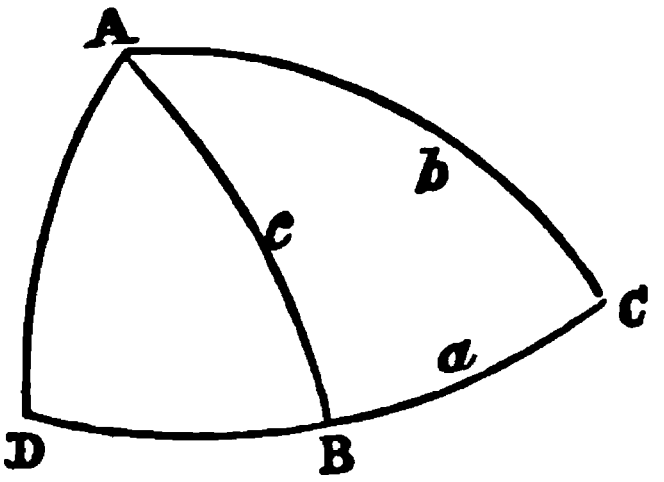
Let $\tan \theta = \tan b \cos C$ $\ldots\ldots (2).$

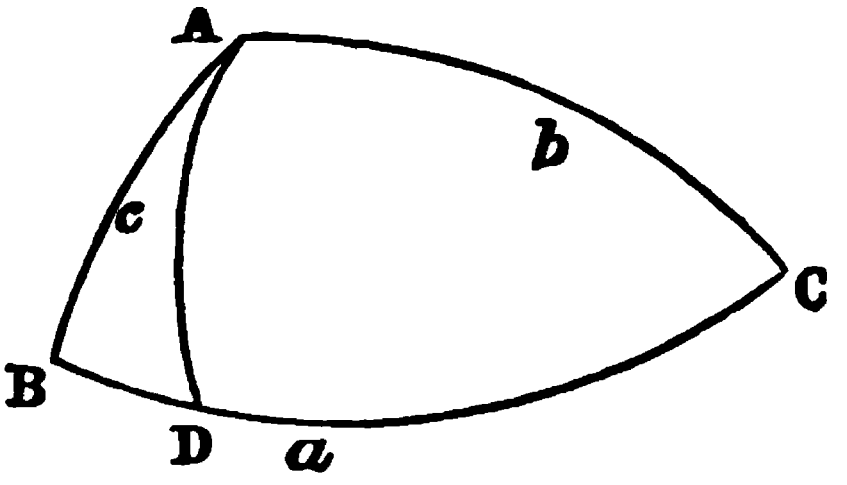
Then, from (1),

$$\cos c = \cos b (\cos a + \sin a \tan \theta) = \frac{\cos b}{\cos \theta} \cos (a - \theta) \ldots\ldots (3).$$

Hence c can be obtained independently by formulæ adapted to logarithmic computation, and there is no ambiguity. Moreover the triangle is always possible.

We may also solve the triangle by resolving it into two right-angled triangles by drawing from A an arc perpendicular to BC, or BC produced.





We have, by Napier's Rules, from the right-angled triangle ACD,

$$\cos C = \cot b \tan CD ; \quad \therefore \tan CD = \tan b \cos C \ldots\ldots\ldots (4).$$

Hence CD is known ; and then, since $BD = BC \sim CD \ldots (5)$,

BD is also determined.

Again, from the right-angled triangle ADB,

$$\cos c = \cos AD \cos DB \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Also we have

$$\cos b = \cos AD \cos CD, \text{ or } \cos AD = \frac{\cos b}{\cos CD} ;$$

$$\therefore \cos c = \frac{\cos b}{\cos CD} . \cos DB \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If we compare (4) with (2), we see that CD is nothing more than θ in the previous assumption. We then have

$$CD = \theta, \text{ and also } BD = a \sim \theta.$$

B may be also found independently of A, as follows :

We have $$\sin CD = \tan AD \cot C \ldots\ldots\ldots\ldots\ldots\ldots (8),$$

and $$\sin BD = \tan AD . \cot ABD \ldots\ldots\ldots\ldots (9).$$

(8) ÷ (9), then $$\frac{\sin CD}{\sin BD} = \frac{\cot C}{\cot ABD}.$$

$$\therefore \tan \text{ABD} = \frac{\sin \text{CD}}{\sin \text{BD}} . \tan \text{C} = \frac{\sin \theta}{\sin (a \sim \theta)} \tan \text{C}.$$

Here ABD is the angle B or its supplement, according as BD is $a - \theta$ or $\theta - a$.

40. Case IV. *Given two angles and the included side* (A, c, B).

By Napier's Analogies

$$\tan \tfrac{1}{2} (a + b) = \frac{\cos \tfrac{1}{2} (\text{A} - \text{B})}{\cos \tfrac{1}{2} (\text{A} + \text{B})} \tan \tfrac{1}{2} c,$$

$$\tan \tfrac{1}{2} (a - b) = \frac{\sin \tfrac{1}{2} (\text{A} - \text{B})}{\sin \tfrac{1}{2} (\text{A} + \text{B})} \tan \tfrac{1}{2} c.$$

Hence a, b are determined.

Then, since $\sin \text{C} = \sin \text{A} \dfrac{\sin c}{\sin a}$,

C is also determined.

As in Case III., there may be an uncertainty as to which of the two obtainable values of C is admissible. We may however generally settle the point on the principle that the greater angle of a triangle is opposite to the greater side.

We may also find C from formulæ where there is no ambiguity, as in the next article.

41. *To find* C *without previously finding the values of* a *and* b.

We have $\cos \text{C} = -\cos \text{A} \cos \text{B} + \sin \text{A} \sin \text{B} \cos c$

$= \cos \text{B} (-\cos \text{A} + \sin \text{A} \tan \text{B} \cos c)$...(1).

Let $\cot \phi = \tan \text{B} \cos c$ .....(2).

Then, from (1), we have $\cos \text{C} = \dfrac{\cos \text{B}}{\sin \phi} \sin (\text{A} - \phi)$...(3),

from which C can be determined without ambiguity.

Moreover the solution is always possible.

We may also solve the triangle, as in the last case, by drawing a perpendicular from A upon BC or BC produced.

The *process* corresponds exactly to that in Art. 39.

42. CASE V. *Given two sides and the angle opposite to one of them* (a, b, A).

We have $$\sin B = \frac{\sin b}{\sin a} \sin A \quad \ldots\ldots (1).$$

Also we have $$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B) \ldots (2).$$

$$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b) \ldots (3).$$

From (1) B is determined. (We shall reserve the discussion of the cases of ambiguity till Art. 47).

And then from (2), (3) we determine C and *c*.

43. *To determine* C *and* c *independently of* B.

We have (Art. 22),

$$\cot a \sin b = \cos b \cos C + \sin C \cot A$$
$$= \cos b \left(\cos C + \frac{\cot A}{\cos b} \sin C\right) \ldots\ldots (1.)$$

Put $$\tan \phi = \frac{\cot A}{\cos b} \quad \ldots\ldots (2).$$

Then $$\cot a \sin b = \cos b (\cos C + \tan \phi \sin C)$$
$$= \cos b \frac{\cos (C - \phi)}{\cos \phi}.$$

$$\therefore \cos (C - \phi) = \cos \phi \cot a \tan b \quad \ldots\ldots (3).$$

From (2) we obtain ϕ, and then from (3), having found $C - \phi$, we obtain C.

$$\text{Again, } \cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$= \cos b (\cos c + \sin c \tan b \cos A) \ldots\ldots (4).$$

$$\text{Put } \tan \theta = \tan b \cos A \quad \ldots\ldots (5).$$

$$\text{Then, } \cos a = \cos b (\cos c + \sin c \tan \theta)$$
$$= \cos b \cdot \frac{\cos (c - \theta)}{\cos \theta}.$$

$$\therefore \cos (c - \theta) = \frac{\cos a \cos \theta}{\cos b} \quad \ldots\ldots (6).$$

From this equation we obtain $c - \theta$, and then having obtained θ from (5), we get c.

Since from (3) the value of $C - \phi$ may be positive or negative, there may be an ambiguity as in the last Article. And the same remark applies to equation (5).

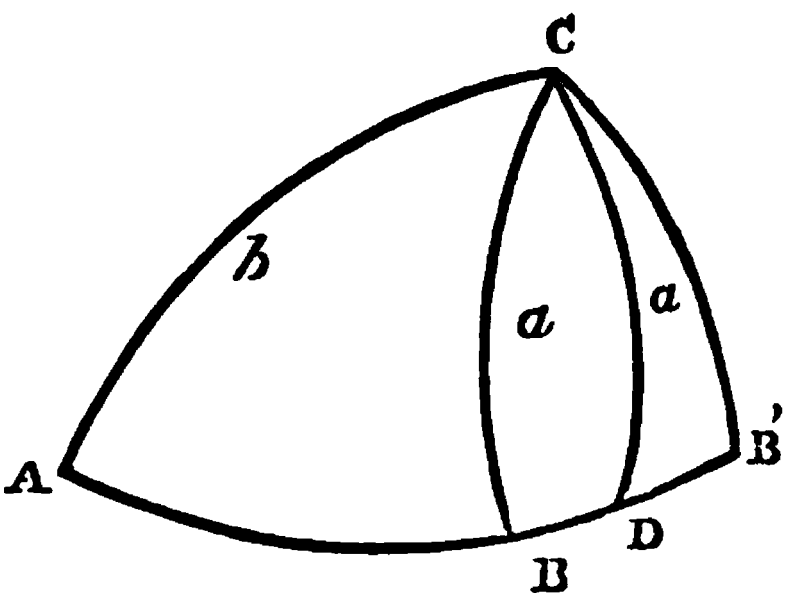
44. *Geometrical illustration of* CASE V.

Draw CA = b, and make the angle CAB = $\angle$ A.

Also draw CB and CB′ each equal to a.

Then we have the two triangles CAB, CAB′ answering the given conditions.

Draw CD perpendicular to AB′. Then evidently



$$BD = DB', \text{ and } \angle BCD = \angle B'CD.$$

From the right-angled triangle ACD, we have

$$\cos b = \cot A \cot ACD, \text{ or } \cot ACD = \frac{\cos b}{\cot A},$$

or $\tan ACD = \dfrac{\cot A}{\cos b}$, which determines $\angle$ ACD..... (1).

Comparing this equation with (2) of the last Art., we find $\angle$ ACD = ϕ.

Again we have $\cos ACD = \tan CD \cot b$,

and $\cos BCD = \tan CD \cot a$.

Hence $$\frac{\cos BCD}{\cos ACD} = \frac{\cot a}{\cot b} = \frac{\tan b}{\tan a};$$

$$\therefore \cos BCD = \cos ACD \cdot \frac{\tan b}{\tan a},$$

which determines $\angle$ BCD or BCD′.

Hence the two values of C are determined.

Again, we have

$$\cos A = \tan AD \cot b, \text{ or}$$

$$\tan AD = \tan b \cos A, \text{ which determines AD ..... (2).}$$

Comparing this equation with (5) of last Art. we find $AD = \theta$.

Also we have $\left.\begin{aligned}\cos b &= \cos CD \cos AD,\\ \cos a &= \cos CD \cos BD;\end{aligned}\right\}$ from which and

$$\cos BD = \frac{\cos a}{\cos b} \cos AD,$$ which determines BD or B′D.

Hence the two values of c are determined.

45. CASE VI. *Given two angles and the side opposite to one of them* (A, B, a).

We have $$\sin b = \frac{\sin B}{\sin A} \sin a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also we have

$$\tan \tfrac{1}{2} C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B) \ldots\ldots(2).$$

and $$\tan \tfrac{1}{2} c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b) \ldots\ldots(3).$$

From (1) we obtain b, and then C and c from (2) and (3). The ambiguities are analogous to those of Case V.

46. *To determine* C *and* c *independently of* b.

We have (Art. 22),

$$\cot A \sin B = \cot a \sin c - \cos c \cos B$$

$$= \cos B \left(\frac{\cot a \sin c}{\cos B} - \cos c\right) \ldots\ldots\ldots\ldots(1).$$

Put $$\cot \theta = \frac{\cot a}{\cos B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Then, from (1),

$$\cot A \sin B = \cos B (\cot \theta \sin c - \cos c)$$

$$= \frac{\cos B}{\sin \theta} \sin (c - \theta);$$

$$\therefore \sin (c - \theta) = \sin \theta \cot A \tan B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence we obtain $c - \theta$, and therefore knowing θ from (2), we get c.

Again $\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$= \cos B (-\cos C + \tan B \cos a \,.\, \sin C)$...(4).

Put $\cot \phi = \tan B \cos a$.....(5).

Then

$$\cos A = \cos B (-\cos C + \cot \phi \sin C) = \frac{\cos B}{\sin \phi} \sin (C - \phi).$$

$$\therefore \sin (C - \phi) = \frac{\sin \phi \cos A}{\cos B} \text{.....(6).}$$

Hence, knowing ϕ from (5), after obtaining $C - \phi$ from (6), we get C.

Since in (3) we obtain $c - \theta$, and in (6) we get $C - \phi$ from their sines, there may be ambiguities.

It may be shown, as in Art. 44, if a perpendicular CD be drawn, that θ corresponds to B′D or BD, and ϕ to $\angle$ BCD or B′CD.

47. *Ambiguities of* CASE V.

We will first take *the particular case* when $a = b$.

Then A = B, and there is no ambiguity as regards B.

The formulæ (Art. 23) now give $\cot \frac{1}{2} C = \cos a \tan A$,

and $\tan \frac{1}{2} c = \tan a \cos A$.

Now $\frac{1}{2}$ C, $\frac{1}{2} c$ are $< \frac{\pi}{2}$, and hence their tangents are positive. But cos a and tan A, and also tan a and cos A, have the *same sign* only when A, a are of the same affection.

Hence the solution *is impossible* when A, a *are not of the same affection.*

If $A = a = \frac{\pi}{2}$, then $\cos a = 0$, $\cos A = 0$, $\tan a = \infty$, $\tan A = \infty$.

Hence the values of $\cot \frac{1}{2} C$, $\tan \frac{1}{2} c$ assume the indeterminate form $0 \times \infty$.

C and c are therefore indeterminate, and there are *an infinite number of solutions.*

In all other cases where A, a *are of the same affection*, cot ½ C and tan ½ *c* will be positive. There will therefore be a possible solution, and *only one*.

We shall now consider the more general case.

We have $\sin B = \frac{\sin b}{\sin a} \sin A$.

When sin *b* sin A > sin *a*, there can be no possible value of B.

When sin *b* sin A *is not greater* than sin *a*, there will generally be two values of B, which we may call B and B′. They are supplementary to each other. Let B be that which is not greater than B′. Now as A may be <, =, or > $\frac{\pi}{2}$, it is convenient to take these three cases in turn.

We may remark

since $\tan \frac{1}{2} C = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)} \cot \frac{1}{2}(A - B)$,

and $\tan \frac{1}{2} c = \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)} \tan \frac{1}{2}(a - b)$,

that C, *c* will be always possible, and not otherwise, when ½ (*a* − *b*), ½ (A − B), and therefore *a* − *b*, A − B, are of the same sign.

Hence we have to see that A − B, *a* − *b*, and also A − B′, *a* − *b* have the same sign.

Let A *be less than a right angle.*

1. Let $b < \frac{\pi}{2}$.

(i.) When *a* < *b*, we must have A < B, or B > A.

Now the equation $\sin B = \frac{\sin b}{\sin a} \sin A$.....(1),

gives us B > A, since sin *a* < sin *b* when $b < \frac{\pi}{2}$.

Hence we have both B, B′ > A, and there are *two solutions.*

(ii.) When $a = b$, we have shown, since here a, A are each $< \frac{\pi}{2}$, that there is *one solution.*

(iii.) When $a > b$, we have $A > B$;

Now if $a + b = \pi$, or $\sin a = \sin b$,
equation (1) above gives $\sin B = \sin A$; that is $B = A$, and $B' > A$.

Neither of them is possible.

Again, if $a + b > \pi$, we have, since $b < \frac{\pi}{2}$, $\sin a < \sin b$.

Hence $\sin B > \sin A$, or $B > A$, and therefore $B' > A$.

Neither of these is possible.

Also, if $a + b < \pi$, we have, since $b < \frac{\pi}{2}$, $\sin a > \sin b$;

$\therefore \sin B < \sin A$, $\therefore B < A$.

Hence B' or $\pi - B$ is $> A$, being greater than a right angle.

The *first* value only of B is therefore admissible. Hence there is *one possible solution.*

2. Let $b = \frac{\pi}{2}$.

(i.) When $a < b$, we have $\sin B > \sin A$.

Hence B, B' are both $> A$, and there are *two solutions.*

(ii.) When $a = b$, as above shown, since A, a are not *now of the same affection*, there will be *no solution.*

(iii.) When $a > b$, we have $\sin a < \sin b$, since $b = \frac{\pi}{2}$.

Therefore $\sin B > \sin A$, and both values of B are $> A$, which is impossible.

There is therefore *no solution.*

3. Let $b > \frac{\pi}{2}$.

(i.) When $a < b$, we must have A < B.

If $a + b = \pi$, sin a = sin b, and therefore sin B = sin A.

Hence, the *only possible* solution is $B' = A$.

If $a + b > \pi$, then (Art. 23), $A + B > \pi$, and there can be only *one solution.*

If $a + b < \pi$, then (Art. 23), $A + B < \pi$, and there may, therefore, be *two solutions.*

(ii.) When $a = b$, as above shown, there is *no solution,* since A, a, are not of the same affection.

(iii.) When $a > b$, we have, since $b > \frac{\pi}{2}$,

$$\sin a < \sin b.$$

Hence, sin B > sin A,

And both B, B′, are > A, which values are impossible, since $a > b$.

Hence there is *no solution.*

Collecting the above results, we have:

I. *When* A *is less than a right angle.*

The number of solutions is seen from the table:

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*two*	*two*	$\begin{cases} a+b < \pi, \text{ two} \\ a+b \geqq \pi, \text{ one} \end{cases}$
$a = b$	*one*	*none*	*none*
$a > b$	$\begin{cases} a+b < \pi, \text{ one} \\ a+b \geqq \pi, \text{ none} \end{cases}$	*none*	*none*

And by similar reasoning we obtain the following tables for the cases where A is equal to, or greater than a right angle.

II. *When* A *is equal to a right angle.*

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*none*	*none*	$\begin{cases} a+b > \pi, \text{ one} \\ a+b \leqq \pi, \text{ none} \end{cases}$
$a = b$	*none*	*infinite*	*none*
$a > b$	$\begin{cases} a+b < \pi, \text{ one} \\ a+b \geqq \pi, \text{ none} \end{cases}$	*none*	*none*

III. *When* A *is greater than a right angle.*

	$b < \frac{\pi}{2}$	$b = \frac{\pi}{2}$	$b > \frac{\pi}{2}$
$a < b$	*none*	*none*	$\begin{cases} a+b > \pi, \text{ one} \\ a+b \leqq \pi, \text{ none} \end{cases}$
$a = b$	*none*	*none*	*one*
$a > b$	$\begin{cases} a+b > \pi, \text{ two} \\ a+b \leqq \pi, \text{ one} \end{cases}$	*two*	*two*

The ambiguities which arise in Case VI. may be discussed *in a similar manner.*

Ex. II.

1. Show that

$$\cos \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b) \sin C = \cos^2 \tfrac{1}{2} c \sin (A + B),$$

and

$$\sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2}(a - b) \sin C = \sin^2 \tfrac{1}{2} c \sin (A - B).$$

2. Prove that, when a triangle ABC is equal and similar to its polar triangle,

$$\sec^2 A + \sec^2 B + \sec^2 C + 2 \sec A \sec B \sec C = 1.$$

3. If $a + b + c = \pi$, show that

$$\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2,$$

$$\tan a = \frac{\sin \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}.$$

4. If C = A + B, and the side AB be bisected in D, then CD, AD, BD are equal to each other.

Prove that:

5. $\sin a \sin b \sin c \sin A \sin B \sin C = 4\,N n$ (see Art. 49),

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{N}{n}.$$

6. $\sin^2 a \sin^2 b \sin^2 c \sin A \sin B \sin C = 8\,n^3$,

$\sin^2 A \sin^2 B \sin^2 C \sin a \sin b \sin c = 8\,N^3$.

7. $\sin^2 s = n \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$.

$\cos^2 S = N \tan \frac{a}{2} \tan \frac{b}{2} \tan \frac{c}{2}$.

8. $\sin^2 \frac{1}{2}(A - B) \sin^2 \frac{1}{2} c + \sin^2 \frac{1}{2}(A + B) \cos^2 \frac{1}{2} c = \cos^2 \frac{1}{2} C$,

$\sin^2 \frac{1}{2}(a - b) \cos^2 \frac{1}{2} C + \sin^2 \frac{1}{2}(a + b) \sin^2 \frac{1}{2} C = \sin^2 \frac{1}{2} c$.

9. $\cos^2 \frac{1}{2}(A - B) \sin^2 \frac{1}{2} c + \cos^2 \frac{1}{2}(A + B) \cos^2 \frac{1}{2} c = \sin^2 \frac{1}{2} C$,

$\cos^2 \frac{1}{2}(a - b) \cos^2 \frac{1}{2} C + \cos^2 \frac{1}{2}(a + b) \cos^2 \frac{1}{2} C = \cos^2 \frac{1}{2} c$.

Solve the six following right-angled triangles, C being the right angle :

10. $a = 62° 45'$, $B = 135° 21'$.

11. $a = 127° 21' 30''$, $b = 80° 32' 27''$.

12. $c = 132° 15' 17''$, $B = 63° 51' 24''$.

13. $c = 56° 21' 35''$, $b = 140° 11' 38''$.

14. $A = 41° 31' 48''$, $B = 118° 56' 10''$.

15. $A = 156° 20' 30''$, $a = 65° 15' 45''$.

16. A ship sails from the Cape of Good Hope in lat. 33° 56′ S., lon. 18° 23′ E., and crosses the Equator in lon. 28° 18′ W. What is the least distance which she can have run in geographical miles?

17. The elevation of the North Pole is 45°, and I observe a bright star on the horizon at an angular distance of 45° from the north part of the horizon. What is its angular distance from the North Pole?

18. Given δ the sun's declination, and a his altitude when due E, what is the latitude of the place of observation?

Solve the following triangles :—

19. $a = 35° 21' 40''$, $b = 126° 25' 32''$, $c = 29° 24' 26''$.

20. $A = 118° 6' 12''$, $B = 56° 29' 14''$, $C = 87° 15' 18''$.

21. $a = 19° 54' 24''$, $b = 56° 29' 16''$, $C = 85° 31' 20''$.

22. $A = 89° 56'$, $B = 61° 28'$, $c = 45° 29'$.

23. $a = 48° 10' 18''$, $b = 64° 15' 20''$, $A = 56° 29' 10''$.

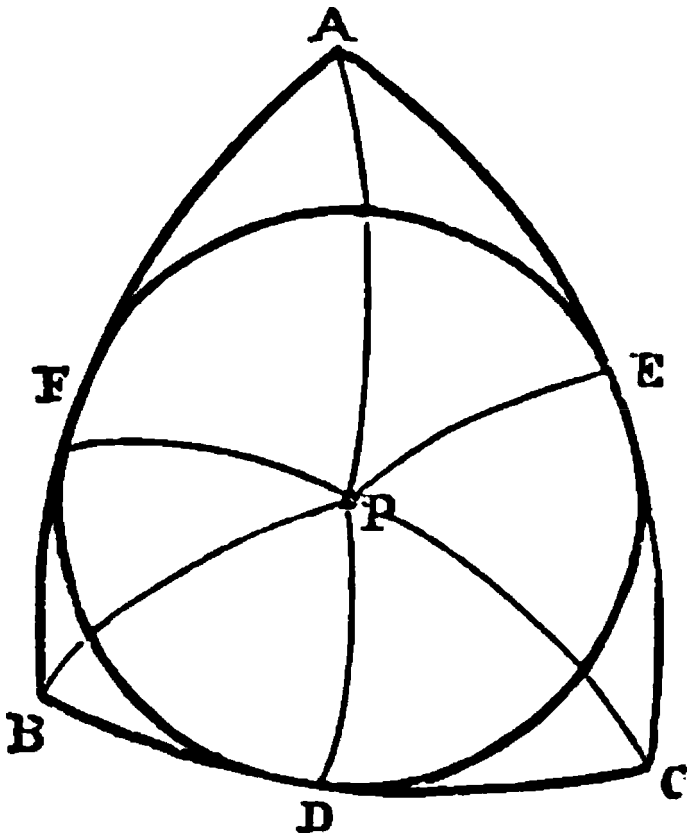
24. $A = 84° 15' 17''$, $B = 29° 13' 11''$, $a = 59° 16' 18''$.

CHAPTER V.

INSCRIBED, CIRCUMSCRIBED, AND ESCRIBED CIRCLES.

Inscribed Circle.

48. *To find the radius of the small circle inscribed in a given spherical triangle.*



Let ABC be the spherical triangle.

Bisect the angles A, B, by arcs meeting in the point P. From P draw the arcs PD, PE, PF perpendicular to the sides. Then it is easily seen that

$$PD = PE = PF,$$

and that P is the pole of the small circle DEF.

Also AE = AF, CD = CE, BD = BF.

Hence $AE + CD + BF = \frac{1}{2}(a + b + c) = s$,

or $AE + CD + BD = s$, $\therefore AE = s - BC = s - a$.

Now, from the right-angled triangle APE, we have

$$\tan PE = \sin AE \,.\, \tan PAE = \sin (s - a) \tan \frac{A}{2};$$

or $$\tan r = \sin (s - a) \tan \frac{A}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence, by Art. 20,

$$\tan r = \sin (s - a) \,.\, \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}},$$

or $$\tan r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}} \quad \ldots\ldots\ldots (2).$$

We may express the radius in terms of the angles thus:

We have again

$$\tan r = \sin(s-a)\tan\tfrac{1}{2}A = \sin\{\tfrac{1}{2}(b+c) - \tfrac{1}{2}a\}\tan\tfrac{1}{2}A\text{; or expanding,}$$

$$= \{\sin\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}A\,.\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(b+c)\sin\tfrac{1}{2}A\,.\sin\tfrac{1}{2}a\}\frac{1}{\cos\frac{1}{2}A};$$

or, by Gauss's Theorem,

$$= \{\cos\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}a\sin\tfrac{1}{2}a\}\cdot\frac{1}{\cos\frac{1}{2}A}$$

$$= \frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}\sin a.$$

Hence (Art. 20), by substitution and reduction,

$$\tan r = \frac{\sqrt{-\cos S\cos(S-A)\cos(S-B)\cos(S-C)}}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

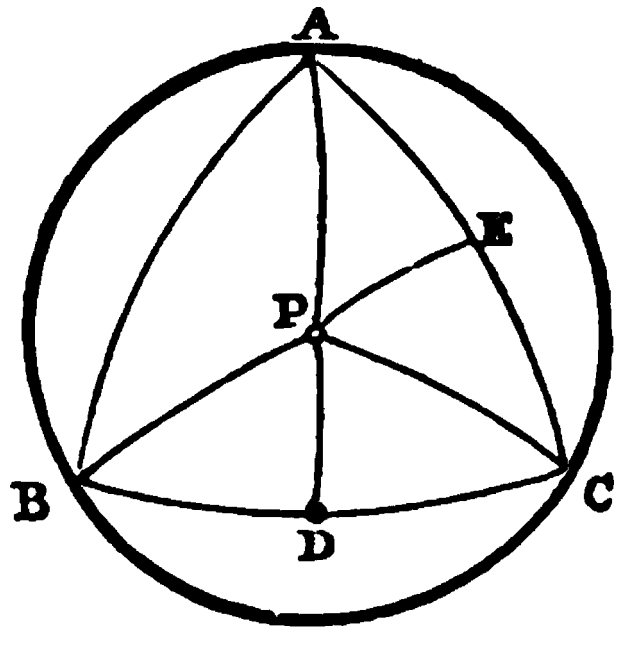
Circumscribed Circle.

49. *To find the radius of the small circle which may be circumscribed about a given triangle.*

Let P be the pole of the circumscribed circle.

Then it may be easily shown that the arcs of great circles drawn perpendicularly from P upon the sides BC, AC also bisect those sides in D, E.

We have also $\angle$ PAB = $\angle$ PBA, $\angle$ PBD = $\angle$ PCD, and $\angle$ PAC = $\angle$ PCA.



Hence it follows that $\angle$ PCB = S − A.

We have from the right-angled triangle PCD,

$$\cos \text{PCD} = \tan \text{CD} \cot \text{PC}.$$

$$\therefore \tan \text{PC} = \frac{\tan \text{CD}}{\cos \text{PCD}}; \text{ or}$$

$$\tan \text{R} = \frac{\tan \frac{1}{2} a}{\cos (\text{S} - \text{A})} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence (Art. 21), by substitution and reduction,

$$\tan \text{R} = \sqrt{\frac{-\cos \text{S}}{\cos (\text{S} - \text{A}) \cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}} \ldots (2).$$

Again, proceeding as in the last Art., we get

$$\tan \text{R} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}} \ldots (3).$$

Cor. 1. Multiplying this last result by (2) of the last Art., we have

$$\tan \text{R} \tan r = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sin s}$$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sin \frac{1}{2} (a + b + c)}.$$

Cor. 2. It is convenient to represent the expressions

$$\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)},$$

and $\sqrt{-\cos \text{S} \cos (\text{S} - \text{A}) \cos (\text{S} - \text{B}) \cos (\text{S} - \text{C})}$,

respectively by n and N.

Then, $$\tan r = \frac{n}{\sin s} = \frac{\text{N}}{2 \cos \frac{1}{2} \text{A} \cos \frac{1}{2} \text{B} \cos \frac{1}{2} \text{C}},$$

and $$\tan \text{R} = \frac{\cos \text{S}}{\text{N}} = \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{n}.$$

Escribed Circles.

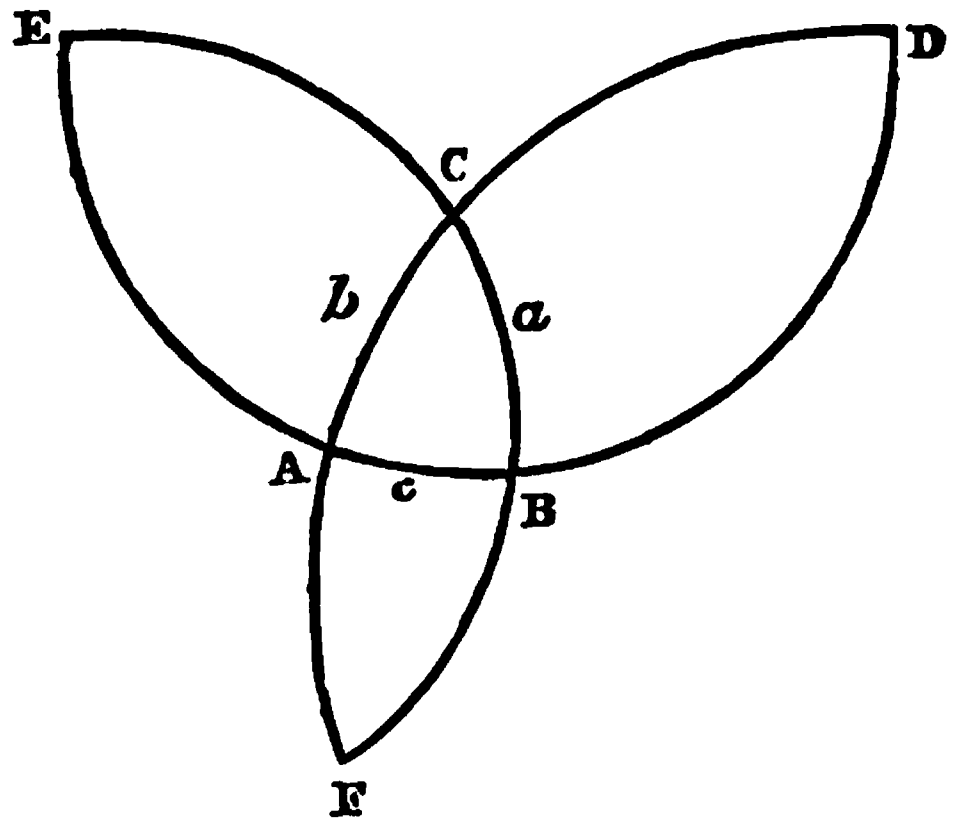
50. *To find the radius of the small circle which touches one side of a spherical triangle* and the other sides produced.

Let ABC be the triangle, and let its sides be produced to meet in D, E, F.

Then evidently the angles of the triangle BCD are equal to

A, $\pi - B$, $\pi - C$;

And the sides are a, $\pi - b$, $\pi - c$.



Now the radius of the small circle touching BC, and AB, AC produced, is the inscribed circle of the triangle BCD.

Let r_a represent the radius of the escribed circle touching the side a; then if we suppose a, b_1, c_1, the sides of BCD, we have

$$\tan r_a = \sqrt{\frac{\sin (s_1 - a) \sin (s_1 - b_1) \sin s_1 - c_1)}{\sin s_1}},$$

But $s_1 = \frac{1}{2}(a + \overline{\pi - b} + \overline{\pi - c})$, $\therefore \sin s_1 = \sin (s - a)$.

And so
$$\sin (s_1 - a) = \sin s$$
$$\sin (s_1 - b_1) = \sin (s - c)$$
$$\sin (s_1 - c_1) = \sin (s - b).$$

Hence, by substitution,

$$\tan r_a = \sqrt{\frac{\sin s \sin (s - b) \sin (s - c)}{\sin (s - a)}} = \frac{n}{\sin (s - a)} \quad \ldots\ldots(1).$$

So
$$\tan r_b = \sqrt{\frac{\sin s \sin (s - a) \sin (s - c)}{\sin (s - b)}} = \frac{n}{\sin (s - b)} \quad \ldots\ldots(2),$$

and
$$\tan r_c = \sqrt{\frac{\sin s \sin (s - a) \sin (s - b)}{\sin (s - c)}} = \frac{n}{\sin (s - c)} \quad \ldots\ldots(3).$$

And similarly we express these radii in terms of the angles in the following form :

$$\tan r_a = \frac{\cos \frac{1}{2} B \cos \frac{1}{2} C}{\cos \frac{1}{2} A} \sin a = \frac{N}{2 \cos \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C} \quad (4).$$

Cor. If R_a, R_b, R_c represent respectively the small circles described about the triangles BCD, CAE, ABF, then we have similarly

$$\left.\begin{aligned} \tan R_a &= \frac{\tan \frac{1}{2} a}{-\cos S} = \frac{\cos (S - A)}{N} \\ &= \frac{\sin \frac{1}{2} a}{\sin \frac{1}{2} b \sin \frac{1}{2} c} \cdot \frac{1}{\sin A} \\ &= \frac{2 \sin \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{n} \end{aligned}\right\}.$$

And similar expressions for $\tan R_b$, $\tan R_c$.

The four triangles, ABC and the three formed by producing the sides to meet in D, E, F, are called **associated triangles**, the triangle ABC being the **fundamental triangle.**

Ex. III.

Prove the following relations :

1. $\tan r \tan r_a \tan r_b \tan r_c = n^2$.
2. $\cot R \cot R_a \cot R_b \cot R_c = N^2$.
3. $\cot r \tan r_a \tan r_b \tan r_c = \sin^2 s$.
4. $\tan r \tan r_b \tan r_c \cot r_a = \sin^2 (s - a)$.
5. $\cot R \cot R_b \cot R_c \tan R_a = \cos^2 (S - A)$.
6. $\tan R \cot R_a \cot R_b \cot R_c = \cos^2 S$.
7. $\cot R \cot R_a + \cot R_b \cot R_c = \sin B \sin C$.
8. $\tan r \tan r_a + \tan r_b \tan r_c = \sin b \cdot \sin c$.
9. The sum of the products of the tangents of the radii of the circles inscribed in the associated triangles, when taken two and two together, is equal to the sum of the products of the sines of the sides of the fundamental triangle when taken *two together.*

10. A similar property belongs to the radii of the circumscribed circles of the associated triangles.

11. In any spherical triangle, if the vertical angle and the difference between the base and the other two sides be given, the radius of the inscribed circle will be *constant*, and the centre *a fixed point.*

12. Prove also the corresponding theorem for the circumscribing circle.

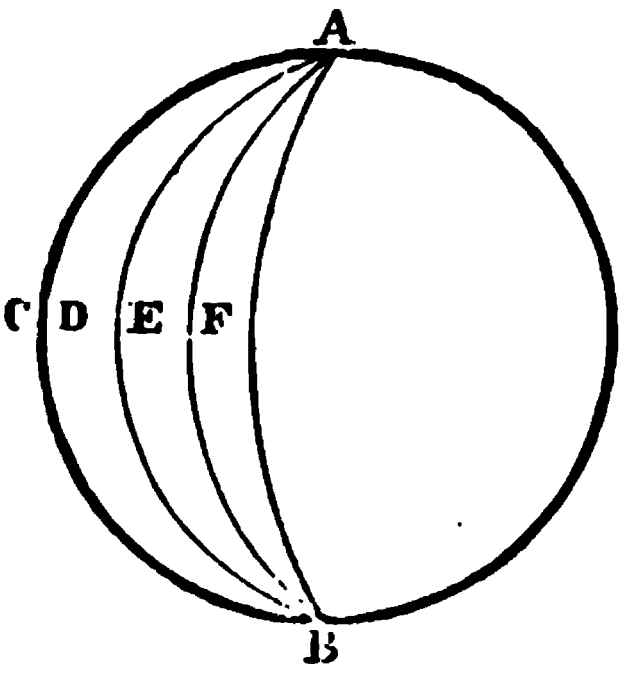
CHAPTER VI.

AREA OF A SPHERICAL TRIANGLE. SPHERICAL EXCESS.

51. *To find the area of a lune.*

DEF. The portion of the surface of a sphere which is included between two great semicircles is called a **Lune.**

Let the lunes ACBD, ADBE, AEBF, have each the same angle A.



Then, since each may by superposition be made to exactly coincide with another, they are equal to each other.

Hence it easily follows that any lune, whatever the area of ACBD, will bear to the area of *any* lune ACBF, the ratio of the angle A to the angle CAB.

Instead of the lune ACBF let us take the whole sphere whose angle is 2π.

Then $$\frac{\text{area of lune whose angle is A}}{\text{area of sphere}} = \frac{A}{2\pi}.$$

If r be the radius of the sphere, then—

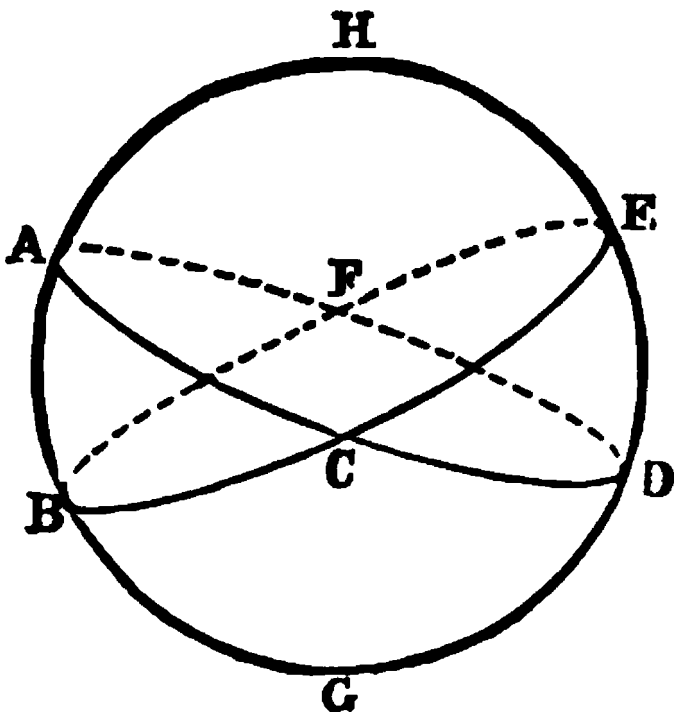
$$\textit{Surface of a sphere} = 4\pi r^2.$$

Hence, $$\textit{Area of lune} = \frac{A}{2\pi} \cdot 4\pi r^2 = 2Ar^2.$$

(The student will remember that A here is the circular measure of the angle.)

52. *Area of spherical triangle.*

Let ABC be the spherical triangle; produce the arcs forming the sides to meet again two and two. This will be the case when each has become a semicircle.



Now in the triangles AFB and DCE we have

$$\angle AFB = \angle DCE.$$

Also $DC = AD - AC$

$$= CAF - AC = AF,$$

and $EC = BE - BC$

$$= CBF - BC = BF.$$

Hence the triangles AFB and DCE are equal.

Let Σ represent the area of the Δ ABC, then

$$\Sigma = ABGDEH - (BGDC + AHEC + DCE)$$

$$= \text{hemisphere} - (\text{lune } AGD - \Delta ABC) - (\text{lune } BAE - \Delta ABC) - \Delta AFB;$$

And $\Delta AFB = \text{lune } CAFB - \Delta ABC$. Hence

$$\Sigma = 2\pi r^2 - (2Ar^2 - \Sigma) - (2Br^2 - \Sigma) - (2Cr^2 - \Sigma)$$

$$= 2r^2(\pi - A - B - C) + 3\Sigma.$$

$\therefore \Sigma = (A + B + C - \pi) r^2$; where A, B, C represent the circular measures of the angles of the triangle.

If A, B, C represent the numbers of degrees, we have

$$\Sigma = \frac{A + B + C - 180}{180} \cdot \pi r^2$$

DEF. The excess of the sum of the angles of a triangle above two right angles is called **the spherical excess.**

Hence, if E be the number of degrees in the spherical excess, we have

$$A + B + C - 180 = E,$$

And

$$\Sigma = \frac{E}{180} \cdot \pi r^2.$$

Cor. Since a spherical polygon may be divided into as many triangles as the polygon has sides, by drawing arcs of great circles from a point within it to the angles, we have, if n be the number of sides,

Area of polygon

$$= \text{area of } n \text{ triangles}$$

$$= (\text{sum of angles of the triangles} - n \,.\, 180°)\, \frac{\pi r^2}{180°}$$

$$= (\text{sum of angles of the polygon} - \overline{n-2} \,.\, 180°)\, \frac{\pi r^2}{180°}$$

53. *Cagnoli's Theorem. To show that*

$$\sin \tfrac{1}{2} \mathrm{E} = \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) (s-c)}}{2 \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b \cos \tfrac{1}{2} c}$$

We have, $\sin \frac{1}{2} \mathrm{E}$

$$= \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B} + \mathrm{C} - \pi) = - \cos \tfrac{1}{2} (\mathrm{A} + \mathrm{B} + \mathrm{C})$$

$$= - \left\{ \cos \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) \cos \tfrac{1}{2} \mathrm{C} - \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) \sin \tfrac{1}{2} \mathrm{C} \right\}$$

$$= \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) \sin \tfrac{1}{2} \mathrm{C} - \cos \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) \cos \tfrac{1}{2} \mathrm{C}.$$

Or, by Gauss's Theorem,

$$= \frac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} c} \cos \tfrac{1}{2} \mathrm{C} \sin \tfrac{1}{2} \mathrm{C} - \frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} c} \sin \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} \mathrm{C}.$$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin \mathrm{C} \text{ ; or, by Art. 20,}$$

$\sin \frac{1}{2} \mathrm{E}$

$$= \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \cdot \frac{2}{\sin a \sin b} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

$$= \frac{\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}.$$

54. *Llhuillier's Theorem. To show that*

$$\tan\tfrac{1}{4}\mathrm{E} = \sqrt{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)}.$$

$$\tan\tfrac{1}{4}\mathrm{E}$$

$$= \frac{2\sin\frac{1}{4}(\mathrm{A}+\mathrm{B}+\mathrm{C}-\pi)\,.\,\cos\frac{1}{4}(\mathrm{A}+\mathrm{B}-\mathrm{C}+\pi)}{2\cos\frac{1}{4}(\mathrm{A}+\mathrm{B}+\mathrm{C}-\pi)\,.\,\cos\frac{1}{4}(\mathrm{A}+\mathrm{B}-\mathrm{C}+\pi)};$$

or,

$$= \frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})-\sin\frac{1}{2}(\pi-\mathrm{C})}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})+\cos\frac{1}{2}(\pi-\mathrm{C})} = \frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})-\cos\frac{1}{2}\mathrm{C}}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})+\sin\frac{1}{2}\mathrm{C}}$$

Or, by Gauss's Theorem, after reduction, .

$$\tan\tfrac{1}{4}\mathrm{E}$$

$$= \frac{\cos\frac{1}{2}(a-b)-\cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b)+\cos\frac{1}{2}c}\cdot\frac{\cos\frac{1}{2}\mathrm{C}}{\sin\frac{1}{2}\mathrm{C}},$$

$$= \frac{\sin\frac{1}{4}(b+c-a)\,\sin\frac{1}{4}(a+c-b)}{\cos\frac{1}{4}(a+b+c)\,\cos\frac{1}{4}(a+b-c)}\cot\tfrac{1}{2}\mathrm{C},$$

$$= \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)}{\cos\frac{1}{2}s\,\cos\frac{1}{2}(s-c)}\cdot\sqrt{\frac{\sin s\,\sin(s-c)}{\sin(s-a)\,\sin(s-b)}},$$

$$= \sqrt{\tan\tfrac{1}{2}s\,\tan\tfrac{1}{2}(s-a)\,\tan\tfrac{1}{2}(s-b)\,\tan\tfrac{1}{2}(s-c)}.$$

Other formulæ, involving the spherical excess, follow in the exercises upon this chapter.

Ex. IV.

Prove the following relations :

1. $\cot\frac{1}{2}\mathrm{E} = \dfrac{\cot\frac{1}{2}a\,\cot\frac{1}{2}b + \cos\mathrm{C}}{\sin\mathrm{C}}.$

2. $\cos\frac{1}{2}\mathrm{E} = \dfrac{1+\cos a+\cos b+\cos c}{4\cos\frac{1}{2}a\,\cos\frac{1}{2}b\,\cos\frac{1}{2}c}.$

3. $\sin\frac{1}{2}\mathrm{E} = \sin\frac{1}{2}a\,\sin\frac{1}{2}b\,\sec\frac{1}{2}c\,\sin\mathrm{C}.$

4. $\sin\frac{1}{4}\mathrm{E} = \dfrac{\sin\frac{1}{2}s\,\sin\frac{1}{2}(s-a)\,\sin\frac{1}{2}(s-b)\,\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\,\cos\frac{1}{2}b\,\cos\frac{1}{2}c}.$

5. $\cos \frac{1}{4} \mathrm{E} = \dfrac{\cos \frac{1}{2} s \cos \frac{1}{2} (s - a) \cos \frac{1}{2} (s - b) \cos \frac{1}{2} (s - c)}{\cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$.

6. $\tan \frac{1}{4} \mathrm{E} \tan \frac{1}{4} \mathrm{E}_a \tan \frac{1}{4} \mathrm{E}_b \tan \frac{1}{4} \mathrm{E}_c = \tan^2 \frac{1}{2} s$, where E, E_a, E_b, E_c are respectively the spherical excesses of the associated triangles.

7. $\cot \frac{1}{4} \mathrm{E} \cot \frac{1}{4} \mathrm{E}_b \cot \frac{1}{4} \mathrm{E}_c \tan \frac{1}{4} \mathrm{E}_a = \cot^2 \frac{1}{2} (s - a)$.

8. If E', E'_a, E'_b, E'_c be the spherical excesses of the polar triangle of the associated triangles, then

$$\cot^2 \tfrac{1}{4} \mathrm{E}' = \cot \tfrac{1}{4} \mathrm{E}_a \cot \tfrac{1}{4} \mathrm{E}_b \cot \tfrac{1}{4} \mathrm{E}_c \tan \tfrac{1}{4} \mathrm{E},$$

$$\cot^2 \tfrac{1}{4} \mathrm{E}'_a = \cot \tfrac{1}{4} \mathrm{E} \cot \tfrac{1}{4} \mathrm{E}_b \cot \tfrac{1}{4} \mathrm{E}_c \tan \tfrac{1}{4} \mathrm{E}_a.$$

$$\&c. = \&c.$$

9. $\tan \frac{1}{4} \mathrm{E} \tan \frac{1}{4} \mathrm{E}' = \tan \frac{1}{4} \mathrm{E}_a \tan \frac{1}{4} \mathrm{E}'_a = \tan \frac{1}{4} \mathrm{E}_b \tan \mathrm{E}'_b = \tan \frac{1}{4} \mathrm{E}_c \tan \frac{1}{4} \mathrm{E}'_c$.

10. $\cot \frac{1}{2} \mathrm{E}$

$$= \frac{\cot r_a + \cot r_b + \cot r_c - \cot r}{\tan r_a + \tan r_b + \tan r_c - \tan r} \sqrt{\tan r \tan r_a \tan r_b \tan r_c}.$$

11. Prove similar formulæ for E_a, E_b, E_c.

12. If E be the excess of the triangle each of whose angles is 120°, and E' that of the polar triangle, then

$$\tan \tfrac{1}{3} \mathrm{E} : \tan \tfrac{1}{3} \mathrm{E}' :: 27 + 8\sqrt{6} : 5.$$

CHAPTER VII.

POLYHEDRONS.

55. By a **Polyhedron** is meant a solid bounded by plane rectilineal figures; and it is called **regular** when the bounding surfaces are similar and equal. We have space for only a few of the more elementary propositions.

56. *In any polyhedron, if* F *be the number of faces,* S *the number of solid angles,* E *the number* of edges, then

$$\mathrm{S} + \mathrm{F} = \mathrm{E} + 2.$$

Take any point within the polyhedron, and draw to this point lines from each of the solid angles. Describe a sphere with this point as centre, and let its surface be cut by these lines. Draw arcs of great circles connecting the points where the lines meet the surface.

Then the surface of the sphere will be divided into as many spherical polygons as there are faces in the solid.

Now, Art. 52, Cor.,

Area of one of these polygons

$$= \{\text{sum of the angles of the polygon} - \overline{n-2}.180^\circ\}\frac{\pi r^2}{180^\circ}.$$

Hence, total area of all the polygons

$$= \{\text{sum of angles of all the polygons} - (2\,E - 2\,F)\,180^\circ\}\frac{\pi r^2}{180^\circ}.$$

$$= \{S\,.\,360^\circ - (2\,E - 2\,F)\,180^\circ\}\frac{\pi r^2}{180^\circ} = 2\,(S - E + F)\,\pi r^2.$$

But the total area of all the polygons

$$= \text{surface of the sphere} = 4\,\pi r^2.$$

Hence $$2\,(S - E + F)\,\pi r^2 = 4\,\pi r^2,$$

or $$S - E + F = 2.$$

$$\therefore S + F = E + 2.$$

Cor. If m be the number of sides in each face, and n the number of plane angles in every solid angle, then in any regular polyhedron

$$2\,E = mF = nS.$$

57. *The sum of all the plane angles which form the solid angles of any polyhedron* $= (S - 2)\,360^\circ$.

Now, Sum of all the plane angles

= sum of all the interior angles of the faces..... (1).

Now, if m be the number of sides in any one face, then *(Euc. I., 32)* Sum of the interior angles of that face $= (m - 2)\,180^\circ$,

And the whole number of sides in the faces is 2 E.

Hence, Total sum of the interior angles of all the faces $= (2\,E - 2\,F)\,180°$.

Hence from (1),

$$\text{Sum of all the plane angles} = (2\,E - 2\,F)\,180°$$
$$= (E - F)\,360°, \text{ or (Art. 56)},$$
$$= (S - 2)\,360°.$$

58. *In any regular polyhedron,*

$$S = \frac{4m}{2(m+n) - mn}, \quad E = \frac{2mn}{2(m+n) - mn},$$

$$F = \frac{4n}{2(m+n) - mn}.$$

We have (Art. 56), $S + F = E + 2$,

and $$2E = mF = nS.$$

The above values of S, E, F at once follow from these equations.

59. *There are only five regular polyhedrons.*

The expressions for S, E, F just found must always be positive integers.

Hence $2(m+n) - mn > 0$, or $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$.

Now n, the number of plane angles in a solid angle, cannot be less than 3.

Hence $\frac{1}{m} > \frac{1}{2} - \frac{1}{3}$, or $> \frac{1}{6}$.

And hence neither m nor n can be less than 3 or greater than 6.

And on trial, the only admissible values of m and n are *3, 4, 5.*

If we substitute these values in the above values of S, E, F, we find the only pairs of values are as follows:

m	n	S	E	F	Name of regular Polyhedron.
3	3	4	6	4	Tetrahedron or Regular Pyramid.
4	3	8	12	6	Hexahedron or Cube.
3	4	6	12	8	Octahedron.
5	3	20	30	12	Dodecahedron.
3	5	12	30	20	Icosahedron.

Cor. Since it is not necessary for the relations

$$2\,E = mF = nS,$$

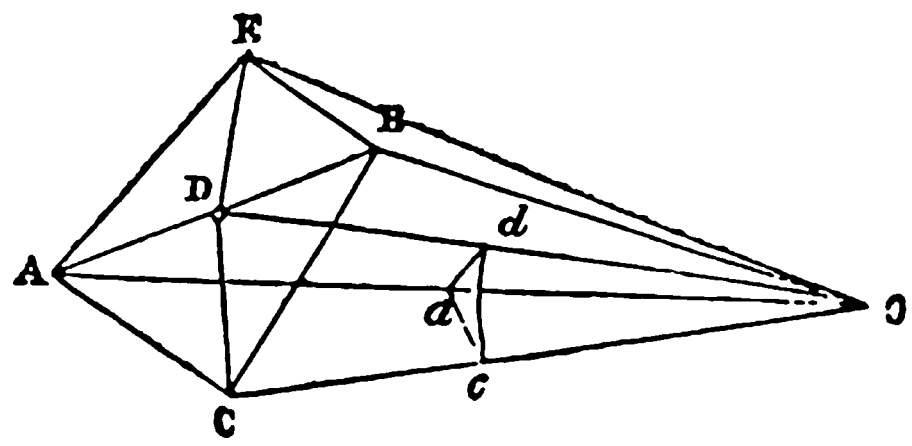
and

$$S + F = E + 2,$$

that the faces should be equilateral and equiangular, and all equal, it follows that we may conclude the following more general proposition.

There can only be five solid bodies each of which has all its faces of the same number of sides, and all its solid angles formed of the same number of plane angles.

60. *To find the inclination of two contiguous faces of a regular polyhedron.*



Let AB be the common edge of two faces; C, E, the

centres of the faces. Bisect AB in D, and join DC, DE. Then evidently CD, CE are perpendicular to AB.

$\therefore \angle$ CDE = I, the inclination of the faces.

Draw CO, EO at right angles to CD, DE respectively, and both in the plane CDE. Join OA, OB, OD.

Let a sphere be described with O as centre, so as to meet OA, OC, OD in *a*, *c*, *d* respectively, and form the spherical triangle *acd*.

Now AB is perpendicular to CD, CE, and it is therefore perpendicular to the plane CDE. Hence the plane AOB in which AB lies is also perpendicular to the plane CDE or the plane COE.

But *ad* lies in the plane AOB, and *cd* lies in the plane COE.

Hence, the angle *adc* of the spherical triangle *acd* is a right angle.

Now, if m be the number of sides in each face of the polyhedron, and n the number of plane angles forming each solid angle,

$$\angle acd = \angle \text{ACD} = \tfrac{1}{2}\,\text{ACB} = \tfrac{1}{2} \cdot \frac{360^\circ}{m} = \frac{180^\circ}{m},$$

And $\angle cad = \tfrac{1}{2}$ one of the equal angles formed upon the sphere round the point *a*.

$$= \tfrac{1}{2} \cdot \frac{360^\circ}{n} = \frac{180^\circ}{n}.$$

By Napier's Rules, we have from the right-angled triangle *acd*

$$\cos cad = \cos cd \sin acd.$$

$$\therefore \cos \frac{180^\circ}{n} = \cos cd \sin \frac{180^\circ}{m} \ldots\ldots\ldots\ldots\ldots(1).$$

But $cd = \angle \text{DOC} = \tfrac{1}{2}\,\text{EOC} = \tfrac{1}{2}(180^\circ - \text{CDE}) = 90^\circ - \tfrac{1}{2}\text{I}$

$$\therefore \cos cd = \sin \tfrac{1}{2}\text{I}.$$

Hence, from (1),

$$\cos\frac{180^\circ}{n} = \sin\tfrac{1}{2}\mathrm{I}\sin\frac{180^\circ}{m}$$

$$\therefore \sin\tfrac{1}{2}\mathrm{I} = \frac{\cos\dfrac{180^\circ}{n}}{\sin\dfrac{180^\circ}{m}}.$$

Cor. The following results are easily obtained :

(1.) For the *Tetrahedron*, $\sin\frac{1}{2}\mathrm{I} = \frac{1}{\sqrt{3}}$; $\therefore \cos\mathrm{I} = \frac{1}{3}$.

(2.) *Hexahedron or Cube*, $\sin\frac{1}{2}\mathrm{I} = \frac{1}{\sqrt{2}}$;

$$\therefore \cos\mathrm{I} = 1, \text{ or } \mathrm{I} = 90^\circ.$$

(3.) *Octahedron*, $\sin\frac{1}{2}\mathrm{I} = \sqrt{\frac{2}{3}}$, $\therefore \cos\mathrm{I} = -\frac{1}{3}$.

(4.) *Dodecahedron*, $\sin\frac{1}{2}\mathrm{I} = \frac{2}{\sqrt{10 - 2\sqrt{5}}}$;

$$\therefore \cos\mathrm{I} = -\tfrac{1}{5}\sqrt{5}.$$

(5.) *Icosahedron*, $\sin\frac{1}{2}\mathrm{I} = \frac{\sqrt{5}+1}{2\sqrt{3}}$;

$$\therefore \cos\mathrm{I} = -\tfrac{1}{3}\sqrt{5}.$$

61. *To find the radii of the spheres inscribed in and described about a regular polyhedron.*

In the last diagram, since OC, OE are at right angles to the planes ABC, ABE, they are equal to each other.

Hence O is the centre of the inscribed and circumscribed spheres.

Let $\mathrm{AB} = a$, $\mathrm{OA} = \mathrm{R}$, $\mathrm{OC} = r$.

From the right-angled triangle *acd*, we have, by Napier's *Rules*,

$$\cos ac = \cot acd \cot cad.$$

But cos ac = cos AOC = $\frac{\text{OC}}{\text{OA}} = \frac{r}{\text{R}}$.

Hence (Art. 60), $\frac{r}{\text{R}} = \cot\frac{180^\circ}{m}\cot\frac{180^\circ}{n}$ ..... (1).

Again,

$$\text{OC} = \text{CD}\tan\text{CDO};\ \text{or}\ r = \text{CD}\tan\frac{\text{I}}{2},$$

and $$\text{CD} = \text{AD}\cot\text{ACD} = \frac{a}{2}\cot\frac{180^\circ}{m}.$$

Hence $$r = \frac{a}{2}\cot\frac{180^\circ}{m}\tan\frac{\text{I}}{2}.$$

And therefore from (1),

$$\text{R} = \frac{a}{2}\tan\frac{180^\circ}{n}\tan\frac{\text{I}}{2}.$$

62. *To find the surface and volume of a regular polyhedron.*

Now as m is the number of sides in each face, we have:

$$\text{Area of each face} = \frac{ma^2}{4}\cot\frac{180^\circ}{m};$$

$$\text{Hence, whole surface} = \frac{\text{F}ma^2}{4}\cot\frac{180^\circ}{m}.$$

Again, Volume of each pyramid, having O for vertex and a face for base,

$$= \tfrac{1}{3}\text{ corresponding prism}$$

$$= \tfrac{1}{3}r\,.\,\frac{ma^2}{4}\cot\frac{180^\circ}{m}$$

$$= \frac{mra^2}{12}\cot\frac{180^\circ}{m}.$$

Hence whole volume $= \frac{\text{F}mra^2}{12}\cot\frac{180^\circ}{m}$.

Ex. V.

1. Show how to fit together a number of equal regular tetrahedrons, and equal regular octahedrons to form a solid mass, without interstices or gaps.

2. A pyramid stands on a square base, and its altitude is half the side of the base. What are the dihedral angles at the vertex?

3. Similar polyhedrons are to one another in the triplicate ratio of their homologous edges, and their convex surfaces are in the duplicate ratio of their edges.

4. Show by a plane construction how to find the angle contained by any two of the planes AOB, BOC, COA, which form a solid angle at O.

5. When two solid angles are contained each by three plane angles which are respectively equal, the dihedral angles of the one will be equal to the dihedral angles of the other, each to each.

6. In a given dodecahedron inscribe a cube.

7. If R, r be the radii of the circumscribed and inscribed spheres, show that:

(1.) For the tetrahedron, $R = 3r$.

(2.) For the cube or octahedron, $R = \sqrt{3} \cdot r$.

8. In the regular icosahedron, the distance of the regular pentagon, which passes through five of the solid angles, from the centre is one-half the radius of the sphere circumscribing the pentagon.

9. In a pyramid on a square base each edge meeting the vertex is twice the length of the base. Find the inclination of two contiguous faces.

10. If α, β, γ be the angles which a line passing through an angle makes with each of the edges of a cube, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

11. If A, B, C, D, be four points in a plane, and A_1, B_1, C_1, D_1 their projections on any other plane, then the volumes of $ABCD_1$ and $A_1B_1C_1D$ are equal to each other.

12. If a, b, c be three edges of a tetrahedron, and d, e, f the three opposite edges, then, if I be the angle between a and d,

$$\cos \mathrm{I} = \frac{(b^2 + e^2) - (c^2 + f^2)}{2ad}.$$

ANSWERS.

ALGEBRA.

I.—Page 13.

1. $\frac{8}{3}\sqrt{165}$.
2. $cx^2 + bx + a = 0$.
3. $(x+\gamma)^2 - p(x+\gamma) + q = 0$.
4. 4.
5. $a(ac'-a'c)^2 + b(a'b-ab')(ac'-a'c) + c(a'b-ab')^2 = 0$.
7. $m \pm n \pm p = 0$.
8. $x^2 - (p + \sqrt{p^2-4q})\,x + p\sqrt{p^2-4q} = 0$.
12. $(3x + 4y + 5)(2x + 3y - 2)$.

II.—Page 17.

1. 27.
2. 1.
3. 3, 5.
4. $\frac{e_1 d_1^2}{e d^2}$.
5. 156 . 8.
6. $s = 16t^2$.
7. $6\frac{3}{4}$ tons
8. $m^2p : 20n^2q$.
9. $\frac{mc}{m+1} \cdot \frac{n^2}{a^2}$, $\frac{c}{m+1} \cdot \left(\frac{n}{o}\right)^{\frac{2}{3}}$.
11. ·067.
12. 7·231 ; 23·717.

III.—Page 26.

1. 10.
2. $\frac{|10}{|2\ |2\ |3}$.
3. $\frac{|12}{|2\ |2}$, $\frac{|7}{|3\ |2}$.
5. $\frac{|9}{(|2)^3\ |3}$.

6. $\frac{\lfloor 9}{2}$, if no person is to have the same neighbours twice.

7. 12. 8. 7. 12. m^n. 16. 10.

17. $\begin{array}{r|l} 3 & 100 \\ \hline 12 & 88 \end{array}$.

IV.—Page 31.

1. 38, 288. 2. $17\frac{1}{4}$. 3. -18.

4. 150. 5. 50^2. 6. $13a$.

7. $10x$. 8. $84\frac{1}{3}$. 9. 5, 8, 11, ...

10. $229\frac{1}{5}$. 11. 2.

13. $a = \dfrac{P(1-q) - Q(1-p)}{p-q}$, $d = \dfrac{P-Q}{p-q}$.

20. $(3n-7)\frac{n}{2}$. 21. 3, 5, 7. 22. 1, 2, 3, 4.

23. An A. P. where p, q, r are the P^{th}, Q^{th}, R^{th} terms respectively. The common differences are the reciprocals of each other.

24. 9 or $-10\frac{2}{3}$.

V.—Page 36.

1. 96. 2. $\frac{7}{125}$. 3. $60\frac{2}{3}$. 4. $\frac{248}{15}\left\{\left(\frac{5}{2}\right)^{12} - 1\right\}$

5. $\frac{2}{3}\left\{1 - \left(-\frac{1}{3}\right)^n\right\}$. 6. $1 - \dfrac{1}{2^{10}}$ 7. 2.

8. $\frac{3}{5}$. 9. $5\frac{5}{11}$. 10. $7\frac{1}{9}$. 11 $\frac{6}{11}$.

12. $\dfrac{a^3b^2}{ab+1}$. 13. $\dfrac{(a-b)(a+b)^2}{a+b-1}$. 14. $\frac{1}{4}(11+5\sqrt{5})$.

15. $2\frac{1}{4}$. 16. $2\frac{5}{7}\frac{6}{2}$. 17. $\frac{1}{4}(\sqrt{3}+1)^3$.

18. $\frac{5}{16}$. 19. $a = -\frac{513}{341}$. 20. $a^{\frac{1}{3}}b^{\frac{2}{3}}$.

23. $a = \frac{2 s_1 s_2}{s_1^2 + s_2}, r = \frac{s_1^2 - s_2}{s_1^2 + s_2}$. 35. $2(2^n - 1) + \frac{1}{2} n (n+1)$.

VI.—Page 41.

1. $\frac{8}{6}1$, $\frac{4}{5}\frac{1}{7}$, $1\frac{8}{5}\frac{1}{7}$, &c. 2. $\frac{2}{1}1$, $\frac{4}{2}\frac{3}{3}$, $\frac{7}{4}$, &c.

3. -6, ∞, 6, 3 and 1, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{2}{3}$.

5. The common difference of the A. P. is $\frac{Q - P}{PQ(p - q)}$.

6. $\frac{2ac - bc - ab}{2b - a - c}$.

10. $\frac{2}{1}\frac{0}{2}\frac{}{3}(4 - \sqrt{7})$, $\frac{1}{4}\frac{5}{1}$, $\frac{2}{1}\frac{0}{2}\frac{}{3}(4 + \sqrt{7})$.

VII.—Page 63.

7. $126 a^5 b^4$, $126 a^4 b^5$. 8. $252 (15x)^5$. 9. $-20 a^6 b^6$.

10. $2^n \cdot \frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{\lfloor n}$. 11. $\frac{135}{8} x^2$, $\frac{45}{32} x^3$.

12. $-35 \cdot 4^4 \cdot 3^3 a^2 b$, $35 \cdot 4^3 \cdot 3^4 \cdot a^{\frac{3}{2}} b^{\frac{4}{3}}$.

13. $\frac{1}{3^{r+\frac{1}{3}}} \cdot \frac{1 \cdot 6 \cdot 11 \ldots (5r - 4)}{\lfloor r} \cdot (5x)^r$.

14. $\frac{\lfloor n}{\lfloor r \quad \lfloor n - r} a^{n-r} b^r x^{n-2r}$. 15. $a^{-\frac{1}{p}(r+1)} b^{\frac{r}{p}}$.

21. 1st term. 22. 1st term.

VIII.—Page 72.

1. $\frac{1}{2} y + \frac{5}{8} y^2 + \frac{2}{1}\frac{5}{6} y^3 +$ &c.

2. $1 + x - 2x^3 - 5x^4 -$ &c.

3. $1 + x + \frac{1}{3} x^2 - x^3 +$ &c.

4. $5x + 5x^2 + 35x^3 +$ &c. 5. $\frac{1}{6} n (n+1)(2n+1)$.

7. $\dfrac{3}{x+3} - \dfrac{2}{x+5}$.

8. $\frac{1}{1\frac{1}{4}}\left\{\dfrac{1}{x^2-6} - \dfrac{1}{x^2+8}\right\}$.

9. $\dfrac{1}{2(x+1)} - \dfrac{1}{x+2} + \dfrac{1}{2(x+3)}$.

10. $\dfrac{1}{x} - \dfrac{1}{(x+1)^2} - \dfrac{2}{x+1} + \dfrac{x}{1+x+x^2}$.

11. $\dfrac{1}{3x-1} + \dfrac{2}{x-1} - \dfrac{1}{x}$.

12. $\dfrac{1}{1+x} - \dfrac{x-2}{1-x+x^2}$.

14. $\dfrac{1}{(a-b)(a-c)(x+a)} + \dfrac{1}{(b-a)(b-c)(x+b)}$
$+ \dfrac{1}{(c-a)(c-b)(x+c)}$.

15. $\dfrac{a}{(a-b)(c-a)(x+a)} + \dfrac{b}{(b-c)(a-b)(x+b)}$
$+ \dfrac{c}{(c-a)(b-c)(x+c)}$.

16. $\dfrac{a^2}{(a-b)(a-c)(x+a)} + \dfrac{b^2}{(b-c)(b-a)(x+b)}$
$+ \dfrac{c^2}{(c-a)(c-b)(x+c)}$.

17. $\frac{1}{7}\left\{\dfrac{1}{x+3} - \dfrac{x}{x^2+3x+7}\right\}$.

18. $\dfrac{a^2-ma+n}{(a-b)(a-c)(x+a)} + \dfrac{b^2-mb+n}{(b-a)(b-c)(x+b)}$
$+ \dfrac{c^2-mc+n}{(c-a)(c-b)(x+c)}$.

X.—Page 90.

1. 16. 2. 111. 3. $1\frac{5}{3}1\frac{7}{0}1\frac{5}{7}x$.

4. -84600. 5. $12\,a_0^2a_1a_3 + 6\,a_0^2a_2^2 + 12\,a_0a_1^2a_2 + a_1^4$.

6. $\frac{1}{2}\frac{9}{7}$. 7. 420. 8. 10.

9. 840. 10. $6a_0^2a_3^2 + 24\,a_0a_1a_2a_3 + 4\,a_1^3a_2$.

11. 0. 12. $\frac{16.15.14.13}{1.2.3.4}$.

13. $1 + x + \frac{2}{3}x^2 - \frac{2}{3}x^3 - \frac{1}{9}x^4$.

14. $1 + \frac{7}{2}x^2 + \frac{1}{2}x^3 - \frac{4}{8}\frac{9}{}x^4$.

XI.—Page 99.

1. £350 $(1{\cdot}035)^{12}$. 2. $n = \frac{\log 2}{\log 1{\cdot}05}$.

3. £1250 $\{(1{\cdot}04)^{10} - 1\}$. 4. $\frac{150{\cdot}76}{1 - (1{\cdot}04)^{-35}}$.

5. $6\,\frac{6}{1}\frac{9}{6}$ months. 6. $n = \frac{\log 5}{\log 1{\cdot}06}$.

7. $6\frac{1}{4}$. 8. See Ex. 2, page 99.

9. $n = 20 . \frac{\log 6}{\log e}$, reckoning 5 per cent. per annum.

10. $96\,(1{\cdot}06)^5$. 11. $\frac{dr\mathrm{R}^a}{\mathrm{R}^c - 1}$.

12. $\frac{\log a - \log (a - br)}{\log (1 + r)}$.

XII.—Page 111.

1. $2 + \frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{19}$. 5. See Art. 72.

12. $x = 7, 12, 17,$ &c.
$y = 2, 5, 8,$ &c.

XIII.—Page 121.

1. $3 + \cfrac{1}{6+}\ \cfrac{1}{6+\ \ldots.}$ 2. $4 + \cfrac{1}{8+}\ \cfrac{1}{8+\ \ldots.}$

3. $4 + \cfrac{1}{1+}\ \cfrac{1}{2+}\ \cfrac{1}{3+\ \ldots.}$ 4. $5 + \cfrac{1}{2+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{2+\ \ldots.}$

5. $1 + \cfrac{1}{2+}\ \cfrac{1}{1+}\ \cfrac{1}{6+\ \ldots.}$ 6. $2 + \cfrac{1}{1+}\ \cfrac{1}{8+\ \ldots.}$

7. $3 + \cfrac{1}{2+}\ \cfrac{1}{3+}\ \cfrac{1}{10+\ \ldots.}$ 8. $3 + \cfrac{1}{1+}\ \cfrac{1}{2+\ \ldots.}$

9. $2 + \cfrac{1}{2+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{2+\ldots.}$ $3 + \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{2+}\ \cfrac{1}{1+\ \ldots.}$

10. $1 + \cfrac{1}{1+}\ \cfrac{1}{12+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{2+\ \ldots.}$

11. $2 + \cfrac{1}{3+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{3+\ \ldots.}$, $\cfrac{1}{4+}\ \cfrac{1}{1+}\ \cfrac{1}{1+}\ \cfrac{1}{3+}\ \cfrac{1}{1+\ \ldots.}$

12. $\cfrac{1}{1+}\ \cfrac{1}{2+}\ \cfrac{1}{5+}\ \cfrac{1}{2+}\ \cfrac{1}{1+\ \ldots.}$, $-\cfrac{1}{2+}\ \cfrac{1}{5+}\ \cfrac{1}{2+}\ \cfrac{1}{1+}\ \cfrac{1}{2+\ldots.}$

13. $5 + \cfrac{1}{1+}\ \cfrac{1}{3+}\ \cfrac{1}{1+}\ \cfrac{1}{3+\ \ldots.}$,

$1 + \cfrac{1}{4+}\ \cfrac{1}{4+}\ \cfrac{1}{1+}\ \cfrac{1}{3+}\ \cfrac{1}{1+\ \ldots.}$

14. $2a \pm \left(a + \cfrac{1}{2a\ +\ \ldots.}\right)$. 15. $\sqrt{26}$.

19. $\cfrac{9\ +\ \sqrt{15}}{7}$.

20. $2, \frac{5}{2}, \frac{7}{3}, \frac{12}{5}$, &c.; $1, 2, \frac{25}{13}, \frac{37}{14}$; $2, \frac{7}{3}, \frac{9}{4}, \frac{16}{7}$.

23. No.

XV. Page 146.

1. $\cfrac{1 + 5x}{1 - 3x + 2x^2}$. 2. $\cfrac{1 + 2x}{1 - x + 2x^2}$.

3. $\dfrac{2+5x}{1-4x+7x^2}$.

4. $\dfrac{3-7x}{1-6x+3x^2}$.

5. $\dfrac{9-2x}{1+x+7x^2}$.

6. $\dfrac{6-2x+x^2}{1-3x+4x^2-x^3}$.

7. $\Sigma = \dfrac{3-4x}{(1-2x)(1+5x)}$.

8. $\Sigma = \dfrac{6-3x}{(1+7x)(1-2x)}$.

9. $\Sigma = \dfrac{1-2x}{(1+x)(1+2x)(1-3x)}$.

10. $\Sigma = \dfrac{1-4x}{(1-3x)(1-x)(1+x)}$.

11. $\Sigma = \dfrac{2-3x}{(1-x)(1+5x)}$.

12. $\Sigma = \dfrac{5+3x}{(1-3x)(1+2x)}$.

13. $\Sigma = \dfrac{3+5x+3x^2}{(1-x)^3}$.

14. $\Sigma = \dfrac{4+5x+x^2}{(1-x)(1-2x)(1-3x)}$.

15. $1-3x+3x^2-x^3$.

16. $\dfrac{a-(a-b)x}{(1-x)^2}$.

19. $\frac{1}{4}n(n+1)(n+2)(n+3)+n(n+1)(n+2)+\frac{3}{2}n(n+1)$.

20. $\frac{1}{4}\{(x+n)(x+n+1)(x+n+2)(x+n+3)-x(x+1)(x+2)(x+3)\}$

21. $\frac{1}{4}\left\{\dfrac{23}{120}-\dfrac{1}{(n+2)(n+4)}-\dfrac{1}{(n+3)(n+5)}\right\}$.

22. $\frac{5}{6}-\dfrac{3n+5}{(n+2)(n+3)}$.

23. $\dfrac{n}{3}(n^2+2)$.

24. $\dfrac{1}{2b}\left\{\dfrac{1}{(a+b)(a+2b)}-\dfrac{1}{(a+\overline{n+1}.b)(a+\overline{n+2}.b)}\right\}$.

25. $\dfrac{n}{3}(4n^2-1)$.

26. $n^2(2n^2-1)$.

27. $1-\dfrac{1}{\lfloor n+1}$.

28. $1 + \dfrac{4}{\lfloor n+2} - \dfrac{3}{\lfloor n+1}$. 29. $\dfrac{1}{n}$.

XVI.—Page 156.

1. 3010, 1443. 2. 744. 3. 2672.
4. 9. 5. 12. 6. 7.
7. 117·7. 8. e 41. 9. e 25·33.
10. 15. 11. 222. 12. $\frac{3}{5}\frac{2}{5}$.

16. 3^7, 1 in one scale, and 3^6, 3^5, 3^4 in the other.

19. $p_1^2 = 4 p_0 p_2$.

XVII.—Page 171.

16. (i.) one ; (ii.) equal to G.C.M. ; (iii.) may be reduced to (i.) and (ii.).

Miscellaneous Examples.—Page 179.

3. $\frac{1}{8}(9 \pm \sqrt{833})$. 6. Transpose, cube, &c.

7. 4, 9. 10. According as it is beyond or between 3 and 1; $2\sqrt{2} - 3$.

19. $(1 + a_0)(1 + a_1) \ldots . (1 + a_n)$.

26. $\dfrac{a^2 - ma + n}{(b-a)(c-a)} \cdot \dfrac{1}{x - a} + \&c.$

28. The rationalized expression is $a^5 - b^6$; $5 - 2\sqrt{3}$.

31. 12789·8 sq. met. 35. $\dfrac{1}{3+}\ \dfrac{1}{3+}\ \dfrac{1}{9+}\ \dfrac{1}{2+\ldots}$.

36. 301·9. 37. $\dfrac{a}{x}$.

43. $\dfrac{B_{2n}}{\lfloor 2n}$, where B_1, B_2, &c., have the following relation :

$$rB_1 + \frac{r(r-1)}{1\,.\,2}B_2 + \ldots . + \frac{r(r-1)}{1\,.\,2}B_{r-2} + rB_{r-2} = 0.$$

44. £3600.

45. $x = y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{7}{15}y^7 + \&c.$

46. $x = 8, y = 6.$ 49. $\dfrac{1}{(a-1)(1+x)}.$

PLANE TRIGONOMETRY.

I.—Page 200.

8. $\sin 7°\frac{1}{2} = \sqrt{\dfrac{2\sqrt{2} - \sqrt{3} - 1}{4\sqrt{2}}}$, $\sin 22°\frac{1}{2} = \sqrt{\dfrac{2-\sqrt{2}}{2}}$,

$\sin 37°\frac{1}{2} = \sqrt{\dfrac{2\sqrt{2} - \sqrt{3} + 1}{4\sqrt{2}}}.$

12. $\cos^{-1}\dfrac{x}{a} - \alpha = \cos^{-1}\dfrac{y}{b} - \beta.$

II.—Page 218.

1. ·01745, ·015705, ·000291, ·00000485.

2. 27° 15′; 10° 15′ 3. 78·54.

9. $2\theta = n\pi + (-1)^n \dfrac{\pi}{4}.$ 10. $\theta = 2n\pi \pm \dfrac{\pi}{3}$, $\theta = \frac{1}{2}n\pi.$

11. $4\theta = 2n\pi \pm \dfrac{\pi}{3}$, $\theta = n\pi.$

12. $(n \pm n)\theta = (4r \pm 1)\dfrac{\pi}{2}.$

13. $\theta = (6n \pm 1)\dfrac{\pi}{3}$, or $(4n \pm 1)\dfrac{\pi}{2}.$

14. $5\theta = n\pi \mp 2\theta.$

15. $\tan\frac{1}{2}(n\pi + \theta)$ has different values according as n is *even* or odd.

III.—Page 232.

1. Put $b = a \tan^2 \theta$, then $a + b = a \sec^2 \theta$, &c.

2. $5\theta = 2n\pi \pm 2\theta$.

IV.—Page 244.

1. $\tan\theta = \dfrac{a \sin\alpha \sin(\beta - \gamma) - b \sin\gamma \sin(\alpha - \beta)}{a \cos\alpha \sin(\beta - \gamma) - b \cos\gamma \sin(\alpha - \beta)}$.

2. $\frac{1}{2}(a + l \cot\theta)$.

3. See Ex. 3, page 243.

4. $\dfrac{\cot\alpha_2 - \cot\beta_2}{\cot\alpha_1 - \cot\beta_1} . h.$

5. $\left\{ \dfrac{v^2 (at_2^2 - bt_1^2)}{a - b} + ab \right\}^{\frac{1}{2}}$.

6. $h = \dfrac{l}{\sin\gamma} \sqrt{\sin^2\alpha + \sin^2\beta - 2 \sin\alpha \sin\beta \cos\gamma}$.

8. $l^2 = a^2 - 2ab$.

9. $\dfrac{a}{\sqrt{2}} \cdot \dfrac{\sin\beta + \cos\beta}{\sin(\beta - \alpha)}, \quad \dfrac{a}{\sqrt{2}} \cdot \dfrac{\sin\alpha + \cos\alpha}{\sin(\beta - \alpha)}$.

11. x is found from the equation

$$\frac{a^2}{x^2} + \frac{a}{x} = \cot\phi (\cot\theta - \cot\phi).$$

12. $\dfrac{a}{\sqrt{\operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta - 2 \operatorname{cosec}\alpha \operatorname{cosec}\beta \cos\gamma}}$.

V.—Page 275.

1. $\cos\dfrac{2n\pi}{n} + \sqrt{-1} . \sin\dfrac{2n\pi}{n}$.

2. $\cos\dfrac{(2n+1)\pi}{n} + \sqrt{-1} \sin\dfrac{(2n+1)\pi}{n}$, where m may have any value from 0 to $n - 1$.

18. $-1, \cos\dfrac{\pi}{3} \pm \sqrt{-1} \sin\dfrac{\pi}{3}, \cos\dfrac{2\pi}{3} \pm \sqrt{-1} \sin\dfrac{2\pi}{3}$.

SPHERICAL TRIGONOMETRY.

II.—Page 350.

10. $A = 71° 47' 45''$, $b = 138° 42' 35''$, $c = 110° 7' 23''$.

11. $A = 126° 58' 51''$, $B = 82° 27' 22''$, $c = 95° 43' 24''$.

12. $a = 158° 10' 3''$, $b = 41° 38' 22''$, $A = 108° 16' 32''$.

13. $A = 123° 40' 40''$, $B = 50° 15' 41''$, $a = 136° 8' 50''$.

14. $a = 31° 11' 47''$, $b = 136° 51' 56''$, $c = 128° 37' 32''$.

15. Impossible.

16. $41° 20' 44''$.

17. $60°$.

18. $\sin \lambda = \dfrac{\sin \delta}{\sin \alpha}$.

19. Impossible.

20. $a = 122° 16' 6''$.

21. $A = 25° 39' 37''$, $B = 79° 28' 39''$.

22. $a = 64° 46' 27''$, $b = 52° 37' 57''$.

23. $\sin b \,.\, \sin A > \sin a$.

24. $b = 24° 56' 40''$.